Ref.TH.2772-CERN

INEQUALITIES ON HEAVY QUARK-ANTIQUARK SYSTEMS

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A B S T R A C T

We obtain inequalities and estimates on the radii and kinetic energies of heavy quark-antiquark systems in the framework of non-relativistic quantum mechanics. We also establish theorems on the mass dependence of the kinetic energy. These are used to derive bounds on the mass difference between the c quark and the b quark from experimental data on the J/ψ and T systems. We also obtain tight upper and lower bounds on the masses of mixed flavour systems, on the mass of the toponium as a function of the t mass, and on the energy gap between a hidden flavour mass and the threshold for flavour pair production.

Ref.TH.2772-CERN
2 November 1979

INTRODUCTION

The description of the $J/\psi, \psi'$ and related states, as well as the family of T states, as non-relativistic heavy quark-antiquark systems, has been a tremendous success, even if relativistic effects may introduce some quantitative but not qualitative changes into the picture. The most disturbing experimental feature, the possible existence of an η_c with mass 2.38 GeV decaying predominantly into two photons, has been eliminated, and a state of mass 2.97 GeV¹⁾ has been found instead.

Though, as we said, the potential model description is very satisfactory, we think it is desirable to obtain results which do not depend on the specific form of the potential but rather to use directly present experimental facts to make predictions. One way of course, is the one which has been followed by Quigg, Rosner and Thacker which consists in reconstructing without prejudice the potential by the inverse scattering method²⁾ and use later on this potential to make predictions. However, this method has its limitations: one must use the information from the $\ell = 0$ states and it is only a posteriori that the correctness of the $\ell = 1$ and 2 energies can be checked; the knowledge of the wave functions at the origin obtained from the leptonic width via the Weisskopf-Van Royen formula³⁾ is essential while we know⁴⁾ that this formula may have to be corrected by an appreciable factor due to relativistic effects; finally, there is the fact that, because only a limited number of levels is available, the potential obtained is unavoidably non-confining. So if one wants to make predictions on different quark systems, assuming flavour independence of the forces for which we have already good evidence⁵⁾, it might be desirable to short-circuit if possible, the construction of the potential even if, in the end, the results turn out to be the same. This is what we try to do here.

Some of the results we present here are valid for completely arbitrary potentials. However, most of the time we shall restrict ourselves to potentials which satisfy some general conditions:

$$(A): \left(\frac{d}{dr}\right)^3 r^2 V \geq 0, \quad \lim_{r \rightarrow 0} r \left[V + \frac{1}{2} r \frac{dV}{dr} \right] \leq 0 \quad (1)$$

which is satisfied in particular by superpositions of attractive power potentials not more singular than the Coulomb potential:

$$\int_{-1}^{\infty} \varepsilon(\alpha) \rho(\alpha) r^\alpha d\alpha, \quad \rho \geq 0, \quad \varepsilon(\alpha) = \alpha/|\alpha|. \quad (2)$$

$$(B): \frac{d}{dr} \frac{1}{r} \frac{d}{dr} \left[V(r) + \frac{1}{2} r \frac{dV}{dr} \right] < 0. \quad (3)$$

(A) $\cap$ (B) is satisfied by

$$V = \int_{-1}^{+2} \varepsilon(\alpha) \rho(\alpha) r^\alpha d\alpha, \quad \rho \geq 0, \quad (4)$$

i.e., potentials rising not faster than the harmonic oscillator well.

$$(C): V'' < 0, \quad \text{i.e. } V \text{ concave}. \quad (5)$$

(A) $\cap$ (C) is satisfied by

$$V = \int_{-1}^{+1} \varepsilon(\alpha) \rho(\alpha) r^\alpha d\alpha. \quad (6)$$

The motivation for restricting the potential to (A) $\cap$ (C) or even (6) is that one expects the short range attraction due to one gluon exchange and possibly attenuated by asymptotic freedom to be at most singular like $1/r$, and on the other hand, the measured leptonic widths of the J/ψ and ψ' and T and T' favour a concave potential⁶⁾ even if some relativistic corrections are made⁷⁾.

The paper is organized as follows. In section 2 we derive inequalities on the mean square radius of a $Q\bar{Q}$ system and on the "conjugate" quantity, the mean kinetic energy. We also derive more realistic estimates, based on the knowledge of the first energy levels.

Section 3 is devoted to general properties of Hamiltonians with fixed potential and variable mass.

Section 4 is devoted to bounds on quark and hadron masses.

Section 5 contains a discussion of spin effects.

2. BOUNDS ON THE RADIUS AND THE KINETIC ENERGY

We remind our notation: a state of given angular momentum is labelled by the number of zeros of its reduced radial wave function including the origin. We begin with the ordering of the mean square radii and refer to Appendix 1 for the proofs

Theorem 1

$$\langle r^2 \rangle_{1S} < \langle r^2 \rangle_{1P} < \langle r^2 \rangle_{2S} \quad (7)$$

The first inequality holds for any potential. The second if conditions (A) and (B) are satisfied.

Remark 1

Not all potentials with $E_{1P} < E_{2S}$ (implied by condition (A)⁸⁾) have $\langle r^2 \rangle_{1P} < \langle r^2 \rangle_{2S}$. A counterexample is the infinite square well, in which what dominates is the repulsion of the P waves from the origin.

Remark 2

We believe that

$$\langle r^2 \rangle_{1S} < \langle r^2 \rangle_{2S} \quad (8)$$

should hold for any monotonous increasing potential including the square well. However we have not succeeded in doing that. In Appendix I we find conditions for which this is true, which contain in particular all attractive power potentials up to r^3 .

Now we get bounds on $\langle r^2 \rangle$ in terms of measurable quantities. This is easily done with the help of the Thomas-Reiche-Kuhn (TRK) sum rule and, in fact for the 1S state the result is already quoted in Ref. 9).

Theorem 2

$$\langle r^2 \rangle_{1S} \leq \frac{3}{m_q(E_{1P} - E_{1S})} \quad (9)$$

where m_q is the common mass of the quark-antiquark system, or, in the general case, 2 times the reduced mass. For the harmonic oscillator (9) becomes an equality because the only non-vanishing matrix element of x , in the TRK sum rule is $\langle 1S|x|1P \rangle$. The result for the 1P state is

Theorem 3

$$\frac{2}{m_q(E_{1P} - E_{1S})} \leq \langle r^2 \rangle_{1P} \leq \frac{3 + \frac{E_{2S} - E_{1S}}{E_{1P} - E_{1S}}}{m_q(E_{2S} - E_{1P})} \quad (10)$$

Here we give the inequality for the case $E_{1P} < E_{2S} < E_{1D}$, for which conditions (A) and (B) are sufficient^{8),10)}. Again, the right-hand side of (10) becomes an equality for the harmonic oscillator, while the left-hand side is saturated by a narrow infinite well at a distance from the origin. The proofs are in Appendix 1.

