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ON THE CHARACTERIZATION OF MONOPOLES IN NON-ABELIAN GAUGE THEORIES

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ABSTRACT

We attempt to help clarify some confusion in the literature on what characterizes monopoles in non-Abelian gauge theories. There is no meaningful classification of monopoles as "magnetic charge multiplets" belonging to representations of a "magnetic gauge group" as suggested by some authors, the "magnetic charge" defined there not being in general conserved. We found no physically meaningful classification other than the topological quantum number or gauge type of bundles over spheres. For a pure $su(n)$ Yang-Mills theory, this can be characterized by the n^{th} roots of unity.

We attempt in this note to clarify the question of what characterizes monopoles in non-Abelian gauge theories. Although the answer is largely contained in a profound and rich paper of Wu and Yang in 1975¹⁾ there is still considerable confusion in the recent literature on the subject.

To introduce our language which follows closely that of Ref. 1) consider first an Abelian theory invariant under simple phase transformations. For most problems, which deal with infinitesimal gauge transformations and small oscillations about the vacuum, only the local^{*)} properties of the gauge group are relevant and so only its Lie algebra needs be specified. In contrast, the study of monopoles requires a specification of the gauge group (i.e., global^{*)} properties) as well. In an Abelian theory, the gauge group is $U(1)$ (compact) if all physical charges are integral multiples of some minimal charge e ; otherwise, if charge is not quantized, the gauge group is non-compact, with group manifold $\mathbb{R}$ (the real line) and not S^1 (the circle) as in $U(1)$ ²⁾.

The gauge potential $A_\mu(\underline{x})$ cannot be non-singular everywhere on a sphere surrounding a monopole (hence the Dirac string), but the sphere can be covered by two co-ordinate patches, which we may take to be the northern and southern hemispheres overlapping along the equator. In each patch, the potentials denoted by A_μ^N and A_μ^S respectively, are non-singular, and in the overlap region, they are related by a gauge transformation:

$$A_\mu^S = A_\mu^N - \frac{i}{e} S \frac{\partial}{\partial x_\mu} S^{-1}, \quad (1)$$

where S is the patching function which must be single valued. (Thus gauge potentials are precisely connection 1-forms defined on a $U(1)$ bundle. The above patching put the bundle over S^2 in the so-called standard form³⁾, which means that we need not consider any other, more complicated, patching of the base manifold S^2 .) The patching function S may be considered as a map of the equator into $U(1)$, in other words a closed curve Γ_S on the $U(1)$ group manifold, namely the circle S^1 . An x dependent gauge transformation in either co-ordinate patch just distorts Γ_S continuously into another closed curve on S^1 . It is clear then that gauges are divided into equivalence classes ("gauge types" or principal $U(1)$ bundles) which are in one-one correspondence with the

^{*)} The terms "local" and "global" are used in the literature with respect both to the gauge group and to space-time. To avoid confusion, we shall use them here exclusively with respect to the gauge group. "Local" and "global" (in space-time) gauge transformations we shall refer to as " x dependent" and " x independent". Also we shall adopt the convention of denoting a group and its Lie algebra by capital and small letters, thus for example, $SU(n)$ and $su(n)$, respectively.

homotopy classes $\pi_1(U(1))$ of closed curves on the $U(1)$ manifold, where by definition, two curves belonging to the same homotopy class are continuously deformable into one another. The classes themselves are labelled by integers n , namely:

$$\pi_1(U(1)) = \mathbb{Z}, \quad (2)$$

n being the number of times the curves in that class wind around S^1 .

Let $A_\mu^{N[n]}$ and $A_\mu^{S[n]}$ be potentials defined in any gauge belonging to the gauge type $[n]$. Then the magnetic charge corresponding to $A_\mu^{[n]}$, defined as:

$$g = \frac{1}{4\pi} \oint_{\text{sphere}} F_{ij}^{[n]} d\sigma_{ij}, \quad (3)$$

$$F_{ij}^{[n]} = \partial_i A_j^{[n]} - \partial_j A_i^{[n]}, \quad (4)$$

can be written as:

$$\begin{aligned} g &= \frac{1}{4\pi} \oint_{\text{equator}} \{A_j^{N[n]} - A_j^{S[n]}\} dx_j \\ &= \frac{1}{4\pi} \oint \left\{ -\frac{\partial}{\partial x_j} \log S \right\} dx_j \\ &= n/2e. \end{aligned} \quad (5)$$

In other words, the magnetic charge g is quantized.

