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A NEW APPROACH TO THE THEORY OF GRAVITATION

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ABSTRACT

We discuss the role of Newton's constant in gravity theory. We show that the natural way to deal with the general covariance of the theory is to notice that the action is independent of any dimensional constant. Newton's constant is introduced in gravity through the classical constant solution $g_{\mu\nu} = (1/\ell^2)\delta_{\mu\nu}$ which displays the dilatation breaking of this Poincaré invariant solution.

Nothing changes for the usual results of classical gravity since Newton's constant is responsible for the low energy limits. However, it is easy to realize that there are other solutions of gravity coupled to matter (Yang-Mills and sigma model) where both fields have the same intensity and Newton's constant does not appear. These "non-Newtonian" solutions are due to the specific non-linear character of general gravitation. They can be relevant to the hadron structure.

The above observations give a hint that the asymptotically small distance behaviour of the gravity theory is independent of Newton's constant and should reflect the general invariance of the theory, provided that infinite classes of quantum diagrams are summed up.

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1. INTRODUCTION

The importance of physical theories in which no dimensional constants (like masses and coupling constants) appear has often been stressed. In recent years a beautiful example has been offered by chromodynamics which has quickly emerged as the leading theory of strong interactions.

The formal invariance of those theories under the group of scale and, more generally, conformal transformations leads to important consequences. On one side it has led to the discovery of classical solutions which are expected to play an important role in the study of the infra-red problem; on the other, the small distance ultra-violet behaviour is controlled by non-perturbative dimensional arguments which are at the basis of the renormalization group analysis. The 15 parameter conformal group has an interesting characterization if one notices that those transformations belong actually to a subset of general co-ordinate transformations.

It then looks natural and interesting to investigate from the point of view of dilatation and conformal invariance those Lagrangian theories whose action is generally invariant. The fundamental example of such theories is, of course, Einstein's general theory of gravitation.

In this context it is somewhat surprising that, in the standard formulation, the Lagrangian of the general theory of relativity contains Newton's constant K whose dimensionality is (in natural units) that of a squared length.

The presence of a dimensional constant in the gravitational Lagrangian seems to suggest that the theory of gravitation is far removed from the dimensionless theories discussed earlier and that its nature is somewhat closer to that of theories containing dimensional coupling constants like the point Fermi theories of weak interactions. On the other hand, gravitation and theories with general invariance under co-ordinate transformations bear a strong analogy with electromagnetism and gauge theories of the Yang-Mills type, where the form of the interaction is governed by local gauge invariance and is realized through the presence of the gauge fields $A_\mu^a(x)$. Furthermore, generally covariant equations exhibit classical meron and instanton solutions for $g_{\mu\nu}(x)$ and for the other fields, in complete analogy to the pure Yang-Mills case.

The aim of this paper is to discuss the implications of the co-ordinate transformations, and in particular dilatations, for the Einstein and for other general invariant theories. We shall show that an alternative interpretation of the Einstein theory exhibits its genuine dilatational properties. No dimensional constant is present in the Lagrangian. Newton's constant appears where it does really belong: in the Newtonian expansion of the $g_{\mu\nu}$ tensor around a flat metric¹⁾.

Our formulation will, of course, be fully equivalent to the conventional one for all standard results concerning gravitational forces at large distances. On the other hand, the fact that Newton's constant does not enter the fundamental Lagrangian suggests that other, non-Newtonian, solutions could exist, in which Newton's constant never appears. Such solutions have indeed been found (see I) and seem to be appropriate to describe the gravitational interaction of confined gluons and quarks. This suggests that the results of low energy gravitation should not be blindly extrapolated to the domain of particle physics where a completely new regime might take place, in which exchange of "gravitons" is as important as that of gluons and other strong interacting particles. It is thus possible that the conventional wisdom ascribing to the "Planck mass" a fundamental role at super-high energies does not have any physical foundation.

Much of the mathematics and some of the main physical ideas exposed in this work have already been expressed in I. Here we want to give a more systematic and consistent discussion of the whole subject.

2. GENERAL PROPERTIES OF GRAVITY

a) Gravity and Newton's constant

Einstein's pure gravitation is embodied in an action which is, of course, invariant under general co-ordinate transformations (GCT). Let us first define infinitesimal transformations:

$$x'^{\mu} = x^{\mu} + f^{\mu} \quad (2.1)$$

The Poincaré infinitesimal transformations are obtained if f^{μ} is subjected to the conditions

$$\partial_{\mu} f_{\nu} + \partial_{\nu} f_{\mu} = 0 \quad (2.2)$$

The broader condition

$$\partial_{\mu} f_{\nu} + \partial_{\nu} f_{\mu} - \frac{1}{2} \partial_{\lambda} f^{\lambda} g_{\mu\nu} = 0 \quad (2.3)$$

leads to the infinitesimal transformations of the conformal group

$$f_{\mu} = \alpha_{\mu} + \beta_{\mu\nu} x^{\nu} + \varepsilon x_{\mu} + 2 \gamma \cdot x x_{\mu} - x^2 \gamma_{\mu} \quad (2.4)$$

An important property of conformal transformations is that a "conformally flat" metric^{*)}

$$g_{\mu\nu} = h^2(x) \delta_{\mu\nu} \quad (2.4a)$$

retains its form; the weight function $h(x)$ transforms as

$$-\delta h(x) = \left\{ \frac{1}{4} \partial f + f_a \partial_a \right\} h(x) \quad (2.4b)$$

Under these general transformations a tensor with N covariant and M contravariant indices behaves as

$$\begin{aligned} \delta T^{\mu_1 \dots \mu_M}_{\nu_1 \dots \nu_N} &= T^{\lambda \mu_2 \dots \mu_M}_{\nu_1 \dots \nu_N} \partial_\lambda f^{\mu_1} + \dots \\ &- T^{\mu_1 \dots \mu_M}_{\rho \nu_2 \dots \nu_N} \partial_{\nu_1} f^\rho - \dots - f^\lambda \partial_\lambda T^{\mu_1 \dots \mu_M}_{\nu_1 \dots \nu_N} \end{aligned} \quad (2.5)$$

Since we are particularly interested in the role of scale transformations, let us write their explicit form:

$$f^\mu = -\varepsilon x^\mu \quad (2.6)$$

which gives

$$\frac{1}{\varepsilon} \delta T^{\mu_1}_{(\nu_1)} = (x \partial - \Delta) T^{\mu_1}_{(\nu_1)} \quad (2.7)$$

The Einstein dimensionality $\Delta = M - N$ of a tensor is given by the difference between the number of contravariant and covariant indices.