We turn now to the kinetic energy which is the conjugate quantity of $\langle r^2 \rangle$. To get a lower bound on $\langle p^2 \rangle$ we can use the "uncertainty" relation⁹⁾

$$\langle r^2 \rangle \langle p^2 \rangle \geq 9/4, \quad (11)$$

saturated by the ground state of the harmonic oscillator. Hence we get

Theorem 6

$$\langle T \rangle_{1S} \geq 3/4 (E_{1P} - E_{1S}), \quad (12)$$

where the equality sign holds for the harmonic oscillator. One can also get

Theorem 7

$$\langle T \rangle_{1P} \geq 5/4 (E_{1D} - E_{1P}). \quad (13)$$

This theorem is proved in Appendix 1. Again the inequality is saturated by the harmonic oscillator potential. There are probably generalizations to higher angular momenta.

The question is now to know how good are inequalities (9), (10), (12) and (13). For potentials not too far from the harmonic oscillator they are excellent, i.e., their left-hand side and right-hand side are close to one another. For instance, we find

for $V = \lambda r$

for $V = \lambda r^3$

$$\frac{m_0}{3} (E_{IP} - E_{IS}) \langle r^2 \rangle_{IS} = 0.994 \quad 0.998$$

$$\frac{4}{3} \frac{\langle T \rangle_{IS}}{E_{IP} - E_{IS}} = 1.016 \quad 1.007$$

and

$$\frac{4}{5} \frac{\langle T \rangle_{IP}}{E_{ID} - E_{IP}} = 1.011 \quad 1.005,$$

which is excellent. However, for potentials singular at the origin the situation changes somewhat. For the Coulomb potential we have a factor $4/3$ between the right-hand side and the left-hand side of (9) and a factor $16/9$ between the left-hand side and the right-hand side of (12) and a factor $36/25$ for (13). We have tried, and, as we shall see succeeded in replacing (9), (12) and (13) by estimates which have been constructed by fitting the Coulomb case and respecting the extremal situation of the harmonic oscillator. These are

$$\langle r^2 \rangle_{IS} \simeq \frac{3}{m_0 (E_{IP} - E_{IS})} \left[1 - \frac{1}{4} C \right] \quad (14)$$

$$\langle T \rangle_{IS} \simeq \frac{3}{4} (E_{IP} - E_{IS}) \left[1 + \frac{7}{9} C \right] \quad (15)$$

$$\langle T \rangle_{IP} \simeq \frac{5}{4} (E_{ID} - E_{IP}) \left[1 + \frac{11}{27} C \right] \quad (16)$$

where

$$C = \left(\frac{E_{2S} + E_{IS} - 2E_{IP}}{E_{2S} - E_{IS}} \right)^2 \quad (17)$$

is positive, vanishes for the case of the harmonic oscillator and is unity for the Coulomb potential. We have carried numerous tests of (14), (15), (16) with first of all power potentials r^α with $-1 < \alpha < +\infty$, superpositions of the form $-A/r + Br$, $A, B > 0$, etc., i.e., what we believe to be a representative (A). We found that estimates (14), (15), (16) are true to within a few per mille. We feel therefore that it is justified to use these estimates in applications to $Q\bar{Q}$ systems which will be presented in section 4. An implication of formula (15) is, for potentials satisfying condition (A) which guarantees $E_{1P} < E_{2S}$,

$$\langle T \rangle_{1S} \leq \frac{4}{3} (E_{2S} - E_{1S}) \quad (18)$$

In Appendix 1 we give an explicit proof of (18) for

$$1) \quad V = - \int_{-1}^0 \rho(\alpha) r^\alpha d\alpha \quad \rho \geq 0$$

$$2) \quad V = \int_2^\infty \rho(\alpha) r^\alpha d\alpha \quad \rho \geq 0$$

$$3) \quad V = -\frac{c}{r} + \lambda \delta V \quad , \quad \lambda \text{ small,}$$

δV satisfying condition (A).

We are convinced that (18) holds for all potential satisfying condition (A).

3. GENERAL PROPERTIES OF THE MASS DEPENDENCE OF THE HAMILTONIAN

In the framework where forces between quarks are described by a flavour independent potential and spin effects are neglected, the only flavour dependent parameter entering into the Schrödinger equation for a system $Q\bar{q}$, where Q and q are quarks sufficiently heavy to be treated non-relativistically, is the reduced mass. We shall prefer to use m , twice the reduced mass, which becomes the quark mass for $Q\bar{Q}$ systems:

$$m = \frac{2 M_Q M_q}{M_Q + M_q} \quad (19)$$

The Hamiltonian becomes

$$H = \frac{\vec{p}^2}{m} + V(r), \quad (20)$$

and one sees that $1/m$ enters linearly into (20). Let us remember now that, in general, if a Hamiltonian has the form

$$H = A + \lambda B \quad (21)$$

The ground state energy $E_1(\lambda)$ is a concave function of λ ⁹⁾

$$\frac{d^2 E_1}{d\lambda^2} < 0. \quad (22)$$

Property (22) is also valid for the sum of the first n energy levels.

Here we get

Theorem 8

The ground state energy is a concave function of $1/m$, i.e.:

$$m \frac{d^2 E}{dm^2} + 2 \frac{dE}{dm} \leq 0. \quad (23)$$

Equivalently one can say that mE is a concave function of m . Noticing, then that by the Feynman-Hellmann theorem ¹¹⁾ we have

$$\frac{dE}{dm} = - \frac{\langle T \rangle}{m} \quad (24)$$

where $\langle T \rangle$ is the expectation value of the kinetic energy, we get as a corollary

Theorem 9

The product of the mass by the kinetic energy is increasing with m :

$$\frac{d}{dm} (m \langle T \rangle) \geq 0. \quad (25)$$

This property is optimal in a certain way. Indeed in the case of an infinite square well, we have precisely, for the ground state

$$m \langle T \rangle = \text{const.}$$

As we shall see in the next section, the combination of Theorems 6 and 9 both completely general gives already some useful information on the quark masses. However, by imposing some restrictions on the potential we get stronger results.

Theorem 10

$\langle T \rangle / m$ decreases with m for potentials satisfying the conditions

$$\frac{d}{dr} \left[V + r \frac{dV}{dr} \right] > 0, \quad \frac{dV}{dr} > 0. \quad (26)$$

Conditions (26) are fulfilled by potentials satisfying condition (A) and therefore by the attractive power superpositions not more singular than the Coulomb potential given by Equation (2). For the Coulomb case $T/m = \text{const.}$ The proof of this theorem, based on the virial theorem and on properties of the ground state wave functions is given in Appendix 2.

With Theorems 9 and 10 we have upper and lower bounds on the kinetic energy for a mass M once we know it for a mass m . However, (25) may not be sufficiently restrictive for our purpose. We have indications in favour of the fact that the quark-antiquark potential is concave from the leptonic widths of $J/\psi, \psi'$, $^{12)} T$ and T' $^{13), 14)}$. We have obtained

Theorem 11

$m^{1/3} T$ is increasing if

$$V'' \leq 0 \quad \text{and} \quad \frac{d}{dr} \left[V + \frac{1}{2} r \frac{dV}{dr} \right] > 0. \quad (27)$$

the second condition (27) is rather innocent. It is satisfied by all superpositions of attractive power potentials (6). In the limiting case of the linear potential we have precisely $m^{1/3} T = \text{const.}$

The proof of Theorem 11 is similar to that of Theorem 10 and is given in Appendix 2.

