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RADIATIVE MUON CAPTURE IN MEDIUM-HEAVY NUCLEI

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A B S T R A C T

Radiative muon capture in nuclei is thoroughly investigated in view of its sensitivity to the pseudoscalar coupling constant g_p . In the photon energy domain of interest ($k \geq 60$ MeV) the effective one-body Hamiltonian in use is shown essentially to hold true, with minor corrections due to the muon propagator in the nuclear field and to the two-step process. The closure approximation is avoided by appropriately integrating the response function calculated in the Fermi gas model, in the energy momentum transfer plane.

Theoretical predictions in medium-heavy nuclei of the high-energy photon spectrum, of photon polarization, and of muon-spin photon angular correlation are given for various values of the pseudoscalar coupling constant.

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1. INTRODUCTION

The process of radiative muon capture (hereafter called RMC)

$$\mu^-(\mu) + p(p_1) \rightarrow n(p_2) + \nu(\nu) + \gamma(k)$$

represents, as is well known, a favourable candidate to test the presence of the pseudoscalar coupling constant g_p . As a matter of fact, about one half of the high-energy part ($k \geq 60$ MeV) of the photon spectrum is accounted for by the presence of this term. The rarity of the elementary process, which is of the order of 10^{-4} times less than ordinary muon capture, has so far prevented its detection. For such a reason we investigate in this work the sensitivity to g_p and the experimental feasibility of nuclear RMC, presenting a systematics in medium-heavy nuclei.

Let us briefly recall the experimental status, which is rather peculiar since nothing is known outside ^{40}Ca . Experiments have been performed by Conversi, Diebold and Di Lella¹⁾ and later by Di Lella, Hammerman and Rosenstein²⁾ with similar techniques. A considerably lower rate of RMC has been found in the second case, explained in terms of discriminated high-energy neutrons which had marred the previous result. Quite recently another experiment has been performed by a group from the William and Mary College³⁾. The result can be summarized in a ratio of radiative to ordinary muon capture of $20.0 \pm 1.4 \times 10^{-6}$, for photon energies greater than 57 MeV, as compared with 15×10^{-6} of Ref. 2.

As to a first orientative comparison with theory, two treatments have been considered by these experimentalists: that of Rood and Tolhoek⁴⁾ in Ref. 2 and that of Rood, Yano and Yano⁵⁾ in Ref. 3. These differ essentially in the treatment of the muon propagator with an ensuing sizeable decrease for the high-energy spectrum in the case of Ref. 3. The spectrum depends, as stated before, rather strongly on the pseudoscalar coupling constant, increasing in a given theory by a factor of two for g_p ranging from $-7g_A$ to $+7g_A$. The combined effect of using a lower theoretical prediction and having a bigger experimental yield (together with other assumptions about the spectrum end point) has led from the previously determined $g_p = -5.9g_A$ to an agreement with the canonical value of g_p .

The main problem a theoretical analysis must face consists in the evaluation of the sum over final nuclear states. All previous treatments were essentially based on the closure approximation, i.e. the total transition strength was taken at a given single mean energy. Let us briefly review them.

The first detailed treatment of RMC was made by Rood and Tolhoek⁴⁾. Its main features are shell-model harmonic oscillator wave functions for the initial nucleus, no summation over partial transitions by introducing an "average" maximum photon energy k_m , and use of a free Green's function for the muon propagator.

The second approach⁶⁾ corrects the shell-model results as far as the dipole part is concerned, taking its energy and strength from photoreactions, and carries over the former calculation of the other multipoles. Also in this case a closure approximation naturally ensues, since the experimental giant dipole energy is the same as the monopole quadrupole shell-model energy used in the calculation of the other multipoles.

The closure approximation is also the essential ingredient of early calculations with the Fermi gas model⁷⁾.

This common feature constitutes in our opinion the central point at issue. As a matter of fact the spectrum end point fixes the lowest possible excitation energy. This does not correspond to some average energy used in closure except in the unrealistic case of a δ excitation function. Even for very rudimentary forms of the excitation spectrum important corrections are to be expected, as recently discussed by the authors⁸⁾ for the case of ^{40}Ca . This applies to the rate where a very reasonable uncertainty upon the value of k_m has dramatic consequences, because of the well-known k_m^6 dependence, but especially to the end of the spectrum which is particularly sensitive to the chosen photon end point, i.e. k_m . This can of course blur the strong dependence of RMC on g_p .