The behaviour of the gravitational tensor $g_{\mu\nu}$ and of its inverse $g^{\mu\nu}$ under GCT follows from (2.5):

$$\begin{aligned} \delta g_{\mu\nu} &= - (g_{\mu\alpha} \partial_\nu f^\alpha + g_{\nu\alpha} \partial_\mu f^\alpha + f \cdot \partial g_{\mu\nu}) \\ \delta g^{\mu\nu} &= (g^{\mu\alpha} \partial_\alpha f^\nu + g^{\nu\alpha} \partial_\alpha f^\mu - f \cdot \partial g^{\mu\nu}) \end{aligned} \quad (2.8)$$

^{*)} For simplicity we shall use a Euclidean formulation.

In particular, for dilatations (2.6), their variations are

$$\frac{1}{\varepsilon} \delta g_{\mu\nu} = (\alpha\partial + 2) g_{\mu\nu} \quad (2.9)$$

$$\frac{1}{\varepsilon} \delta g^{\mu\nu} = (\alpha\partial - 2) g^{\mu\nu} \quad (2.10)$$

so that

$$\begin{aligned} \dim g_{\mu\nu} &= -2 \\ \dim g^{\mu\nu} &= +2 \end{aligned} \quad (2.11)$$

The assignment of dimensionality $-2, +2$ to $g_{\mu\nu}$ and $g^{\mu\nu}$ is the key to all our discussion. Indeed, the simplest generally invariant action, the Einstein action

$$A_E = -\frac{1}{4} \int d^4x \sqrt{g} R \quad (2.12)$$

is dilatation invariant and no dimensional constant (like Newton's constant) should be present in the definition of the physical action. Where has Newton's constant gone? This constant enters naturally in the theory by requiring that the field $g_{\mu\nu}$ should be asymptotically flat. In other words, gravitational effects are related to the deviation of $g_{\mu\nu}$ from the "flat value" proportional to $\delta_{\mu\nu}$. Here again Eq. (2.1) plays the fundamental role. Since $g_{\mu\nu}$ has dimensionality -2 , the flat solution must involve a dimensional constant ℓ which is related to Newton's constant^{*)}:

$$[g_{\mu\nu}]_{\text{flat}} = \frac{1}{\ell^2} \delta_{\mu\nu} \quad (2.13)$$

$$[g^{\mu\nu}]_{\text{flat}} = \ell^2 \delta^{\mu\nu} \quad (2.14)$$

The weak field expansion will now take the form

*) The connection between ℓ and Newton's constant K is $\ell = \sqrt{4\pi(K\hbar/c^2)}$. Numerically, $\ell = 5.72 \times 10^{-38}$ cm. Often the notation $\kappa \equiv 2\ell$ has been used. We prefer to introduce the symbol ℓ because it reminds us of the dimensionality of a length, which is useful for our purposes.

$$g_{\mu\nu} = \frac{1}{e^2} (\delta_{\mu\nu} + h_{\mu\nu}) \quad (2.15)$$

and all quantities are to be expressed in terms of $h_{\mu\nu}$. Once this has been done, the expansion in any order of $h_{\mu\nu}$ will no longer be generally invariant nor scale invariant but just Poincaré invariant.

b) Gravity and matter in a flat limit.

Let us now extend our discussion to the problem of gravitation coupled to matter fields. The general theory of gravity provides all the classical recipes to re-write a given "flat" action in a form generally invariant, introducing the gravitational field. In particular, from the point of view of dilatations, invariance is achieved by inserting the appropriate powers of $g_{\mu\nu}$, $g^{\mu\nu}$ and g according to the Einstein dimensionality of the field tensors.

In order to have an explicit illustration, let us concentrate on the specific examples of a neutral spinless field $\phi(x)$ and of a coloured Yang-Mills field A_μ^a interacting with gravitation:

$$A = \int d^4x \sqrt{g} \mathcal{L} ; \mathcal{L} = \mathcal{L}_E + \mathcal{L}_M \quad (2.16)$$

where

$$\mathcal{L}_E = -\frac{1}{4} R, \quad \mathcal{L}_M = \mathcal{L}_\phi + \mathcal{L}_{YM}$$

and

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad (2.17)$$

$$\mathcal{L}_{YM} = -\frac{1}{4} \sum_a F_{\mu\nu}^a F_{\rho\sigma}^a g^{\mu\rho} g^{\nu\sigma} \quad (2.18)$$

One can, of course, add to the "minimal" Lagrangian (2.16), (2.17), (2.18) other "mass" or interaction terms; for example, one can add to $\mathcal{L}_\phi$ an arbitrary polynomial in ϕ ,

$$P(\phi) = \lambda_2 \phi^2 + \lambda_3 \phi^3 + \lambda_4 \phi^4 + \dots \quad (2.19)$$

Our action is generally (and dilatation) invariant. The fields are true tensors, thus^{*)}

$$\dim \phi = 0, \quad \dim A_\mu = -1, \quad \dim F_{\mu\nu} = -2 \quad (2.20)$$

The constants e and λ_i are pure numbers.

If we now insert the weak field expansion in our Lagrangian, general invariance will be spontaneously broken and we will be only left with the Poincaré group. The presence of the dimensional constant ℓ in the new effective Lagrangian containing the matter fields and $h_{\mu\nu}$ is a measure of how dilatation invariance is spontaneously broken.

A detailed study of the violation of conformal invariance in the flat limit is interesting. Indeed, we know that in this limit Lagrangian theories with field dimensions defined canonically (not necessarily coincident with Einstein dimensions) and containing only dimensionless parameters are conformal invariant. Our approach clarifies in a simple way the transition to the flat limit.

It is easy to show (see Appendix) that the spontaneous breaking of conformal invariance in the flat limit is proportional to

$$\mathcal{T} = \int d^4x \, \theta_{\mu\nu}(x) \, a(x) \quad (2.21)$$

where

$$\theta_{\mu\nu} = -\frac{2}{e^2} \left[\frac{1}{\sqrt{g}} \frac{\delta A_M}{\delta g^{\mu\nu}} \right]_{g_{\mu\nu} = \frac{1}{e^2} \delta_{\mu\nu}} \quad (2.22)$$

is the energy-momentum tensor in the flat limit and

^{*)} For the fact that the Einstein dimensionality of $g_{\mu\nu}$ is different from the one usually attributed, see for instance the lectures by J. Wess²⁾ where it is remarked that the usual dimensionality is the combined result of Einstein dilatations plus Weyl rescaling.

$$a(x) = \partial_\mu f^\mu \quad (2.23)$$

is in the case of the conformal group a polynomial of first order in x^μ *)

$$a(x) = 4\varepsilon + \delta x \quad (2.24)$$

Let us discuss the vector and scalar Lagrangians on the basis of this condition. In the vector case we have

$$\mathcal{Q}_{\mu\nu} = \sum_a \left(F_{\mu\rho}^a F_{\nu\rho}^a - \frac{1}{4} \delta_{\mu\nu} F_{\alpha\beta}^a F_{\alpha\beta}^a \right) \quad (2.25)$$

obeying the trace condition

$$\mathcal{Q}_{\mu\mu} = 0$$

This ensures that the Yang-Mills theory is conformal invariant in the flat limit. This could be argued more directly, at least in the case of dilatations. Indeed, if one inserts directly the ansatz (2.13) in Eq. (2.18) the dimensional constant ℓ disappears.