4. BOUNDS ON QUARK AND HADRON MASSES

1) Bounds on $M_b - M_c$

We want to exploit the experimental information on the J/ψ spectrum and the T spectrum to obtain limits on the difference between the b quark mass and the c quark mass. A somewhat different approach $^{15)}$ assuming conditions (A)

and (B) on the potential had already given the result

$$M_b - M_c > 3.28 \text{ GeV}. \quad (28)$$

Also the inverse scattering method led Quigg, Rosner and Thacker²⁾ to propose

$$M_b - M_c = 3.46 \text{ GeV}. \quad (29)$$

Here the general strategy will be the following: we have

$$\begin{aligned} M_{J/\psi} &= 2M_c + E(M_c) \\ M_\gamma &= 2M_b + E(M_b), \end{aligned} \quad (30)$$

and integrating (24)

$$E(M_c) - E(M_b) = \int_{M_c}^{M_b} \frac{T(m)}{m} dm. \quad (31)$$

We have information on $T(m)$ for $m = M_c$ and $m = M_b$ if we use the inequalities and estimates of Section 2. The theorems of Section 3 give us some control on the intermediate values.

First we start with inequalities which hold for completely arbitrary potentials and make use only of Theorems 6 and 9:

$$\begin{aligned} E(M_c) - E(M_b) &\geq M_c T(M_c) \left(\frac{1}{M_c} - \frac{1}{M_b} \right) \\ T(M_c) &\geq \frac{3}{4} [E_{IP}(M_c) - E_{IS}(M_c)]. \end{aligned} \quad (32)$$

Hence

$$M_b - M_c \geq \frac{M_\gamma - M_{J/\psi}}{2 - \frac{3}{4} \frac{E_{IP}(M_c) - E_{IS}(M_c)}{M_b}} \quad (33)$$

using experimental numbers^{12),13)}, and weighing the P states with $2J + 1$ to get the mean P state mass and then taking $M_b = 5 \text{ GeV}$ in the right-hand side of (33)

$$M_b - M_c > 3.29 \text{ GeV}. \quad (34)$$

Clearly the result is rather insensitive to the choice of M_b , and is already better than (28), which was not free from assumptions.

We can do better if we take seriously the estimate (15) for $T(M_c)$, which gives

$$T(M_c) \approx 0.37 \text{ GeV}, \quad (35)$$

and the bound (18) which gives

$$T(M_b) < 0.75 \text{ GeV}, \quad (36)$$

assuming that the potential satisfies condition (A). This latter estimate is very poor because of our ignorance of the P state energies of the T system. Condition (A) also allows us to use Theorem 10 and we shall take for granted that V is concave and use Theorem 11 to get the lower bound

$$M_b - M_c > \frac{1}{2} (M_\gamma - M_{J/\psi}) + \frac{3}{2} T(M_c) \left[1 - \left(\frac{M_c}{M_b} \right)^{1/3} \right]. \quad (37)$$

If we want an upper bound on $E(M_c) - E(M_b)$ we integrate (31) from M_c to a value M_0 using

$$T(m) \leq \frac{m}{M_c} T(M_c), \quad (38)$$

by Theorem 10, and from M_0 to M_b using

$$T(m) \leq \left(\frac{M_b}{m} \right)^{1/3} T(M_b), \quad (39)$$

by Theorem 11, where M_0 is defined by the equality of (38) and (39) (see Fig.1)

$$M_0 = \left[M_c M_b^{1/3} \frac{T(M_b)}{T(M_c)} \right]^{3/4}. \quad (40)$$

This leads to

$$M_b - M_c < \frac{1}{2} (M_\gamma - M_{J/\psi}) + \frac{1}{2} T(M_c) \left(\frac{M_0}{M_c} - 1 \right) + \frac{3}{2} T(M_b) \left[\left(\frac{M_b}{M_0} \right)^{1/3} - 1 \right] \quad (41)$$

The bounds one gets on $M_b - M_c$ in this way are slightly dependent on what is assumed for M_c which varies between 1.1 GeV and 1.65 GeV, with the value of 1.25 GeV favoured by the sum rules of the ITEP group^{16),17)}.

We give on Table 1 a list of mass differences for masses of the charmed quark varying from $M_c = 1.1$ to $M_c = 1.7$ GeV. For any given value of M_c the uncertainty on $M_b - M_c$ does not exceed ± 150 MeV. The extreme values for

$$1.1 < M_c < 1.7 \text{ GeV}$$

are

$$3.36 < M_b - M_c < 3.69 \text{ GeV} \quad (42)$$

Table 1

M_c	$M_b - M_c$ in GeV				
	Lower bound	Upper bound	fit to J/ψ and T data		
1.1	3.39	3.69	3.46 ²⁾	3.55 ²⁰⁾	
1.25	3.38	3.64	3.40 ^{16),18)}	3.42 ²¹⁾	3.53 ²³⁾
1.5	3.37	3.58	3.39 ¹⁹⁾	3.41 ²²⁾	
1.7	3.36	3.54			

It is very remarkable, but not unexpected, that the determination of Quigg, Rosner and Thacker²⁾ for $M_b - M_c = 3.455$ GeV via the inverse-scattering method falls into these limits and that the value $M_b - M_c = 3.4$ GeV of the ITEF group¹⁶⁾⁻¹⁸⁾ obtained by sum rules from Q^2 duality, fits very well. Furthermore models giving a reasonable fit of present data (we discard models using obsolete values for the T mass) also agree.

We feel that these limits are very generous because they correspond to taking rather extreme cases for the potential and also an extreme value for the upper bound of the kinetic energy of the T systems. The upper bound on $M_b - M_c$ is obtained in a situation where $T(m)$ has a maximum somewhere between $m = M_c$ and $m = M_b$. We have reasons to believe that at least for potentials of the type (2) this cannot happen. These reasons are explained in Appendix 2. When we put all this together we incline to believe that the Quigg, Rosner and Thacker²⁾ value is very close to the truth, in spite of the fact that their potential is not confining.

It is clear that a decisive information will be the knowledge of the energy of the P state of the T system, and this we may hope to get soon from CESR.

2) Bounds on the mass of toponium

The procedure of the previous chapter can be applied to any heavy quarks, thus also to the top quark but unfortunately toponium which is the input is not found yet. What we do instead is starting from the opposite direction. We use the above inequalities for bounds on the mass of toponium, M_t , as a function of M_c which certainly is of interest for future experiments.

Since the P states of the T family are not measured yet we can only assign the kinetic energy T_T the upper limit (18) which corresponds to a pure Coulomb potential and gives $T_T < 0.75$ GeV [Eq. (36)]. This at least provides us the opportunity to integrate the lower bound from M_b on which is much more restrictive than to start from M_c .

$$M_Y + (M_t - M_b) \left[2 - \frac{T_T}{M_b} \right] < M_\Phi \quad (43)$$

For the upper bound we still have to begin with M_c and obtain

$$M_\Phi < M_{J/\psi} + 2(M_t - M_c) - 3 \frac{T_{J/\psi}}{M_c} \left[1 - \left(\frac{M_c}{M_t} \right)^{1/3} \right] \quad (44)$$

The numerical results are drawn in Fig. 2. The bounds are quite narrow and can be even improved, especially the lower bound, once the P states of the T family are measured.

3) Bounds on the energy splitting $M_{\text{threshold}} - M_{Q\bar{Q}}$.