The main purpose of this work in which we give the first theoretical predictions for medium-heavy nuclei, consists therefore primarily in avoiding the closure approximation and in extending techniques already successfully used in muon capture⁹⁾, where the rate has also a strong ν^4 dependence on the average neutrino energy.

Two methods were used to predict ν . First the average neutrino energy was predicted from the ratio of energy-weighted to non-energy-weighted sum rules plus energy conservation. Alternatively the more sophisticated method of integrating over the energy in the E', ν plane with the usual energy conservation constraint was used.

Here, the problem is in principle exactly the same, apart from calculational complications, in the way of treating the variables k , ν , and E' . For any photon momentum, the same line integral of the excitation function is performed on the variables E', ν . It is therefore clear that, in principle, to every photon energy must correspond a different nuclear excitation energy. In particular, muon capture is recovered, as regards the spectrum, in the soft photon limit.

The nuclear model used in this article will be the Fermi gas model. Although it is known from its successful application to ordinary capture that it gives a good description of the gross structure of the energy excitation, some influence of the shape of the spectrum must however be expected, in particular in so far as the low E' region influences high k 's. For this reason we will be concerned with medium-heavy nuclei only, where unphysical low-energy excitations are absent also in the Fermi gas spectrum.

Aside from the main point of the nuclear model, many other effects deserve a careful theoretical scrutiny. First it is mandatory to use a reliable Hamiltonian. This applies both to the free proton amplitude which can be checked via low-energy theorems and to the nuclear amplitude. For the latter it must be checked which are the corrections to the impulse approximation in a correct treatment of the process in which weak and electromagnetic currents act at different points. This effect of "non-locality" is analysed through the usual insertion of an intermediate set of states. It is shown that for low photon energies the effective Hamiltonian in use breaks down completely and that its range of validity is limited to very energetic photons. The correct use of an amplitude describing the two-step process lends itself naturally also to the discussion of exchange effects or influence by the medium. Whereas no problem exists for the electromagnetic and vector current,

owing to the knowledge of the divergence, the situation is unsettled as regards the axial vector current. Exchange effects would show up as a renormalization of the form factors g_A and g_P appearing in our expressions. However this renormalization has no reason to be the same as that of the axial and pseudoscalar coupling constants in ordinary capture¹⁰⁾. In the absence of a theoretical prediction we have therefore provisionally used the nucleonic coupling constant values.

Finally there is the problem of the treatment of the muon propagator in the nuclear field, which is critical both in that it is expected that the approximation of using a free one gets worse as the nuclear charge increases and in that it affects only the part of the amplitude due to the radiation by the muon. The problem has been solved by the authors in a preceding paper¹¹⁾.

In summary, a thorough treatment of RMC has been given in the present paper. We therefore conclude that we are in a position to pinpoint nuclear anomalies, all other effects being under control from the theoretical side. Major discrepancies among experiments and our predictions with nucleonic values of the axial-vector coupling constants should hence hint at a renormalization of the axial current.

The plan of this work is the following. In Section 2 the non-relativistic reduction of the free proton low-energy amplitude is performed and compared with the Rood-Tolhoek effective Hamiltonian. Section 3 is devoted to the discussion of two-body correlations and of exchange effects, with final remarks on the nuclear effective Hamiltonian. Section 4 gives some preliminary results by applying the closure approximation. In Section 5 numerical results for ordinary capture are briefly recalled. Section 6 explains our procedure to overcome the closure approximation and gives the results for the spectrum of medium-heavy nuclei. Section 7 is concerned with muon-spin photon angular correlation and photon polarization predictions and in Section 8 conclusions are drawn. Finally in Appendix A,

the effective Hamiltonian for radiative muon capture is given, and in Appendix B an analytic solution of the photon spectrum is explicitly worked out in the Fermi gas model for illustrative purposes, in the simplified case of constant couplings.