In the case of the scalar field theory the situation is different. The trace of the $\theta_{\mu\nu}$ tensor is

$$\mathcal{Q}_{\mu\mu} = -\frac{1}{e^2} (\partial\phi)^2 + \frac{4}{e^4} P(\phi)$$

so that the integral (2.21) does not vanish. On the other hand again, plugging the ansatz (2.13) in the scalar Lagrangian we see the constant ℓ does not disappear, a clear sign that even dimensional invariance is broken.

One indication of how this difficulty can be overcome follows from the fact that the field ϕ has zero Einstein dimensionality, whereas the "canonical dimensionality" of a scalar field is -1 (for vector fields Einstein and canonical dimensionalities coincide). One is thus led to define the new scalar field

*) In the case $\alpha = \text{constant}$, the condition $J = 0$ is simply the requirement that in the flat limit the constant ℓ must disappear; that is, it requires invariance under the "Weyl transformation of first kind"

$$\hat{\phi} = g^{\frac{1}{2}} \phi \quad (2.26)$$

which transforms with the general law

$$\delta \hat{\phi} = - \left(f \cdot \partial \hat{\phi} + \frac{1}{4} (\partial f) \hat{\phi} \right) \quad (2.27)$$

which for dilatations becomes

$$\delta \hat{\phi} = (\lambda \partial + 1) \hat{\phi} \quad (2.28)$$

$\hat{\phi}$ transforms thus as a canonical field of dimensionality -1. The wisdom of this choice is shown by the fact that if we now substitute the ansatz (2.13) in the scalar Lagrangian the constant λ disappears from the kinetic term and also from the polynomial if it reduces to the renormalizable ϕ^4 term, as expected:

$$P(\phi) = \lambda_4 \phi^4 \quad (2.29)$$

We can go back now to Eqs (2.21) and (2.22) where now $\delta A / \delta g^{\mu\nu}$ is computed for $\hat{\phi}$ constant. We obtain

$$\begin{aligned} \theta_{\mu\nu} &= \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} - \frac{1}{4} \delta_{\mu\nu} (\partial \hat{\phi})^2 + \\ & e^{-4} \delta_{\mu\nu} \left(P - \frac{1}{4} \hat{\phi} \frac{\partial}{\partial \hat{\phi}} P \right) \end{aligned} \quad (2.30)$$

so that

$$\theta_{\mu\mu} = \frac{1}{e^4} \left(4 - \hat{\phi} \frac{\partial}{\partial \hat{\phi}} P \right) \quad (2.31)$$

so that when $P(\phi)$ has the form (2.22), Eq. (2.14) is satisfied.

We have discussed in some detail the case of the scalar field because it is the example of a general phenomenon: the difference between the Einstein dimensionality and the "canonical" one which is obtained by redefining the field in such a way that in the flat limit the kinetic term is independent of λ . In the spin 1 case, exceptionally, $\hat{A}_\mu = A_\mu$ whereas in the spin 1/2 case it is not hard to check that

$$\hat{\psi} = g^{\frac{3}{16}} \psi$$

Let us go back to our original Lagrangian including $P(\phi)$ of (2.19). In the flat limit it becomes

$$[\mathcal{L}_{YM}]_{\text{flat}} = -\frac{1}{4} \sum_a F_{\mu\nu}^a F_{\mu\nu}^a \quad (2.32)$$

$$[\mathcal{L}_\phi]_{\text{flat}} = -\frac{1}{2} \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} + \frac{\lambda_2}{e^2} \hat{\phi}^2 + \frac{\lambda_3}{e} \hat{\phi}^3 + \lambda_4 \hat{\phi}^4 + \dots \quad (2.33)$$

It is interesting to notice that the definition $\hat{\phi} = \phi/\ell$ which leads back to the familiar kinetic term in the flat limit, gives a mass $\lambda_2^{1/2}/\ell$ of the order of Planck's mass (if λ_2 is any reasonable number; remember that λ_2 is dimensionless). Also the super renormalizable term, $\propto \phi^3$, would get an exceedingly large dimensional coupling constant $O(1/\ell)$. Such a field, if quantized in the quadratic limit, would correspond to quanta endowed with Planck's mass. Such quanta would have few possibilities of producing effective results when virtual, and would require an exceedingly large energy to be produced as real particles.

Now let us deal with the next order, namely $g_{\mu\nu} = 1/\ell^2 (\delta_{\mu\nu} + h_{\mu\nu})$. Adopting for simplicity the harmonic gauge

$$\partial_\mu h_{\mu\nu} = \frac{1}{2} \partial_\nu h_{\mu\mu} \quad (2.34)$$

we have

$$A_E = -\frac{1}{8\ell^2} \int d^4x \left\{ (\partial_\alpha h_{\mu\sigma})^2 - \frac{1}{2} (\partial_\alpha h)^2 \right\} \quad (2.35)$$

which suggests the natural redefinition

$$h_{\mu\nu} = \ell \varphi_{\mu\nu} \quad (2.36)$$

and hence

$$A_E = -\frac{1}{8} \int d^4x \left\{ (\partial_\alpha \varphi_{\mu\nu})^2 - \frac{1}{2} (\partial_\alpha \varphi)^2 \right\} \quad (2.37)$$

As for the next term, describing the interaction between matter and gravitation^{*)} one obtains as expected:

$$\mathcal{L}_{int} = 2\ell \theta_{\mu\nu} \varphi_{\mu\nu} \quad (2.38)$$

where $\theta_{\mu\nu}$ is defined in Eq. (2.22).

The comparison of our approach with the conventional one is now very easy. In the conventional approach the flat solution has the form

$$g_{\mu\nu} = \delta_{\mu\nu}$$

This forces $g_{\mu\nu}$ to have zero dimensionality. This, of course, leads to the presence of the dimensional Newton's constant in the gravitational Lagrangian. As already pointed out in I, the translation from ours to the conventional language is

$$\begin{aligned} [g_{\mu\nu}]_{conv} &= e^2 [g_{\mu\nu}]_{ours} \\ [A_\mu]_{conv} &= [A_\mu]_{ours} \\ [\phi]_{conv} &= e^{-1} [\phi]_{ours} \\ [\psi]_{conv} &= e^{-\frac{3}{2}} [\psi]_{ours} \end{aligned} \quad (2.39)$$

We wish at this point to insist on the fact that our approach represents something more than a simple rescaling of fields. Indeed, our procedure exhibits clearly the fundamental scale invariance of Einstein's theory, which is somewhat obscured in the conventional formalism. As shown in the next chapter it allows the discovery of new classical solutions which are invariant under sub-groups other than the Poincaré group. The form of such non-Newtonian solutions (which will not contain Newton's constants) will be dictated by invariance under Einstein's transformation which leads to what we think is the correct definition of dimensionality. At the

^{*)} Of course, we should not forget the contribution of the gravitational field $\phi_{\mu\nu}$ to $\theta_{\mu\nu}$. This leads (at the same order in ℓ) to a cubic contribution to $\mathcal{L}_{INT}$ describing self-interaction of gravitation.

same time, our approach clearly exhibits the phenomenon of theories which are not scale invariant in flat space but which recover scale and conformal invariance when coupled with $g_{\mu\nu}$. The dimensionality -2 of $g_{\mu\nu}$ performs the miracle. We shall see in the next chapter that just those models are extremely interesting as possible candidates for theories of strong interactions.

c) Quantization

Let us now discuss briefly how the previous arguments can be extended to the quantized case. Since the Lagrangian is fully independent of the Newton constant, both the equations of motion and the fundamental quantization properties will turn out to be ℓ independent.