The continuum for a heavy $Q\bar{Q}$ system begins where the quarks can dissociate into heavy meson pairs $Q\bar{q} + \bar{Q}q$. Although a system heavy plus light quarks cannot be expected to obey a non-relativistic description, its dependence on M_Q is very simple^{24),25)}

$$M_{Q\bar{q}} = M_Q + A \quad (45)$$

where A is a constant independent of the heavy quark which behaves like a fixed centre. The energy gap from the first resonance to continuum

$$\Delta_Q = 2 M_{Q\bar{q}} - M_{Q\bar{Q}} = 2A - E_{Q\bar{Q}} \quad (46)$$

is, when multiplied with M_Q , a convex function of M_Q according to Theorem 8. Benefiting from this fact we minimize by a straight line which is fixed by the values

$$\Delta_c = 0.645 \text{ GeV} \quad \text{from} \quad M_{c\bar{u}} = M_D = 1.87 \text{ GeV}^{26)}$$

and

$$\Delta_b = 1.14 \text{ GeV} \quad \text{from} \quad M_{b\bar{u}} = M_B = 5.3 \text{ GeV}^{27)}$$

This implies the lower bound on top-threshold

$$1.33 \text{ GeV} - \frac{0.85}{M_t} < \Delta_t. \quad (47)$$

Top-quark masses below 15-16 GeV are excluded according to PETRA results²⁸⁾ thus

$$1.28 \text{ GeV} < \Delta_t. \quad (48)$$

For an upper bound on Δ_t we start from Eq. (24) and minimize the right-hand side by its value at M_b

$$\frac{dE}{dm} = -\frac{T(m)}{m} > -\frac{T(M_b)}{M_b} . \quad (49)$$

Integration up to the top-quark,

$$E_{t\bar{t}} - E_{b\bar{b}} > -T(M_b)\left(\frac{M_t}{M_b} - 1\right) , \quad (50)$$

together with the difference

$$\Delta_t - \Delta_b = -(E_{t\bar{t}} - E_{b\bar{b}}) , \quad (51)$$

finally yields the upper bound

$$\Delta_t < \Delta_b + T_\gamma \left(\frac{M_t}{M_b} - 1 \right) . \quad (52)$$

Numerically

$$\Delta_t < \begin{matrix} 2.96 & \text{for } M_t = 16 \text{ GeV} \\ 3.60 & \text{for } M_t = 20 \text{ GeV} . \end{matrix} \quad (53)$$

Again, we want to emphasize that this bound will improve once the upsilon P states are measured and will be saturated once M_t is high enough. This comes from the fact that high M_t systems behave more and more Coulombic and the pure Coulomb potential generates this upper bound.

4) Bounds on systems with unequal quark masses

Such systems are heavy $Q\bar{q}$, $\bar{Q}q$ mesons which originate from the dissociation of the heavy $Q\bar{Q}$ systems once enough energy is available. In deriving bounds we assume the lighter quark q ($M_q < M_Q$) to be still heavy enough to justify the non-relativistic approach.

We begin with

Theorem 12

$$\frac{1}{2} (E_{Q\bar{Q}} + E_{q\bar{q}}) < E_{Q\bar{q}} , \quad (54)$$

for any potential and for ground state energies.

This is a direct consequence, of Theorem 8 on concavity since $2/m = 1/M_q + 1/M_{\bar{q}}$. This immediately leads, by adding quark masses to a lower bound on $M_{Q\bar{q}}$

$$\frac{1}{2}(M_{Q\bar{q}} + M_{q\bar{q}}) < M_{Q\bar{q}}. \quad (55)$$

For an upper bound we return to the basic relation (24) once more but split its integration at $m_{Q\bar{q}}$

$$2E_{Q\bar{q}} - E_{Q\bar{q}} - E_{q\bar{q}} = - \int_{m_{q\bar{q}}}^{m_{Q\bar{q}}} \frac{T(m)}{m} dm + \int_{m_{Q\bar{q}}}^{m_{Q\bar{q}}} \frac{T(m)}{m} dm. \quad (56)$$

Applying the "Coulomb" condition (Theorem 10: $T(m)/m \searrow$) for each integral separately and adding quark masses implies

$$M_{Q\bar{q}} < \frac{1}{2}(M_{Q\bar{q}} + M_{q\bar{q}}) + \frac{1}{4}T(m_{Q\bar{q}}) \frac{(M_Q - M_q)^2}{M_Q M_q}. \quad (57)$$

Allowance for concave potentials only restricts the kinetic energy to (Theorem 11)

$$T(m_{Q\bar{q}}) < \left(\frac{m_{Q\bar{q}}}{m_{Q\bar{q}}} \right)^{1/3} T(m_{Q\bar{q}}), \quad (58)$$

which finally leads to the result

$$M_{Q\bar{q}} < \frac{1}{2}(M_{Q\bar{q}} + M_{q\bar{q}}) + \frac{1}{4}T(M_Q) \frac{(M_Q - M_q)^2}{M_Q M_q} \left(\frac{M_Q + M_q}{2M_q} \right)^{1/3}. \quad (59)$$

Alternatively one can proceed as in Chapter 4.2 to obtain

$$M_{Q\bar{q}} < M_{q\bar{q}} + (M_Q - M_q) - 3T(M_q) \left[1 - \left(\frac{M_Q + M_q}{2M_Q} \right)^{1/3} \right]. \quad (60)$$

Both bounds are rather sensitive to the difference in the quark masses which for numerical calculations should be known quite accurately.

Let us begin as an experiment with the $c\bar{s}$ system, the F, F^* mesons, whose masses are measured quite accurately $M_F = 2.03 \pm 0.02 \text{ GeV}$ ²⁶⁾ while the F^* , which is 1^{--} , is at $2.14 \pm 0.06 \text{ GeV}$ ²⁶⁾. Using rather high values for the strange quark mass $M_s \approx 0.45 \text{ GeV}$ in order to keep the non-relativistic approach reliable we get using the ϕ and J/ψ masses

$$2.05 < M_{c\bar{s}} < 2.20 \text{ GeV}, \quad (61)$$

which agrees with the data very nicely.

For the $b\bar{s}$ system the difference in M_b and M_s is already too large to make above bounds meaningful. Here we help ourselves by the argument used already in Chapter 4.3 where $M_{Q\bar{q}} = M_Q + A$. This applied to the $b\bar{s}$ and $c\bar{s}$ systems gives

$$M_{b\bar{s}} = M_{c\bar{s}} + M_b - M_c \quad (62)$$

taking for $M_{c\bar{s}}$ the mass of the F and the bounds (42) results in

$$5.39 < M_{b\bar{s}} < 5.72 \text{ GeV} \quad (63)$$

For experimental search we favour the value

$$M_{b\bar{s}} \sim 5.5 \text{ GeV} \quad (64)$$

Finally, the $b\bar{c}$ system subjected to above bounds has its mass within

$$6.28 < M_{b\bar{c}} < 6.66 \text{ GeV} \quad (65)$$

we propose here

$$M_{b\bar{c}} \sim 6.4 \text{ GeV}. \quad (66)$$

5. A DISCUSSION OF SPIN EFFECTS

In the previous section, little or no attention has been paid to spin effects. However, these exist; for instance $M_{J/\psi} - M_{\eta_c} \approx 100 \text{ MeV}$ if we make the most likely identification for the state discovered at SLAC ¹⁾ and $M_{F^*} - M_F \approx 130 \text{ MeV}$ ²⁶⁾. Our bounds are so accurate that one cannot disregard these

spin effects, which as everybody knows are a pain in the neck for amateurs of rigour because the associated potentials are singular and strictly speaking the corresponding Hamiltonians are meaningless. However, the common belief, which we shall follow, is that this catastrophe comes from a somewhat illegal non-relativistic reduction and that one should content oneself to treat these effects in lowest order, as it was done long ago in Atomic Physics. It was difficult to take this point of view two years ago, when the splitting between the J/ψ and the η_c was believed to be 265 MeV.