Moreover, a potential $A_\mu^{[n_1]}$ defined at time $t = t_1$ in the gauge type $[n_1]$ cannot change continuously into another potential $A_\mu^{[n_2]}$ defined at $t = t_2$ in another gauge type $[n_2]$. For a classical theory, this means already that the magnetic charge is conserved. That the same conclusion holds also in a quantum theory can be seen as follows. The transition amplitude from the state at $t = t_1$ labelled by $A_\mu^{[n_1]}$ to the state at $t = t_2$ labelled by $A_\mu^{[n_2]}$ may be written symbolically as:

$$\langle 2|1 \rangle = \int \delta A_\mu e^{i\delta[A_\mu]}, \quad (6)$$

where the integral is to be taken over all field paths satisfying $A_\mu(\underline{x}, t = t_1) = A_\mu^{[n_1]}(\underline{x})$ and $A_\mu(\underline{x}, t = t_2) = A_\mu^{[n_2]}(\underline{x})$. The fact that there is no continuous field path satisfying these boundary conditions means that the action S is infinite, or that the transition amplitude $\langle 2|1 \rangle$ is zero.

We note that the preceding considerations depend only on the fact that the gauge group is $U(1)$; in other words, these considerations are purely geometric. For example, nowhere have we used an explicit form for the Lagrangian. The beauty of a magnetic charge defined in this way, which is both quantized and conserved, is just that it is so primitive, and therefore presumably fundamental.

Consider now non-Abelian gauge theories. We have first to specify the gauge group by examining as in the Abelian case all the representations to which the physical fields of the theory belong. For example, if a theory is invariant under infinitesimal transformations of the Lie algebra of $su(n)$, and the physical fields of the theory include some transforming as the fundamental representation, then the gauge group is $SU(n)$. However, for pure gauge theories containing only gauge fields, the gauge group is $SU(n)/Z_n$ ^{*}). That this is so can be seen as follows. Gauge potentials A_μ (where $A_\mu = A_\mu^i T_\mu^i$, T_μ^i being the generators of the Lie algebra) belong to the adjoint representation, which, under the action of an element g belonging to the group $SU(n)$, transforms as:

$$A_\mu \rightarrow A'_\mu = g A_\mu g^{-1}. \quad (7)$$

They thus remain invariant under the operation of all group elements which commute with the generators, namely those belonging to the centre $Z(n)$ of $SU(n)$ which is isomorphic to the cyclic group Z_n of order n . It follows that any two elements g_1 and g_2 in $SU(n)$ related by $g_1 = c g_2$ for $c \in Z(n)$ are indistinguishable in any theory containing only gauge fields, and that the actual gauge group is the group obtained by identifying these elements in $SU(n)$, namely $SU(n)/Z_n$.

As in Abelian theories, gauges are divided into equivalence classes (gauge types) which are in one-one correspondence with the homotopy classes of closed curves on the gauge group manifold, namely $\pi_1(G)$. This is in accordance with the fact that the principal G bundles over S^2 are classified by $\pi_1(G)$.

^{*}) Although we study here only the explicit examples of $SU(n)$ and $SU(n)/Z_n$ much of our consideration applies in general to groups having a finite cyclic centre, e.g., all the connected groups corresponding to the classical and exceptional Lie algebra except $Spin(4n)$ ⁴⁾.

Now the manifolds for $SU(n)$, ($n > 1$), are all simply connected in the sense that all closed curves on the manifolds can be continuously shrunk to a point, namely

$$\pi_1(SU(n)) = 0. \quad (8)$$

Hence for $SU(n)$ theories, all gauges belong to the same gauge type. The group $SU(n)/\mathbb{Z}_n$ on the other hand, is n -tuply-connected:

$$\pi_1(SU(n)/\mathbb{Z}_n) = \mathbb{Z}_n, \quad (9)$$

since a curve joining the identity of $SU(n)$ to another element of the centre $Z(n)$, though not a closed curve in $SU(n)$, is a closed curve in $SU(n)/\mathbb{Z}_n$. There are then n gauge types for $SU(n)/\mathbb{Z}_n$.

Suppose now that we were to attempt to define "magnetic charges" for non-Abelian theories in analogy to the Abelian case just discussed by considering a special class of potentials $\underline{A}_\mu$ pointing in a fixed direction in isospace, thus

$$\underline{A}_\mu(\underline{x}) = \underline{M} A_\mu(\underline{x}) \quad (10)$$

as it is done in the literature. Since $[\underline{A}_\mu(\underline{x}), \underline{A}_\nu(\underline{x}')] = 0$, then

$$\underline{F}_{ij}(\underline{x}) = \partial_i \underline{A}_j(\underline{x}) - \partial_j \underline{A}_i(\underline{x}), \quad (11)$$

and following the same argument as in the Abelian case !