Newton's constant will appear in the vacuum expectation value^{*)}

$$\langle 0 | g_{\mu\nu}(x) | 0 \rangle = \frac{1}{\ell^2} \delta_{\mu\nu} \quad (2.40)$$

which is the quantum analogue of the classical equations (2.13) and (2.14).

Let us discuss the meaning of Eq. (2.40). It clearly gives a violation of dilatation invariance through a vacuum state which is not invariant under dilatations. At first sight one would be tempted to argue that, since we are dealing with spontaneous breaking of dilatation symmetry, the Goldstone theorem should apply leading to the presence of a massless scalar "dilaton". This is indeed not the case as it can be checked in the framework of quantum gravity.

The reason is that in general theory of relativity general co-ordinate transformation in particular dilatations are completely analogous to "gauge" transformations leading from one to another gravitational "gauge"⁴⁾. Quantization is done in a particular gravitational gauge, i.e., for a particular choice of the system of co-ordinates. Since there is no operator leading from one to another co-ordinate system, there is no physical operator generating dilatations. Therefore, the general arguments for the proof of the Goldstone theorem do not apply to dilatations in the gravitational case.

*) The idea of introducing Newton's constant through a vacuum expectation value of a field is not new. Generally the proposal is to introduce a scalar field such that $\langle \phi \rangle \propto \ell^{-1}$, which leads to a modified theory of gravitation³⁾. We want to emphasize the difference from the present approach where it is the expectation value of the Einstein field that determines the gravitational constant.

The analogy with gauge theory in electrodynamics is quite instructive. Indeed, quantization is again performed in a particular gauge and again there are no operators leading from a gauge to another. It is not hard to recognize that those gauge generators would correspond to a component of the electromagnetic field describing unphysical photons. Likewise, it would not be hard to verify that (in the linear approximation) the generator of dilatations is related to a component of the $g_{\mu\nu}$ field corresponding to an unphysical "scalar graviton".

Going back to our general discussion we see that the fact that the Newton constant ℓ^2 enters quantum gravity only through the definition of the vacuum state is very important. All fundamental properties of the Green functions which depend only on the commutators will be ℓ independent. On the other hand it is conceivable that other "phases" could exist, corresponding to different "vacuum" states with completely different expectation values of $g_{\mu\nu}$. The search for such new phases is of course related to the search for other "non-Newtonian" classical solutions of the field theoretical equations.

3. OUTLOOK ON THE CLASSICAL SOLUTIONS

In the last section we have discussed, from our point of view, the role of the Newton constant in the framework of the general theory of relativity. This constant does not appear in the field theoretic Lagrangian but is associated with a special kind of classical solution

$$g_{\mu\nu}(x) = \frac{1}{\ell^2} \delta_{\mu\nu} \quad (3.1)$$

which is the starting point of the developments leading to Newton's theory of gravitation.

Our point of view opens the way for the use of other solutions of gravitational theories which do not contain the Newton constant and could lead to sizeable gravitational effects in the field of particle physics.

Solutions of this kind have been found in an earlier investigation; they do indeed generalize the instanton and meron solutions which appear in conformal invariant field theories.

The most interesting achievement would be to construct a theory which on one side should exhibit the flat solution (3.1), thus retaining all the results of standard gravitational theory, and on the other side would possess some of the non-Newtonian classical solutions which could contain some different scale parameter, for instance, of the order of those involved in ordinary hadron physics,

thus giving rise to gravitational non-Newtonian effects of a completely new kind. Of course, the ideal thing would be to find a more general kind of solution, interpolating between the two. For the moment, in the absence of that, already a model displaying the peaceful coexistence of two completely different solutions in the framework of the same Lagrangian theory would be sufficiently interesting to deserve further study. However, it is not trivial to achieve this program.

For instance, it is important to notice that not all general invariant models admit the solution (3.1). In general, the presence of a cosmological term in the Lagrangian forbids the solution (3.1); this circumstance leads, indeed, to an extremely stringent experimental bound on the cosmological constant. On the other hand, not all general invariant theories admit meron or instanton solutions. In general one needs Lagrangians in which gravitation is coupled with matter fields, and in some cases, non-vanishing values of the cosmological constant will be required.

As in our previous investigations, our search will be guided by group theoretic considerations. The starting point will, of course, be general invariance. The simplest solutions which will be discussed in this paper actually follow from the invariance of the theories under the 15 parameter conformal subgroup.

As a consequence of the discussion in Section 2a our solutions have the general form

$$g_{\mu\nu} = h^2(x) \delta_{\mu\nu} \quad (3.2)$$

The different classes of solutions will be determined by the subgroup of the conformal group under which the solution is invariant. The form of the classical solution will be determined by the differential equations (following from Eq. (2.4b))

$$\left\{ f^a \partial_a + \frac{1}{4} (\partial f) \right\} h(x) = 0 \quad (3.3)$$

where the functions f_α characterize the subgroup under consideration.

a) Poincaré subgroup

In this case, translation invariance forces all classical values of the field to be constant. Moreover, four dimensional rotation invariance allows (for example, for the general Lagrangian discussed in Section 2) non-vanishing values only for $h(x)$ and $\phi(x)$ which we denote

$$h(x) = \frac{1}{e} \quad ; \quad g_{\mu\nu} = \frac{1}{e^2} \delta_{\mu\nu} \quad (3.4)$$

and

$$\phi(x) = \phi_0 \quad (3.5)$$

The classical energy momentum tensor is

$$\Theta_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta A_M}{\delta g^{\mu\nu}} = \frac{1}{e^2} P(\phi_0) \delta_{\mu\nu} \quad (3.6)$$

with

$$P(\phi) = \lambda_0 + \lambda_2 \phi^2 + \lambda_3 \phi^3 + \dots$$

In order to have a flat solution we must have

$$P(\phi_0) = 0 \quad (3.7)$$

In the customary case $\phi_0 = 0$, this requires the vanishing of the cosmological constant λ_0 . Actually, the present state of experimental evidence allows an exceedingly small, if not zero, value of the cosmological constant, of the order of 10^{-120} . If $\phi_0 \neq 0$, these limitations hold for the "effective cosmological constant" $P(\phi_0)$.