What is believed to be flavour independent is the spin independent part of the potential. If we follow the previous philosophy we see that in the combination $1/4M_1^{--} + 3/4M_0^- +$ the diagonal parts of all spin dependent forces in the $L = 0$ state are eliminated. The same happens in the $L = 1, S = 1$ states for the combination $1/9 (J = 0 + 3 \cdot J = 1 + 5 \cdot J = 2)$. For the $J/\psi - \eta_c$ the hyperfine splitting, due to lowest order to a $(\vec{\sigma}_1 \cdot \vec{\sigma}_2) \delta^3(\vec{r})/M^2$ term is about 100 MeV and so

$$M_{c\bar{c}}(\text{NO SPIN}) \simeq 3.01 \text{ GeV} . \quad (67)$$

For the $b\bar{b}$ system we expect the hyperfine splitting to be much smaller. If we make the approximation of a linear potential we have $|\psi(0)|^2 \sim M$, and hence

$$M(b\bar{b}, 0^{++}) \simeq 9.43 \text{ GeV} \quad (68)$$

$$M_{b\bar{b}}(\text{NO SPIN}) \simeq 9.44 \text{ GeV} . \quad (69)$$

In this way we are led to revise our extreme limits (42) for $M_b - M_c$ as

$$3.44 < M_b - M_c < 3.77 \text{ GeV}, \quad (70)$$

for

$$1.1 < M_c < 1.7 \text{ GeV} .$$

The upper bound on the mass of the toponium will be lowered by 95 MeV, because $M_{J/\psi}$ has to be replaced by the mean $c\bar{c}$ mass while the lower bound will not change because we use the information from the T system where spin effects are very small. The mass of the $0^{++} b\bar{s}$ system can be obtained by using (70):

$$5.47 < M_{b\bar{s}}(0^{++}) < 5.80 \text{ GeV} . \quad (71)$$

The mean mass of the $b\bar{c}$ system becomes

$$6.24 < M_{b\bar{c}} < 6.62 \text{ GeV}, \quad (72)$$

and this mean mass is very close to both the 0^{++} and 1^{--} masses because of the smallness of hyperfine splitting for large masses.

6. FINAL REMARKS

We believe that it is clear now that the constraints that we have obtained, from the assumption that heavy quark systems can be described by an effective Schrödinger equation, from a few general principles and some widely accepted assumptions on the potential, are remarkably tight and predictive. Thus our results join the large amount of predictions on quarkonia already obtained by non-relativistic treatment - for a detailed discussion we refer to the review articles²⁹⁾⁻³⁵⁾. Our approach complements and consolidates the approaches in which either one makes a best fit with an adjustable potential to all existing data or one uses a systematic procedure of reconstruction of the potential. We know that as new experimental information becomes available the constraints will become correspondingly tighter, especially if we learn something about the P states of the T system.

We would like to acknowledge many stimulating questions and remarks from our colleagues, in particular Chris Michael, Dave Jackson, Margarete Krammer John Ellis and Harald Grosse.

APPENDIX 1

The main technique of the following proofs applied already successfully by Common^{7),36)}, Grosse¹⁰⁾ and one of us (A.M.)^{8),15),16)} can be found summarized in Ref. 29).

1) Proof of Theorem 1

i) left-hand side of (7)

$$\langle r^2 \rangle_{1S} = \int_0^\infty U_{1S}^2 r^2 dr, \quad (1.1)$$

$$\langle r^2 \rangle_{1P} = \int_0^\infty U_{1P}^2 r^2 dr. \quad (1.2)$$

From the Schrödinger equations

$$-U_{1S}'' + V U_{1S} = E_{1S} U_{1S} \quad (1.3)$$

$$-U_{1P}'' + \frac{2}{r^2} U_{1P} + V U_{1P} = E_{1P} U_{1P} \quad (1.4)$$

it is easy to prove

$$\frac{d}{dr} [U_{1P}/U_{1S}] > 0, \quad U_{1S} > 0.$$

Hence $U_{1P}^2(r) - U_{1S}^2(r)$ vanishes once at $r = r_0$.

$$\langle r^2 \rangle_{1P} - \langle r^2 \rangle_{1S} = \int_0^\infty (U_{1P}^2 - U_{1S}^2)(r^2 - r_0^2) dr > 0. \quad (1.5)$$

Q.E.D.

ii) right-hand side of (7): we have

$$\langle r^2 \rangle_{2S} - \langle r^2 \rangle_{1P} = \int_0^\infty r^2 (U_{2S}^2 - U_{1P}^2) dr \quad (1.6)$$

we know⁸⁾, that under condition (A) $E_{2S} > E_{1P}$, hence, using the virial theorem

$$\int_0^{\infty} (u_{2S}^2 - u_{1P}^2) \left[V + \frac{1}{2} r \frac{dV}{dr} \right] dr > 0 \quad (1.7)$$

where

$$\int_0^{\infty} (u_{2S}^2 - u_{1P}^2) dr = 0. \quad (1.8)$$

We know also⁸⁾, that $u_{2S}^2 - u_{1P}^2$ vanishes only twice. Under condition (B) $V + \frac{1}{2} r (dV/dr)$ is an increasing concave function of r^2 ; this means that one can find λ and μ , $\mu > 0$, such that

$$V + \frac{1}{2} r \frac{dV}{dr} - \lambda - \mu r^2$$

vanishes only twice; at the two zeros of $u_{2S}^2 - u_{1P}^2$. Then one gets

$$\int_0^{\infty} (u_{2S}^2 - u_{1P}^2) \left[V + \frac{1}{2} r \frac{dV}{dr} - \lambda - \mu r^2 \right] < 0, \quad (1.9)$$

and using (1.7) and (1.8) we get that (1.6) is positive, which is the desired result.