$$\begin{aligned} \underline{G} &= \frac{1}{4\pi} \oint\!\!\!\oint_{\text{sphere}} \underline{F}_{ij} d\sigma_{ij} \\ &= \frac{1}{4\pi} \oint_{\text{equator}} (\underline{A}_j^N - \underline{A}_j^S) dx_j \end{aligned} \quad (12)$$

$$\begin{aligned}
 &= \frac{1}{4\pi} \frac{i}{e} \oint \left(- \frac{\partial}{\partial x_j} \log S \right) dx_j \\
 &= \frac{1}{4\pi} \frac{i}{e} \left[- \log S(\phi=2\pi) + \log S(\phi=0) \right]. \quad (12)
 \end{aligned}$$

cont

The patching function S , being single-valued, must have the same value for the azimuthal angle $\phi = 2\pi$ and $\phi = 0$. Hence,

$$\exp -i 4\pi e \underline{G} = S^{-1}(\phi=2\pi) S(\phi=0) = 1, \quad (13)$$

which is the "quantization condition" of Englert and Windey⁵⁾. In case the gauge group is compact, connected, which is true in nearly all theories considered in the literature, $\underline{G}$ can be rotated via an x_i independent gauge transformation to lie in the Abelian subalgebra of a maximal torus of the group, thus:

$$\underline{G} \rightarrow \underline{G}' = \sum_r \beta_r \underline{t}^r, \quad (14)$$

where $r = 1, \dots, R$, R being the rank of the group, and $[\underline{t}^r, \underline{t}^s] = 0$. The numbers β_r so defined, called "magnetic weights" by Goddard et al.,⁶⁾, must therefore be quantized.

These quantities $\{\beta\}$, however, have a different significance to that of magnetic charges in Abelian theories, since

- a) they are defined only for a very special class of potentials,
- b) they are not in general conserved, as we shall shortly explain.

They do not therefore constitute a particularly useful concept.

The magnetic charge for Abelian theories was conserved because there was no continuous field path connecting two potentials with different magnetic charges. In the non-Abelian case, however, two potentials $\underline{A}_\mu^{\{\beta_1\}}$ and $\underline{A}_\mu^{\{\beta_2\}}$ with different "magnetic weights" $\{\beta_1\}$ and $\{\beta_2\}$ as defined in (14) may belong to gauges $\underline{G}_1$, and $\underline{G}_2$ of the same gauge type; indeed for gauge group $SU(n)$, the gauge type must be the same. In that case, one can transform the two potentials by non-singular gauge transformations to the same gauge, say $\underline{G}'$. In this gauge, the linear combination

$$\tilde{B}'_{\mu} = \alpha \tilde{A}'_{\mu}^{\{\beta_1\}} + (1-\alpha) \tilde{A}'_{\mu}^{\{\beta_2\}} \quad (15)$$

is also a potential since^{*)}

$$\tilde{B}'^3_{\mu} = S' \tilde{B}'^N_{\mu} S'^{-1} - S' \frac{\partial}{\partial x_{\mu}} S'^{-1}. \quad (16)$$

As α varies in time from 1 to 0, $\tilde{B}'_{\mu}$ describes a continuous field path connecting $\tilde{A}'_{\mu}^{\{\beta_1\}}$ to $\tilde{A}'_{\mu}^{\{\beta_2\}}$. Now for a classical theory, the existence of some such continuous paths implies in general that a potential with "magnetic weights" $\{\beta_1\}$ can evolve into a potential with "magnetic weights" $\{\beta_2\}$, since an action principle only selects out of the set of continuous paths the one with stationary action^{**)}. For a quantum theory, on the other hand, it means that the transition amplitude (6) from the state with $\tilde{A}_{\mu}(\underline{x}) = \tilde{A}_{\mu}^{\{\beta_1\}}(\underline{x})$ at $t = t_1$ to the state with $\tilde{A}_{\mu}(\underline{x}) = \tilde{A}_{\mu}^{\{\beta_2\}}(\underline{x})$ at $t = t_2$ has contributions in the functional integral from some continuous paths with finite action and is therefore in general finite. In either case, we can conclude that the "magnetic weights" $\{\beta\}$ are not conserved quantities in general.

To illustrate further what happens, consider the explicit example of an isolated monopole in a theory with gauge group $SU(2)$ ¹⁾. The potential:

$$\begin{aligned} A^1_{\mu} &= A^2_{\mu} = 0 \\ A^N_t &= A^N_r = A^N_{\theta} = 0, \quad A^N_{\phi} = -\frac{1}{e} \frac{1}{r \sin \theta} (1 - \cos \theta) \\ A^S_t &= A^S_r = A^S_{\theta} = 0, \quad A^S_{\phi} = \frac{1}{e} \frac{1}{r \sin \theta} (1 + \cos \theta), \end{aligned} \quad (17)$$

^{*)} This is a particular case of the general result that convex sums of connections on a bundle are also connections on the same bundle. Linear combinations of connections on different bundles, on the other hand, are not in general connections.