It has been stressed that the above limit on the magnitude of the cosmological constant is in contradiction with the results of spontaneously broken gauge theories^{*)}. Since the cosmological term can be interpreted as the energy of the vacuum one can explicitly compute it in the framework of unified theories of weak and electromagnetic interactions. The value obtained for λ comes out many orders of magnitude larger than experimentally allowed and its removal could perhaps be achieved by delicate manipulations at each order in perturbation theory. This fact represents a serious problem^{**)}.

*) See, for instance, Ref. 5).

**) A recent discussion of this discrepancy can be found in Ref. 6).

b) $O(5)$ invariance subgroup (instanton)

In this case the form of h is

$$h(x) = \frac{c}{x^2 + a^2}, \quad g_{\mu\nu} = \frac{c^2}{(x^2 + a^2)^2} \delta_{\mu\nu} \quad (3.8)$$

For the Yang-Mills field we have

$$A_\mu = i \sigma_{\mu\nu} \partial_\nu \lg h(x) \quad (3.9)$$

and we have a constant for the scalar field:

$$\phi = \text{constant}$$

This implies :

$$\hat{\phi} = \phi g^{\frac{1}{2}} = \frac{\text{const}}{x^2 + a^2} \quad (3.10)$$

c) $O(4) \times O(1,1)$ invariance subgroup (meron)

A guidance for this type of solution is obtained from dilatation invariance.

We have ($r = \sqrt{x^2}$)

$$h(x) = \frac{c}{r}, \quad g_{\mu\nu} = \frac{c^2}{r^2} \delta_{\mu\nu} \quad (3.11)$$

$$A_\mu = i \sigma_{\mu\nu} \partial_\nu \lg h(x) \quad (3.12)$$

$$\phi = \text{const.} \quad \hat{\phi} = \frac{\text{const}}{r} \quad (3.13)$$

This is the form of the elementary meron, singular at the origin and at infinity in the Euclidean domain. By a conformal transformation the singularities may be placed at $x_\mu = \pm a_\mu$:

$$h(x) \rightarrow \frac{2a}{\sqrt{x_+^2 x_-^2}} \quad \text{where} \quad x_\pm = x \pm a \quad (3.14)$$

Now choosing $a_\mu = (0001)$ and performing the Wick rotation $x_4 = ix_0$, we see that this solution is finite with all derivatives and regular in the Minkowski domain. It is $O(4) \times O(2)$ invariant and the form of h is

$$h = \left\{ (1+t_+^2)(1+t_-^2) \right\}^{-\frac{1}{2}} \quad (3.15)$$

with

$$t_\pm = x_0 \pm |\vec{x}|$$

In the following we shall always adopt for convenience the simple forms (3.11)-(3.13) of the meron solutions. One should, however, keep in mind that in the explicit calculation of physical quantities (like the action or dynamical variables) the solution (3.15) has to be used.

General multimeron solutions, regular in the Minkowski domain, can be obtained by exploiting the general invariance of the theory; this has been seen in detail in I.

We want to point out, finally, a feature common to all these solutions. They are first determined, apart from a constant, by the requirement of invariance under a maximal symmetry subgroup so that, owing to the general invariance of the theory, the x dependence of the equations of motion is automatically watched. This reduces the problem to an algebraic one, i.e., to the determination of the over-all constant in terms of the parameters appearing in the Lagrangian. The nature and solutions of the ensuing numerical equations have then to be studied case by case.

After this general discussion about the classification of the interesting solutions we shall report the situation for some interesting models (some of them already treated in I).

4. LAGRANGIANS AND SOLUTIONS

a) General remarks

We are now ready to discuss the different classical solutions in the framework of well defined Lagrangian models.

Since we shall be interested in Lagrangians involving gravitation coupled with matter fields, we shall have a large degree of arbitrariness in the choice of the models. Such freedom of choice will, of course, be strongly limited if we work in the framework of the supergravity which requires gravity and matter fields to belong to a single supersymmetric multiplet.

The study of classical solutions in this more general framework is a fascinating program for the future. In the present preliminary investigation we shall simply consider two kinds of models which come from the general invariant generalization of the Lagrangians which are at present the most popular: non-linear sigma models and gauge theories.

Many of the results discussed in this section are already contained in our previous paper, I, and will be reported again here for the sake of completeness. We wish, however, to insist on the fact that, although the mathematical expressions are the same, the physical background is totally different. In our preceding work we had taken the prudent attitude to identify $g_{\mu\nu}$ with a spin 2 hadronic field whose presence would make both the non-linear σ -model and gauge theory generally invariant, leading to possible new candidates for strong interaction theories.

In the present approach we simply start from the fact that any matter field is naturally coupled to gravitation. Classical field configurations corresponding to conformal subgroups other than the Poincaré group do, in general, require forms of the gravitational field which are fundamentally non-Newtonian, corresponding to strong gravitational effects. This effect will be particularly evident in the case of the meron solutions which in all cases give rise to a non-vanishing classical energy momentum tensor.

In our new framework, the two models under discussion exhibit a fundamental difference since, as repeatedly stated, the presence of a flat solution is now mandatory. The general invariant non-linear sigma model exhibits in addition to meron and instanton, also the flat solution. It is thus acceptable under our present requirements and very little has to be added to the previous discussion.

The general invariant gauge theory discussed in our previous paper is unacceptable, at least in its original form. Indeed, the new non-Newtonian classical solutions require the presence of a sizeable cosmological constant which forbids the existence of a flat solution. We shall, however, show that the appropriate introduction of extra scalar fields might overcome this difficulty, and the indication is that, as in the case of electroweak Lagrangians, it seems to be hard to avoid the presence of scalar fields. Let us now discuss in detail the different models.

b) Non-linear sigma model

Let us discuss the general invariant extension of the celebrated sigma model (well known in pion physics). In its simplest form ($O(5)$ invariance) we introduce five real scalar fields ϕ_a constrained by:

$$\sum_a^5 (\phi^a)^2 = f^2 \quad (4.1)$$

coupled to the gravitational field by the action

$$A = A_E + \int d^4x \sqrt{g} \mathcal{L}_E \quad (4.2)$$

$$\mathcal{L}_E = -\frac{1}{2} \left\{ \sum_a g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a - N(x) (\sum_a \phi^{a2} - 1) \right\}$$

Note that because of the zero Einstein dimensionality of the scalar field, f is a dimensionless constant. Its value will be fixed according to the solutions we require the model to have. We see that no dimensional constant appears in the action and no cosmological constant is needed, either.