2) Sufficient conditions for $\langle r^2 \rangle_{2S} > \langle r^2 \rangle_{1S}$ [Eq. (8)]

We have obtained two disjoint conditions

i)

$$V'' < 0. \quad (1.10)$$

then it is known⁶⁾ that $|u'_{2S}(0)|^2 < |u'_{1S}(0)|^2$ and that $u_{2S}^2(r) - u_{1S}^2(r)$ vanishes only once in $0 < r < \infty$ at $r = r_0$, being positive for large r , then

$$\langle r^2 \rangle_{2S} - \langle r^2 \rangle_{1S} = \int_0^{\infty} (u_{2S}^2 - u_{1S}^2) (r^2 - r_0^2) dr > 0, \text{ Q.E.D.}$$

ii)

$$V'' > 0 \text{ and } \left(\frac{d}{dr^2} \right)^2 \left(\frac{dV}{dr} \right) < 0. \quad (1.11)$$

then we have $|u'_{2S}(0)|^2 > |u'_{1S}(0)|^2$ ⁶⁾, and hence

$$\int_0^{\infty} (u_{2S}^2 - u_{1S}^2) \frac{dV}{dr} dr > 0. \quad (1.12)$$

$u_{2S} - u_{1S}$ vanishes twice, and like in subsection 1)ii) one shows that there exist λ and $\mu, \mu > 0$ such that

$$\int_0^{\infty} (u_{2S}^2 - u_{1S}^2) \left(\frac{dV}{dr} - \lambda - \mu r^2 \right) dr < 0, \quad (1.13)$$

and using the normalization condition and (1.12) we get the desired result.

The union of conditions (1.10) and (1.11) shows that $\langle r^2 \rangle_{2S} > \langle r^2 \rangle_{1S}$ for potentials of the form $\varepsilon(\alpha)r^\alpha$, $\alpha \leq 3$.

3) Proof of theorems 2 and 3

Let $\vec{r} = (x, y, z)$ consider first

$$\begin{aligned} \langle 1S | [H, x] x | 1S \rangle &= \sum_n \langle 1S | [H, x] | n \rangle \langle n | x | 1S \rangle \\ &= \sum_n (E_{1S} - E_n) |\langle n | x | 1S \rangle|^2. \end{aligned} \quad (1.13)$$

On the other hand

$$[H, x] = \frac{1}{m} [p_x^2, x] = -\frac{2}{m} \frac{\partial}{\partial x}$$

and one easily gets

$$\langle 1S | [H, x] x | 1S \rangle = -\frac{1}{m_q}.$$

Noticing that the lowest intermediate n state possible is a $1P$ state we get

$$(E_{1P} - E_{1S}) \sum_n |\langle n | x | 1S \rangle|^2 \leq \frac{1}{m_q} \quad (1.14)$$

and since

$$r^2 = x^2 + y^2 + z^2$$

we obtain

$$\langle r^2 \rangle_{1S} \leq \frac{3}{m_q (E_{1P} - E_{1S})}. \quad (1.15)$$

Generally speaking the only states n contributing to the sum are the P states. However, in the special case of the harmonic oscillator $x|1S\rangle$ is precisely one of the three $|1P\rangle$ states. Hence it is orthogonal to all higher P states, the sum reduces to a single term and the inequality becomes an equality. The right-hand side of theorem 3 is obtained by the same technique: we have, considering the $1P$ state oriented along the x axis

$$\sum_n (E_n - E_{1P}) |\langle n|x|1P_x\rangle|^2 = \frac{1}{m} \quad (1.16)$$

i.e.,

$$\sum_{E_n > E_{1P}} (E_n - E_{1P}) |\langle n|x|1P_x\rangle|^2 = \frac{1}{m} + (E_{1P} - E_{1S}) |\langle 1S|x|1P_x\rangle|^2$$

and hence

$$\left[\inf(E_{2S} - E_{1P}, E_{1D} - E_{1P}) \right] \sum |\langle n|x|1P_x\rangle|^2 \leq \frac{1}{m} \left[2 + \frac{\inf(E_{2S} - E_{1P}, E_{1D} - E_{1P})}{E_{1P} - E_{1S}} \right]; \quad (1.17)$$

on the other hand we have

$$\sum_{E_n > E_{1P}} (E_n - E_{1P}) |\langle n|y|1P_x\rangle|^2 = 1 \quad (1.18)$$

and a similar equation for z . Hence

$$m_q \langle n^2 \rangle_{1P} \leq \frac{2}{E_{1P} - E_{1P}} + \frac{1}{E_{1P} - E_{1S}} + \frac{2}{\inf[E_{2S} - E_{1P}, E_{1D} - E_{1P}]} \quad (1.19)$$

Again the inequality is saturated by the harmonic oscillator.

The left-hand side of Theorem 3 is obtained by a completely different method. First let us prove the inequality

$$\langle r^2 \rangle_P > \frac{\int_0^\infty U_S U_P r^2 dr}{\int_0^\infty U_S U_P dr} \quad (1.20)$$

We have

$$\begin{aligned} & \int_0^\infty U_S U_P (\langle r^2 \rangle_P - r^2) dr \\ &= \int_0^\infty U_P^2 \left[\frac{U_S}{U_P}(r) - \frac{U_S}{U_P}(\sqrt{\langle r^2 \rangle_P}) \right] [\langle r^2 \rangle_P - r^2] dr, \end{aligned} \quad (1.21)$$

using the definition of $\langle r^2 \rangle_P$. u_S/u_P is a decreasing function of r and the product of the two brackets is positive, which proves (1.20). Then we use

$$(E_{1P} - E_{1S}) \int_0^\infty U_S U_P dr = 2 \int_0^\infty \frac{U_S U_P}{r^2} dr, \quad (1.22)$$

which follows from the Schrödinger equation. By Schwarz inequality

$$(E_{1P} - E_{1S}) \int_0^\infty U_S U_P dr \geq 2 \frac{\left(\int_0^\infty U_S U_P dr \right)^2}{\int_0^\infty U_S U_P r^2 dr} \quad (1.23)$$

which gives, by combination with (1.20)

$$\langle r^2 \rangle_P \geq \frac{2}{E_{1P} - E_{1S}} \quad (1.24)$$

The factor 2 cannot be improved.

4) Proofs of Theorems 6 and 7

Theorem 6 is established in the text via the uncertainty relation $\langle r^2 \rangle \langle p^2 \rangle \geq 9/4$. However, we need a direct proof if we want to make generalizations to higher states. We calculate

$$\begin{aligned} \langle n | [H, x] | 1s \rangle &= (E_n - E_{1s}) \langle n | x | 1s \rangle \\ &= -2 \langle n | \frac{\partial}{\partial x} | 1s \rangle \frac{1}{m_g} \end{aligned} \quad (1.25)$$

Hence

$$\begin{aligned} \frac{1}{m_g} \langle 1s | p_x^2 | 1s \rangle &= \frac{1}{m_g} \sum_n |\langle n | \frac{\partial}{\partial x} | 1s \rangle|^2 \\ &= \frac{m_g}{4} \sum_n (E_n - E_{1s})^2 |\langle n | x | 1s \rangle|^2 \end{aligned} \quad (1.26)$$

Comparing with (1.14) we get

$$\frac{1}{m_g} \langle 1s | p_x^2 | 1s \rangle \geq \frac{1}{4} (E_{1p} - E_{1s}) \quad (1.27)$$

and

$$\langle T \rangle_{1s} \geq \frac{3}{4} (E_{1p} - E_{1s}) \quad (1.28)$$

Notice that if $V = r^2$ the sum reduces to a single term and that if $V = r^2 + \lambda \delta V$

$$\sum_{E_n > E_{1p}} (E_n - E_{1s})^2 |\langle n | x | 1s \rangle|^2 \text{ and } \sum_{E_n > E_{1p}} (E_n - E_{1s}) |\langle n | x | 1s \rangle|^2$$

are of the order of λ^2 .