^{**)} It may of course happen that for some exceptional form of the Lagrangian there exists no continuous path with stationary action connecting any two such states, but there is no guarantee that this is the case for any Lagrangian of physical interest.

according to (10)-(14) represents a monopole at the origin with "magnetic weight" $\beta = -1/e$ in the gauge g with patching function $S = \exp 2i\phi T_3$. However, there being only one gauge type for $SU(2)$, one should be able to transform the potential into the gauge where the patching function is trivial. Explicitly indeed, by means of the $SU(2)$ gauge transformations:

$$\begin{aligned} \xi &= \exp [-i\theta (T_1 \sin \phi - T_2 \cos \phi)] = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} e^{-i\phi} \\ -\sin \frac{\theta}{2} e^{i\phi} & \cos \frac{\theta}{2} \end{pmatrix} \\ \eta &= \exp [-i(\pi-\theta)(T_1 \sin \phi - T_2 \cos \phi)] \exp(-i\pi T_2) \\ &= \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} e^{i\phi} \end{pmatrix}, \end{aligned} \quad (18)$$

in respectively the northern and southern hemispheres the patching function S is transformed into:

$$S' = \eta S \xi = 1. \quad (19)$$

In this gauge g' , the potential in (17) becomes

$$A'_\mu{}^k = 0, \quad A'_i{}^k = -\frac{1}{e} \frac{1}{r^2} \varepsilon_{ikj} x^j \quad (20)$$

over the whole sphere.

According to our considerations then, there are continuous field paths connecting this potential (20) to the vacuum $A_\mu{}^i = 0$, for example:

$$\underline{B}'_\mu = \alpha \underline{A}'_\mu \quad (21)$$

with α varying continuously from $\alpha = 1$ at $t = t_1$ to $\alpha = 0$ at $t = t_2$. Let us follow what happens to the monopole as it evolves in time along the path (21). The field tensor corresponding to the potential (21) is

$$E_i^k = i f_{i4}^{\prime k} = \frac{1}{e} \frac{d\alpha}{dt} \frac{1}{r^2} \varepsilon_{ikj} x^j, \quad (22)$$

$$H_i^{\prime k} = -\frac{1}{2} \varepsilon_{irs} f_{rs}^{\prime k} = \frac{1}{e} (-2\alpha + \alpha^2) \frac{x_i x_k}{r^4}. \quad (23)$$

The "magnetic" field H_i was defined in gauge g as the component of $H_i^k = -\frac{1}{2} \varepsilon_{irs} f_{rs}^k$ pointing along the 3-axis in isospace. Hence, in gauge g' , it is the component of H_i^k pointing in the radial direction, namely:

$$H_i' = H_i^k \hat{x}_k = \frac{1}{e} (-2\alpha + \alpha^2) \frac{x_i}{r^3}. \quad (24)$$

Now H_i' in (24) is divergence free for all α at every point $\underline{x}$ except the origin where it has "magnetic weight"^{*)}:

$$\beta = \frac{1}{4\pi} \oint\!\!\!\oint_{\text{sphere}} H_i^{\prime k} d\Omega^i = \frac{1}{e} (-2\alpha + \alpha^2). \quad (25)$$

Thus, as α varies from 1 to 0, the magnitude of the "magnetic weight" decreases continuously from $1/e$ to 0 without at any time radiating off "magnetic" sources. On the other hand, the "electric" fields E_i^k are not in general divergence free. The "magnetic charge" at the origin has thus been made to disappear simply by continuous changes of "electric sources" at the origin and in the surrounding space.

In general, in non-Abelian theories then, an isolated "magnetic" monopole of the form (10) and (17) is an unstable object which will decay eventually to the state with the lowest energy in the same gauge type. The decay can proceed by the emission of "electric sources" alone. For theories with gauge group $SU(n)$, there being just one gauge type which contains the vacuum, the "magnetic charge" will simply disappear, whereas for gauge group $SU(n)/\mathbb{Z}_n$, it will decay into the ground state in one of the n gauge types.

^{*)} Notice that for $\alpha \neq 1, 0$ the potential $B_\mu^{\prime k}$ does not belong to the special class (10) and need not therefore have a quantized "magnetic weight" β as in (14).