We now list the meron and instanton solutions:

Meron:

$$f = 1 \quad g_{\mu\nu} = \frac{c}{r^2} \delta_{\mu\nu}$$

$$\phi^\mu = \frac{x_\mu}{r} \quad (\mu = 1 - - 4) \quad (4.3)$$

$$\phi^5 = 0$$

Instanton:

$$f = \sqrt{3/2} \quad g_{\mu\nu} = \frac{c a^2}{(x^2 + a^2)^2} \delta_{\mu\nu}$$

$$\sqrt{\frac{2}{3}} \phi^\mu = \frac{2 a x^\mu}{(x^2 + a^2)} \quad (4.4)$$

$$\sqrt{\frac{2}{3}} \phi^5 = \frac{a^2 - x^2}{a^2 + x^2}$$

The flat limit, $g_{\mu\nu} = (1/\ell^2) \delta_{\mu\nu}$, requires a redefinition of the scalar field:

$$\hat{\phi}^a = \frac{1}{e} \phi^a \quad (4.5)$$

We have the Lagrangian

$$\mathcal{L}_{flat} = -\frac{1}{2} \partial_\mu \hat{\phi}^a \partial_\mu \hat{\phi}^a \quad (4.6)$$

At the same time the constraint equation (4.1) becomes

$$\sum_a (\phi^a)^2 = \left(\frac{f}{e}\right)^2 \quad (4.7)$$

with the $O(4)$ invariant constant solution

$$\phi^5 = \frac{f}{e} \quad \phi^a = 0 \quad (4.8)$$

As expected, in the flat limit we are led back to something very similar to the celebrated sigma model of pion physics, that is obviously non-scale invariant.

The only important difference is that the analogue of f_π , i.e., f/e , is now of the order of the Planck mass rather than of the pion mass. As a consequence "pion" interactions are negligible in the region of energy smaller than the Planck mass. Of course, one can be induced to think that already at much lower energy the non-Newtonian types of solutions can play the main role.

c) Gauge theory

The first step for a gauge and generally invariant theory, discussed in I, is to investigate the interaction of a gravitational field with an $SU(2)$ gauge field A_μ in the absence of other fields. We have already seen in I that the existence of non-Newtonian solutions requires a non-zero cosmological constant. We write the Lagrangian as

$$\mathcal{L} = \mathcal{L}_E + \mathcal{L}_{YM} - \frac{3}{8} \lambda^2 \quad (4.9)$$

where

$$\mathcal{L}_{YM} = -\frac{1}{4e^2} \int_a (F_{\mu\nu})^a (F_{\rho\sigma})^a g^{\mu\rho} g^{\nu\sigma} \quad (4.10)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (4.11)$$

$$A_\mu = \frac{\sigma^a A_\mu^a}{2i} \quad (4.12)$$

and both λ^2 (the cosmological constant) and the gauge constant e are dimensionless.

In I we have obtained a general family of non-Newtonian solutions under the condition

$$\lambda^2 = e^2 \quad (4.13)$$

connecting the cosmological constant to the colour charge. This family is given by the conformally flat form

$$g_{\mu\nu} = h^2(x) \delta_{\mu\nu} \quad (4.14)$$

and A_μ is given by

$$A_\mu = \epsilon \sigma_{\mu\nu} \partial_\nu \log h(x) \quad (4.15)$$

provided that h is a solution of

$$\square h + \frac{1}{2} \lambda^2 h^3 = 0 \quad (4.16)$$

In particular, the meron solution is given by

$$h(x) = \frac{\sqrt{2}}{e} \frac{1}{r} \quad (4.17)$$

The instanton solution is an exceptional case since the constraint (4.13) is not required^{*)}. It is given by

$$h(x) = \frac{4a}{\lambda^2} \frac{1}{x^2 + a^2} \quad (4.18)$$

Now we turn to the flat limit. In order to have the flat solution for $g_{\mu\nu}$ we must put $\lambda = 0$. If we do so, all that we have discussed in the preceding section applies and we are in the presence of flat Yang-Mills theory where the fields A_μ and $F_{\mu\nu}$ do not need to be redefined. However, the non-Newtonian solutions are lost.

This shows that the theory of pure Yang-Mills field plus gravity does not allow the existence of both flat and non-flat types of solutions. On the other hand, if we keep the cosmological term in order to have non-Newtonian solutions, we cannot recover the Newtonian limit. In conclusion, the theory of pure Yang-Mills coupled with gravity is not satisfactory from our point of view.

The most obvious way of overcoming this difficulty is to introduce an extra scalar field $\phi(x)$ and to go back essentially to the Lagrangian of Section 2:

$$\mathcal{L} = \mathcal{L}_E + \mathcal{L}_{YM} + \mathcal{L}_\phi \quad (4.19)$$

where

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + P(\phi) \quad (4.20)$$

and $P(\phi)$ is the usual polynomial in ϕ .

Conformal invariance (dilatation is enough) compels one to take for ϕ a constant solution; hence, at the classical level only $P(\phi)$ plays a role in ϕ .

Let us suppose that there are two stable constant solutions, ϕ_1 and ϕ_2 with the property

^{*)} This constraint is a consequence of Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R + \frac{3}{2} \lambda') = -2 \Theta_{\mu\nu}$$

In the instanton case $\Theta_{\mu\nu} = 0$.

$$P(\phi_1) = 0 \quad (4.21)$$

$$P(\phi_2) = 0 \quad (4.22)$$

$P(\phi_1)$ is the "effective" cosmological constant.

We see, therefore, that the solution ϕ_1 can be relevant to the flat solution and ϕ_2 to the non-Newtonian solutions (merons or instantons). The simplest case would be

$$P(\phi) = \frac{1}{2} \lambda_2 \phi^2 + \lambda_3 \phi^3 + \lambda_4 \phi^4$$

such that $\phi = 0$ is a minimum of $P(\phi)$. Now in P powers different from ϕ^4 must also be present for such a situation to develop. As a consequence of the presence of these powers, the flat limit is not conformal invariant, as discussed in Section 2 and contains in an essential way the constant ℓ ; the mass associated with the flat limit scalar field ϕ would be $\sqrt{\lambda_2}/\ell$, i.e., of the order of the Planck mass.

We have described a somewhat cumbersome and non-economical way of making gauge theory acceptable from our viewpoint. A more elegant Lagrangian will now be proposed which, as we shall see later, shows other important advantages.

d) Improved gauge theory

An amusing way of avoiding the explicit appearance of a cosmological term is to introduce a scalar field and to write the Lagrangian as

$$\mathcal{L} = \mathcal{L}_E - \frac{3}{8} (\varphi^2 + \frac{1}{\varphi^2} I) + \mathcal{L}_\varphi \quad (4.23)$$

where

$$I = \frac{2}{3} \sum_a F_{\mu\nu}^a F_{\rho\sigma}^a g^{\mu\rho} g^{\nu\sigma} \quad (4.24)$$

and $\mathcal{L}_\varphi$ is a kinetic term in the new field φ , whose specific form will be discussed later.

It is to be noticed that our "improved" Lagrangian reduces to the one of Eqs (4.9) and (4.10) when the scalar field φ is substituted by the constant $\lambda = e$.