We now prove Theorem 7. Here

$$\frac{1}{m_q} \langle 1P_x | p_y^2 | 1P_x \rangle = \frac{m_q}{4} \sum_{E_n \geq E_{1D}} (E_n - E_{1P})^2 |\langle n | y | 1P_x \rangle|^2, \quad (1.29)$$

with

$$1 = \sum_{E_n \geq E_{1D}} (E_n - E_{1P}) |\langle n | y | 1P_x \rangle|^2. \quad (1.30)$$

However, from the Wigner-Eckart theorem

$$\frac{\langle 1P_x | p_x^2 + p_y^2 + p_z^2 | 1P_x \rangle}{\langle 1P_x | p_y^2 | 1P_x \rangle} = 5 \quad (1.31)$$

and we get, noticing that $E_n - E_{1P} \geq E_{1D} - E_{1P}$ in (1.30):

$$\langle T \rangle_{1P} \geq \frac{5}{4} (E_{1D} - E_{1P}) \quad (1.32)$$

which is what we want to prove.

5) Support for inequality (18)

i) Take

$$V = - \int_{-1}^0 \rho(\alpha) r^2 d\alpha$$

condition (A) is then automatically satisfied. Consider the difference

$$4E_{1P} - E_{1S} \quad (1.33)$$

and take

$$V = -\frac{1}{r} + \lambda^2 V_c(\lambda r) \quad (1.34)$$

We have

$$4E_{IP}(\lambda=0) - E_{IS}(\lambda=0) = 0$$

and¹¹⁾

$$\frac{d}{d\lambda} [4E_{IP}(\lambda) - E_{IS}(\lambda)] = \int_0^\infty (4u_{IP}^2 - u_{IS}^2) [2V_c + \lambda^2 r V_c'] dr \quad (1.35)$$

If we have

$$4E_{IP}(\lambda) - E_{IS}(\lambda) = 0$$

it follows by the virial theorem that

$$0 = \int_0^\infty (4u_{IP}^2 - u_{IS}^2) \left[-\frac{1}{r} + \lambda^2 [2V_c(\lambda r) + \lambda r V_c'(\lambda r)] \right] dr \quad (1.36)$$

and combining (1.35) and (1.36) we get

$$\begin{aligned} \frac{d}{d\lambda} [4E_{IP}(\lambda) - E_{IS}(\lambda)] = \\ \lambda \int_0^\infty (4u_{IP}^2 - u_{IS}^2) \frac{2V_c(\lambda r) + \lambda r V_c'(\lambda r) - \frac{r_0}{r} [2V_c(\lambda r_0) + \lambda r_0 V_c'(\lambda r_0)]}{1 - r_0^2 \lambda^2 [2V_c(\lambda r_0) + \lambda r_0 V_c'(\lambda r_0)]} dr, \end{aligned} \quad (1.37)$$

where we choose r_0 to be the unique zero of $4u_{1P}^2 - u_{1S}^2$. If

$$r [2V_c(r) + rV_c'(r)]$$

is negative and increasing we see that the right-hand side of (1.37) is positive. This means that $4E_{1P}(\lambda) - E_{1S}(\lambda)$ cannot vanish if it is positive for small λ . All these conditions are fulfilled if

$$V_c = - \int_{-1}^0 \rho(x) r^2 dx, \quad \rho \geq 0. \quad (1.38)$$

We conclude that

$$4E_{1P} > E_{1S}$$

and

$$4(E_{1P} - E_{1S}) > -3E_{1S} \quad (1.39)$$

Now, applying the virial theorem to potentials of the type (1.38) we get $\langle T \rangle_{1S} < -E_{1S}$ easily. So we conclude

$$\langle T \rangle_{1S} < \frac{4}{3}(E_{1P} - E_{1S}) < \frac{4}{3}(E_{2S} - E_{1S}), \quad (1.40)$$

Q. E. D.

ii) To treat the case

$$V = \int_2^\infty \rho(x) r^2 dx \quad (1.41)$$

we start from²⁹⁾

$$E_{1D} > \frac{7}{3}E_{1S} \quad (1.42)$$

Hence

$$3(E_{1D} - E_{1S}) > 4E_{1S}, \quad (1.43)$$

and, for the class (1.41) we have²⁹⁾ $\langle T \rangle_{1S} \leq E_{1S}$. Therefore

$$\langle T \rangle_{1S} < \frac{3}{4} (E_{1D} - E_{1S}). \quad (1.44)$$

However, for the class (1.41) we know¹⁰⁾ that $E_{1D} < E_{2S}$, therefore

$$\langle T \rangle_{1S} < \frac{3}{4} (E_{2S} - E_{1S}). \quad (1.45)$$

iii) We can also consider general perturbations to $-1/r$. For a pure Coulomb potential we have

$$T_{1S}(m) = \frac{4}{3} [E_{2S}(m) - E_{1S}(m)] \quad (1.46)$$

If we perturb the potential by δV we want to calculate

$$\Delta = \delta T_{1S}(m) - \frac{4}{3} [\delta E_{2S} - \delta E_{1S}] \quad (1.47)$$

Now from Eq. (24)

$$\delta T_{1S}(m) = -m \frac{d}{dm} (\delta E_{1S}) \quad (1.48)$$

Hence

$$\Delta = \int_0^\infty dr \left[2m U_{1S}(m, r) \frac{d}{dm} U_{1S}(m, r) - \frac{4}{3} [U_{2S}^2(m, r) - U_{1S}^2(m, r)] \right] \cdot \delta V(r) \quad (1.49)$$

and, using

$$\begin{aligned} U_{1s} &= 2 m^{3/2} r \exp(-mr) \\ U_{2s} &= \frac{1}{\sqrt{2}} m^{3/2} r \left[1 - \frac{mr}{2} \right] \exp\left(-\frac{mr}{2}\right), \end{aligned} \quad (1.50)$$

we get, letting $m = 1$

$$\Delta = \int_0^\infty r^2 \delta V f(r) dr \quad (1.51)$$

with

$$f(r) = -\frac{4}{3} \exp(-2r) [5 - 6r] - \frac{2}{3} \left(1 - \frac{r}{2}\right)^2 \exp(-r). \quad (1.52)$$

It can be checked that, as expected

$$\int_0^\infty r^2 f(r) dr = 0 \quad \int_0^\infty r f(r) dr = 0 \quad (1.53)$$

On the other hand it is not terribly difficult to prove, by successive multiplications of well chosen factors and differentiations that $f(r)$ has only two zeros, with $f(0) < 0$, $f(r \rightarrow \infty) < 0$. It is then easy to show that Δ is negative if $r\delta V$ is a convex function of r , using the equations (1.53). If δV is taken to satisfy condition (A), or representation (2), this is precisely the case.