The behaviour of such isolated monopoles in pure $su(n)$ Yang-Mills theory has been studied in some detail by Brandt and Neri⁷⁾. (The gauge group is $SU(n)/\mathbb{Z}_n$, not $SU(n)$, as explained above.) Here, static isolated monopoles, namely

$$A_\mu(x, t) = A_\mu(x) \quad (26)$$

for $A_\mu(x)$ of form (10) and (17) with $1/e$ replaced by n/e are solutions of the Yang-Mills equations, or in other words local extrema of the action in functional space. Such solutions have physical significance only if they are local minima of the action. Among the potentials (10) and (17), n such solutions were found, one for each of the n gauge types, corresponding to the potential in that gauge type with the lowest energy, as expected also in our previous considerations. All other solutions of form (26) were found to be unstable under small variations and are therefore of no particular physical interest. However, it should be emphasized that even had they turned out to be stable solutions in this sense, they should still be interpreted only as metastable excitations of the n ground state monopoles rather than as distinct monopoles with higher magnetic charges, since they can decay to the ground state via the emission of "electric" sources alone, as noted before.

It seems therefore that the "magnetic weights" $\{\beta\}$ as defined in (14) do not have much significance in non-Abelian theories apart from being labels for the special class of potentials (10). These $\{\beta\}$ were shown by Goddard et al.⁶⁾, to lie on the weight lattice of a group dual to the gauge group G , and based on this observation they suggested that monopoles in non-Abelian theories form magnetic charge multiplets belonging to representations of a "magnetic gauge group" $\tilde{G}$ dual to G . However, in view of the fact that the "weights" $\{\beta\}$ are not conserved quantities and are defined and quantized only for a very special class of potentials, it would appear that this suggestion cannot be sustained.

For Abelian theories, of course, the magnetic charge is both the "magnetic weight" $\{\beta\}$ and the gauge type. What seems to generalize the concept naturally to non-Abelian theories, however, is the gauge type and not the "magnetic weight". Since a potential belonging to one gauge type cannot change continuously into a potential belonging to another, the gauge type is clearly a conserved quantity. For theories with gauge group $SU(n)$ the statement is trivial since all potentials belong to the same gauge type. For theories with gauge group $SU(n)/\mathbb{Z}_n$, the gauge types are characterized by (9), or equivalently by a quantum number which can take values among the n^{th} roots of unity $\{\omega_r\}$, $r = 0, \dots, n-1$, $\{\omega_r\}^n = 1$.

In terms of the potential A_μ , these numbers can be computed as follows. We first express $A_\mu = A_\mu^{i\bar{i}}$ as an $n \times n$ matrix α_μ in the fundamental representation of $SU(n)$. The matrices in the two patches are related by

$$\alpha_\mu^S = S \alpha_\mu^N S^{-1} - S \frac{\partial}{\partial x_\mu} S^{-1}, \quad (27)$$

where S is the patching function S expressed again as a matrix of the fundamental representation. We then form the integral

$$g = \frac{1}{4\pi} \oint (S \alpha_\mu^N S^{-1} - \alpha_\mu^S) dx_\mu = \frac{1}{4\pi} \oint S \frac{\partial}{\partial x_\mu} S^{-1} dx_\mu \quad (28)$$

around the equator. For A_μ of the special form (10), this is just the operator $\tilde{G}$ in (12) expressed in the fundamental representation. Clearly then the same argument applies, so that

$$\exp -4\pi i e g = \tilde{S}(\phi = 2\pi) \tilde{S}(\phi = 0). \quad (29)$$

The fact that S is single valued in $SU(n)/Z_n$ now implies that $S(\phi = 2\pi) = S(\phi = 0) \times C$, where C is an element in the centre of the group of $SU(n)$ of $n \times n$ unimodular unitary matrices. In other words:

$$\exp -4\pi i e g = \omega_r, \quad \text{for some } r, \quad 0 \leq r \leq n-1. \quad (30)$$

These quantum numbers ω characterizing monopoles in theories with gauge groups $SU(n)/Z_n$ are defined for all potentials, gauge invariant, and necessarily conserved as already explained, and are multiplicative in the sense that for any two potentials $A_{1\mu}$ and $A_{2\mu}$ characterized by quantum numbers ω_{r_1} and ω_{r_2} respectively, their sum

$$A_\mu = A_{1\mu} + A_{2\mu} \quad (31)$$

is characterized by the quantum number

$$\omega_r = \omega_{r_1} \cdot \omega_{r_2} \quad (32)$$

calculated according to (27)-(30). Moreover, it shares with the magnetic charge in Abelian theories the attractive feature of being purely geometric and independent of the detailed form of the Lagrangian.

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