It is easy to see that, apart from the term yet to be determined, the Lagrangian $\mathcal{L}$ has a simple transformation property under common scaling of the φ and $g_{\mu\nu}$ fields. If we perform the transformation

$$\begin{aligned}\varphi &\rightarrow c \varphi \\ g_{\mu\nu} &\rightarrow c^{-2} g_{\mu\nu}\end{aligned}\tag{4.25}$$

the Lagrangian transforms as

$$\mathcal{L} \rightarrow c^2 \mathcal{L}\tag{4.26}$$

The total Lagrangian will then be invariant under such transformation if we define $\mathcal{L}_\varphi$ as

$$\mathcal{L}_\varphi = -\frac{1}{2} g^{\mu\nu} \frac{1}{\varphi^2} \partial_\mu \varphi \partial_\nu \varphi\tag{4.27}$$

Of course, the transformation properties (4.26) mean that the classical field equations are invariant under (4.25). Let us now look for a classical solution of our new field theoretical equations in correspondence with a constant value

$$\varphi(x) = \lambda\tag{4.28}$$

for the scalar field. Because of the invariance property (4.26), (4.27), the value of this constant is completely arbitrary.

Let us now consider the classical field equations. The kinetic term will not play any role; the equations coming from the variations of A_μ and $g_{\mu\nu}$ will be the same as in Section c and will suggest the same solutions

$$A_\mu = i \sigma_{\mu\nu} \partial_\nu \ln h(x), \quad g_{\mu\nu} = h^2(x) \delta_{\mu\nu}\tag{4.29}$$

subjected to

$$\square h + \frac{1}{2} \lambda^2 h^3 = 0 \quad (4.30)$$

The new feature introduced by our improved Lagrangian is an extra equation obtained by varying φ :

$$\varphi^4 = \lambda^4 = \underline{I} \quad (4.31)$$

Introducing Eqs (4.29) into (4.31) one obtains for $h(x)$ the extra constraint

$$\begin{aligned} \underline{I} = \frac{4}{3} \left\{ 6 \left(\partial_\sigma \frac{1}{h} \partial_\sigma \frac{1}{h} \right)^2 - \frac{6}{h} \square \frac{1}{h} \partial_\sigma \frac{1}{h} \partial_\sigma \frac{1}{h} + \right. \\ \left. + \frac{1}{h^2} \left(\square \frac{1}{h} \right)^2 + \frac{2}{h^2} \partial_\sigma \partial_\rho \frac{1}{h} \partial_\sigma \partial_\rho \frac{1}{h} \right\} = \lambda^4 \end{aligned} \quad (4.32)$$

Making the assumption that $h = h(r) \equiv 1/f(r)$ we then obtain the condition

$$\left\{ f \left(f - \frac{2f}{r} \right) + \frac{\lambda^4}{4} \right\}^2 = \frac{\lambda^4}{16} \quad (4.32')$$

Looking at the form of $I(h)$ we see that:

- a) Only the meron and instanton solutions lead to a constant value of $I(h)$. This is due to the special group theoretical properties of those solutions (i.e., invariance under $O(4) \times O(1,1)$ and $O(5)$ transformations, respectively).
- b) In those two privileged cases we have

$$\begin{aligned} I_{\text{meron}} &= \lambda^4 \\ I_{\text{inst}} &= \frac{1}{2} \lambda^4 \end{aligned} \quad (4.33)$$

so that only the meron is a solution of the Lagrangian (4.23).

It is also easy to show that by a slightly different choice of the parameters it is possible to recover the instanton solution. Indeed, the Lagrangian

$$\mathcal{L}' = \mathcal{L}_E - \frac{3}{8} \left(\frac{\varphi^2}{2} + \frac{1}{\varphi^2} \underline{I} \right) + \mathcal{L}_\varphi \quad (4.34)$$

has an instanton solution, but now the meron solution is absent. As in the case of the non-linear sigma model, both meron and instanton solutions exist, alone, in correspondence with different values of the parameters in the Lagrangian.

As we shall see in our next paper, the absence (in the framework of the Lagrangian (4.23)) of all conformally flat solutions different from the meron is a very important feature because it ensures the dynamical stability of the meron solution and allows the use of a quantization procedure in which the different fields are expanded around such solutions.

In this connection it is interesting to notice that in our case also the gauge coupling constant $e = \lambda$ does not appear in the starting Lagrangian but is related to the expectation value of the field φ :

$$\langle 0 | \varphi | 0 \rangle = e \quad (4.35)$$

Let us finally come to a still unsolved question: the existence of a flat solution

$$g_{\mu\nu} = \frac{1}{e^2} \delta_{\mu\nu} \quad (4.36)$$

Although no cosmological constant appears explicitly in the Lagrangian, the solution (4.36) would require a vanishing value of the field φ . But the Lagrangian is singular just in correspondence of such value of φ^* .

The existence of a flat solution of the Lagrangian is an important problem in itself. We hope to be able in the future to gain a better understanding of this question.

5. CONCLUSIONS

In this paper we discuss some physical properties of the theory of gravitation interacting with matter.

*) The nature of the problem might be better seen by writing

$$\varphi = e^\sigma \quad (4.37)$$

leading to the perfectly regular Lagrangian

$$\mathcal{L} = \mathcal{L}_E - \frac{3}{8} (e^{2\sigma} + e^{-2\sigma} I) - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \quad (4.38)$$

The starting point is the invariance of the theory under general co-ordinate transformations. These, of course, include the 15 parameter conformal group and in particular, dilatations.

A fundamental consequence of the dilatation invariance is that it is possible to write the Lagrangian of the theory in a form in which no dimensional constant appears.

The Newton constant - which has the dimension of a $(\text{length})^2$ - appears solely in the vacuum expectation value

$$\langle 0 | g_{\mu\nu}(x) | 0 \rangle = \frac{1}{e^2} \delta_{\mu\nu} \quad (5.1)$$

which leads to the expansion

$$g_{\mu\nu} = \frac{1}{e^2} \delta_{\mu\nu} + \frac{1}{e} \phi_{\mu\nu} \quad (5.2)$$

The fact that the Newton constant does not play any role in the fundamental Lagrangian and in the quantum commutation relations tells us that the role of the Planck mass depends on the range of validity of the expansion (5.2).

This expansion is indeed very good at low energy; the smallness of gravitational effects as compared to the ones due to other interactions is a direct consequence of the smallness of Newton's constant.

The question thus arises about the energy at which the expansion (5.2) might start being invalid and new non-Newtonian effects might be expected.

One of the main results of this paper is that when matter fields develop sizeable classical values which represent a large non-perturbative effect those solutions require completely new forms of gravitational fields which differ radically from anything given by an expansion based on Eq. (5.2).

In this new dynamical situation, gravitational effects are far from negligible. In the next paper we shall study those new classical solutions as a starting point for quantization; it will be shown that in this new framework emission or absorption of "gravitons" occurs with the same rate as for "gluons".

Since the non-perturbative classical solutions (instanton and meron) seem to be important for the problem of confinement⁷⁾, our discussion suggests that gravitation plays a role in the confinement dynamics. If our point of view is correct, then the meeting point between gravitation and other interactions might be much lower than the celebrated Planck mass.