APPENDIX 2

1) Preliminary remark on the 1S wave functions

Consider $u_{1S}(m, r)$ satisfying

$$-\frac{1}{m} \frac{d^2}{dr^2} u_{1S} + V(r) u_{1S} = E_{1S} u_{1S} \quad (2.1)$$

It is easy to show that

$$u_{1S}(M, r) / u_{1S}(m, r), \quad M > m,$$

is a decreasing function of r if $V(r)$ is monotonous increasing. Indeed

$$u'(M, r) u(m, r) - u'(m, r) u(M, r) = \int_0^r [(M - m)V + c] u(M, r) u(m, r) dr \quad (2.2)$$

is negative for $r < r_1$ such that $(M - m)V(r_1) + c = 0$ and for $r > r_1$ where one takes the integral from r to ∞ . Since

$$\int_0^\infty u_{1S}^2(M, r) dr = \int_0^\infty u_{1S}^2(m, r) dr \quad (2.3)$$

it follows that

$$u_{1S}^2(M, r) - u_{1S}^2(m, r)$$

has a unique zero, $r = r_0$ and that

$$|u'_{1S}(M, 0)|^2 > |u'_{1S}(m, 0)|^2. \quad (2.4)$$

2) Proof of Theorems 10 and 11

We shall take a finite mass difference and get Theorems 10 and 11 as a limiting case. In what follows we omit the symbol 1S. We have, using the virial theorem

$$T(M) - T(m) = \int_0^a [u^2(M, r) - u^2(m, r)] \left[\frac{r}{2} \frac{dW}{dr} - \frac{r_0}{2} \frac{dV(r_0)}{dr} \right] dr \quad (2.5)$$

when we have used the normalization condition. Similarly

$$E(M) - E(m) = \int_0^a [u^2(M, r) - u^2(m, r)] \left[V(r) + \frac{r}{2} \frac{dV}{dr} - \left(V(r) + \frac{r}{2} \frac{dV}{dr} \right)_{r_0} \right] dr \quad (2.6)$$

Assume now that

$$\frac{d}{dr} \left[V(r) + \frac{r}{2} \frac{dV}{dr} \right] > 0. \quad (2.7)$$

Then from the previously established properties, the integrand of (2.6) is ≤ 0 . We can write the integrand of (2.5) as the integrand of (2.6) multiplied by

$$\phi(r, r_0) = \frac{\frac{r}{2} \frac{dV}{dr} - \left(\frac{r}{2} \frac{dV}{dr} \right)_{r_0}}{V(r) + \frac{1}{2} r \frac{dV}{dr} - \left(V(r) + \frac{1}{2} r \frac{dV}{dr} \right)_{r_0}} \quad (2.8)$$

and use the mean value theorem:

$$T(M) - T(m) = (E(M) - E(m)) \phi(\bar{r}, r_0), \quad (2.9)$$

but by an extension of Rolle's theorem we have

$$\phi(\bar{r}, r_0) = \frac{\frac{d}{dr} \left(\frac{r}{2} \frac{dV}{dr} \right)}{\frac{d}{dr} \left(V(r) + \frac{r}{2} \frac{dV}{dr} \right)} \bigg|_{r=\bar{r}}, \quad (2.10)$$

$$\bar{r} \leq \bar{r} \leq r_0.$$

Hence,

$$\inf \frac{\frac{d}{dr} \left(\frac{r}{2} \frac{dV}{dr} \right)}{\frac{d}{dr} \left[V(r) + \frac{r}{2} \frac{dV}{dr} \right]} < \frac{T(m) - T(M)}{E(m) - E(M)} < \sup \frac{\frac{d}{dr} \left(\frac{r}{2} \frac{dV}{dr} \right)}{\frac{d}{dr} \left[V(r) + \frac{r}{2} \frac{dV}{dr} \right]} \quad (2.11)$$

We turn now specifically to Theorem 10: assume

$$\frac{d}{dr} \left[V(r) + \frac{r}{2} \frac{dV}{dr} \right] > 0$$

and

$$\frac{dV}{dr} > 0$$

It follows that

$$\frac{d}{dr} \left[V(r) + \frac{r}{2} \frac{dV}{dr} \right] > 0$$

and from the left-hand side of (2.11), that

$$\frac{T(m) - T(M)}{E(m) - E(M)} > -1 \quad (2.12)$$

and

$$\frac{dT}{dm} < - \frac{dE}{dm} \quad (2.13)$$

Inserting into (24), we get

$$\frac{dT}{dm} < \frac{T}{m}, \text{ i.e. } \frac{d}{dm} \left(\frac{T}{m} \right) < 0. \quad (2.14)$$

We turn now to Theorem 11: if

$$\frac{d}{dr} \left[V + \frac{r}{2} \frac{dV}{dr} \right] > 0 \quad \text{and} \quad \left(\frac{d}{dr} \right)^2 V \leq 0$$

we get, from the right-hand side of (2.11)

$$\frac{T(m) - T(M)}{E(m) - E(M)} < \frac{1}{3} \quad (2.15)$$

or

$$\frac{dT}{dm} > \frac{1}{3} \frac{dE}{dm}$$

and inserting in (24)

$$\frac{dT}{dm} > -\frac{1}{3} \frac{T}{m}, \quad \text{or} \quad \frac{d}{dm} (m^{1/3} T) > 0. \quad (2.16)$$

3) Why we believe that $T(m)$ has no maximum

We shall assume that

$$\frac{d}{dr} r \frac{d}{dr} r \frac{dV}{dr} > 0 \quad (2.17)$$

For attractive superpositions

$$\int_{-1}^{\infty} E(\alpha) p(\alpha) r^{\alpha} d\alpha \quad p \geq 0, \quad (2.18)$$

this quantity,

$$\int_{-1}^{\infty} E(\alpha) \alpha^3 p(\alpha) r^{\alpha} d\alpha$$

is indeed positive. We recall the expression of $T(m)$

$$T(m) = \int_0^{\infty} u^2(m, r) \frac{r}{2} \frac{dV}{dr} dr \quad . \quad (2.19)$$

When $(r/2) \cdot (dV/dr)$ is monotonous increasing (decreasing) it is clear, from (2.5) that $T(m)$ is monotonous decreasing (increasing).

If $\rho(\alpha) \neq 0$ for some range of $\alpha < 0$ and for some range of $\alpha > 0$ in (2.18) $(r/2) \cdot (dV/dr)$ has a minimum for some value $r_{\min}$. When m varies from 0 to ∞ , the location of the maximum of $u(m, r)$, because of the decrease of $u(M, r)/u(m, r)$, moves monotonously from ∞ to 0. When m is very small the integral (2.19) is dominated by the large r region and $T(m)$ is decreasing. When m is very large it is dominated by the small r region and it is increasing. What happens in between ?

i) If the maximum of $u^2(m, r)$ is sharp enough it behaves like a delta function, and since the location of the maximum is a monotonous function of m , $T(m)$ has a unique minimum.

ii) in fact it would be sufficient, to get this result to prove that any linear combination of $u^2(M_1, r)$, $u^2(M_2, r)$, $u^2(M_3, r)$ has only two zeros, with M_1, M_2, M_3 infinitesimally close. We believe that this property is true and very general but also incredibly difficult to prove. It can be proved for pure power potentials, and in particular, it is obvious for the Coulomb and harmonic oscillator potentials. However, pure power potentials are precisely such that $T(m)$ is monotonous. This argument is therefore somewhat reminiscent of the drunk man looking for his key under a street light while he knows that the key is not there !

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FIGURE CAPTIONS

- Fig. 1 : The upper bound on the mass dependence of the kinetic energy $T(m)$
as used for deriving inequality (41).
- Fig. 2 : Upper and lower bounds on the mass of toponium M_t as a function
of m_t corresponding to inequalities (43) and (44).

FIG.1

FIG. 2