Another point which requires further study concerns the behaviour of the theory at very small distances. This problem is strongly related to that of the ultimate renormalizability of the theory.

Standard research in the framework of quantum gravity is based on a perturbative expansion in powers of the Newton constant. Since this constant has the dimension of a (length)² this series shows features similar to those of a non-renormalizable theory.

On the other hand, it has been repeatedly emphasized that Newton's constant plays a completely different role in gravitation from that played by the electric charge in electrodynamics. A closer analogy, strongly advocated by S. Weinberg⁸⁾ comes from chiral dynamics, where the "pion coupling" $1/f_\pi$ plays a similar role to the "gravity coupling" ℓ .

Indeed, the analogue of Eq. (5.1) is in this case given by the zero expectation value of the (π, σ) field^{*)} ϕ_α

$$\langle 0 | \phi_\alpha | 0 \rangle = f_\pi \delta_{\alpha 0} \quad (5.3)$$

This strongly suggests that both power series, in $1/f_\pi$ for pions, in ℓ for gravitons, are simply low energy expansions whose validity at high energy is open to questions.

A possible key to a better understanding of the whole question comes from the observation that "the ultra-violet divergences of a theory respect the symmetry of the Lagrangian even if the vacuum does not"⁹⁾.

If this point of view could be applied to our case one would infer that Newton's constant (which appears only in the definition of the vacuum) should play no role in the ultra-violet problem. This might imply that a resummation of an infinite number of diagrams with different powers in ℓ might lead to a completely different small distance behaviour, restoring some of the broken symmetry and thus improving the ultra-violet behaviour of the theory¹⁰⁾.

A limit in that direction comes from the non-polynomial nature of the gravitational Lagrangian. Indeed, the general invariance of the theory comes essentially from the presence of $g_{\mu\nu}$, $g^{\mu\nu}$ and $\sqrt{g}$; if those were considered independent, the Lagrangians would have a finite number of terms. The constraints then compel one to express the fields in terms of a single gravitational field,

*) Recall, however, the different behaviour of the two theories for what concerns the Goldstone boson (Section 2c).

$\phi_{\mu\nu}$. This can be done in many ways and in all cases requires expansion in an infinite number of terms. Those infinite expansions thus implement the general invariance and give the non-polynomial character of the gravitational Lagrangians which contain vertices with any number of gravitation lines. The truncation of the infinite expansion will destroy most of the simple transformation properties of $g_{\mu\nu}$ and $g^{\mu\nu}$ and will thus break most of the general invariance properties of the gravitational Lagrangian.

In some privileged cases the benign nature of the theory might reveal itself by assigning (because of general invariance reasons) vanishing values to seemingly divergent diagrams. This is what happens, at least for the lowest loop contributions, both for self-coupled gravity and for supergravity, and it is not excluded that such a pleasant trend might extend to higher orders.

It is clear that our arguments are highly speculative. We do, however, believe that a systematic study of the general invariance properties of quantum gravity is a fascinating avenue for the future.

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APPENDIX

Let the action be given by

$$A = A_E + A_M \quad (A.1)$$

and let us perform an infinitesimal conformal transformation over the fields; since this operation is an invariance of A , we must have

$$0 = \delta A = \int d^4x \left\{ \frac{\delta A_E}{\delta g^{\mu\nu}} + \frac{\delta A_M}{\delta g^{\mu\nu}} \right\} \delta g^{\mu\nu} + \frac{\delta A_M}{\delta \phi_i} \delta \phi_i \quad (A.2)$$

where ϕ_i symbolize the fields other than gravity. Now take $g^{\mu\nu}$ around a flat metric, $\eta^{\mu\nu}$. In this case from Eq. (2.1)

$$\delta g^{\mu\nu} = -\epsilon^2 (\partial_\mu f_\nu + \partial_\nu f_\mu) \quad (A.3)$$

and using the identity (1.3) we obtain

$$\delta g^{\mu\nu} = \frac{1}{2} \epsilon^2 \partial f \delta_{\mu\nu} \quad (A.4)$$

Now around a flat metric

$$\left(\frac{\delta A_E}{\delta g^{\mu\nu}} \right)_{flat} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \quad (A.5)$$

Eq. (A.2) takes thus the form

$$\int d^4x \frac{\delta (A_M)_{flat}}{\delta \phi_i} \delta \phi_i = \frac{1}{4} \int d^4x \theta_{\mu\nu} \partial f \quad (A.6)$$

where we have defined the "flat" canonical tensor

$$\theta_{\mu\nu} = -\frac{2}{\epsilon^2} \left(\frac{1}{\sqrt{g}} \frac{\delta A_M}{\delta g^{\mu\nu}} \right)_{flat} \quad (A.7)$$

This shows that in the flat limit the quantity

$$\mathcal{J} = \int d^4x \theta_{\mu\mu} \partial f \quad (\text{A.8})$$

gives the indication of the breaking of conformal invariance.

Now let us compute the flat canonical tensor. For the Yang-Mills field we have

$$\theta_{\mu\nu} = \sum_a \left(F_{\mu\alpha}^a F_{\nu\alpha}^a - \frac{1}{4} \delta_{\mu\nu} F_{\alpha\beta}^a F_{\alpha\beta}^a \right) \quad (\text{A.9})$$

For the scalar field with Lagrangian

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + P(\phi) \quad (\text{A.10})$$

we obtain

$$\theta_{\mu\nu} = \frac{1}{e^2} \left[\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \delta_{\mu\nu} (\partial\phi)^2 \right] + \frac{1}{e^4} \delta_{\mu\nu} P(\phi) \quad (\text{A.11})$$

and

$$\theta_{\mu\mu} = -\frac{(\partial\phi)^2}{e^2} + \frac{4}{e^2} P(\phi)$$

If we now redefine

$$\phi = g^{-\frac{1}{8}} \hat{\phi} \quad (\text{A.12})$$

the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu (g^{-\frac{1}{8}} \hat{\phi}) \partial_\nu (g^{-\frac{1}{8}} \hat{\phi}) + P(g^{-\frac{1}{8}} \hat{\phi}) \quad (\text{A.13})$$

Now performing the functional differentiation with respect to $g_{\mu\nu}$ keeping $\hat{\phi}$ fixed we get the canonical tensor of Eq. (2.30)

$$\theta_{\mu\nu} = \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} - \frac{1}{4} \delta_{\mu\nu} (\partial \hat{\phi})^2 + e^{-4} \delta_{\mu\nu} \left(P - \frac{1}{4} \hat{\phi}^2 \frac{\partial}{\partial \hat{\phi}} \right) P$$

Its trace is

$$\theta_{\mu\mu} = \frac{1}{e^4} \left(4P - \hat{\phi}^2 \frac{\partial}{\partial \hat{\phi}^2} P \right)$$

(A.14)

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