2. RADIATIVE CAPTURE BY A FREE PROTON AND THE EFFECTIVE NON-RELATIVISTIC HAMILTONIAN

The problem of the effective nuclear Hamiltonian for RMC has been considered by a number of authors. We will compare our results with Rood and Tolhoek's work⁴⁾. There the amplitude for the process is obtained, in accordance with previous formulations, by attaching a γ in every possible way to ordinary capture on a free proton.

An improvement upon this procedure has come from the application of low-energy theorems by Adler and Dothan¹²⁾, later rechecked by Christillin and Servadio¹³⁾.

The underlying idea is that since both the weak and the electromagnetic (e.m.) currents transfer a four-momentum bound by the muon mass, relatively far from the N^* resonance, it is legitimate to try an expansion in $q = \mu - \nu$ and k by attaching both currents to the bare process. The essential ingredient of this technique is the knowledge of the non-radiative amplitude represented in this case by elastic weak form factors and radiative pion capture. The difference between this procedure and Rood and Tolhoek's consists, broadly speaking, in that the first one is a consistent expansion to the same order whereas the second is a truncated one. This is relevant in order to establish a reliable Hamiltonian, because of the well-known ambiguity $\bar{u}ku = \bar{u}q u$.

Following the notation of Ref. 13, we report the amplitude and the corresponding graphs for the process

$$T = T_a + T_b + T_c + T_d + T_e + T_f + T_g + T_h + T_i \quad (2.1)$$

$$T_a \sim - \bar{u}(h_2) (V_2 + A_2) u(h_2) \bar{u}(\nu) \gamma_2 (1 + \gamma_5) \frac{1}{i(\not{X} - \not{k}) + m_\mu} i \not{\epsilon} u(\mu) \quad (2.2)$$

$$T_b \sim \epsilon_\mu^* L_\nu \bar{u}(h_2) (V_\nu + A_\nu) \frac{1}{i(k_2 - q) + M} i \left[\gamma_\mu + \frac{K_1}{2M} \sigma_{\mu 3} K_3 \right] u(h_1) \quad (2.3)$$

$$T_c \sim \epsilon_\mu^* L_\nu \bar{u}(h_2) i \frac{K_2}{2M} \sigma_{\mu 3} K_3 \frac{1}{i(k_1 + q) + M} (V_\nu + A_\nu) u(h_1) \quad (2.4)$$

$$T_d + T_e \sim \epsilon_\mu^* L_\nu \frac{2M g_A(0)}{g_r(0)} \frac{\bar{u}(h_2) \gamma_5 u(h_1)}{(q-k)^2 + m_\pi^2} g_r(q-k^2) \times \left[- \frac{2q_\mu q_\nu}{q^2 + m_\pi^2} + \delta_{\mu\nu} \right] \quad (2.5)$$

$$T_f \sim -i \epsilon_\mu^* L_\nu \bar{u}(h_2) \frac{f_2^V(0)}{2M} \sigma_{\nu\mu} u(h_1) \quad (2.6)$$

$$T_g \sim i \epsilon_\mu^* L_\nu \bar{u}(h_2) \left\{ 2g_A'(0) (q_\mu \gamma_\nu + \gamma_\mu k_\nu - K \delta_{\mu\nu}) \gamma_5 - \right. \\ \left. - i r(0) \delta_{\mu\nu} \gamma_5 + 2f_3^{V'}(0) \left[\gamma_\mu k_\nu + q_\mu \gamma_\nu - \frac{K+q}{2} \delta_{\mu\nu} \right] + \right. \\ \left. + \frac{M g_A(0)}{g_r(0)} \sigma_{\mu\nu} \right\} u(h_1) \quad (2.7)$$

$$T_h \sim -i \epsilon_{\mu}^* L_{\nu} g_A^{(0)} \frac{q_{\nu}}{q^2 + m_{\pi}^2} \bar{u}(h_2) \gamma_5 \sigma_{\mu\zeta} k_{\zeta} \frac{f_2^S(0)}{2M} u(h_1) \quad (2.8)$$

$$T_i \sim -i \epsilon_{\mu}^* L_{\nu} g_A^{(0)} \frac{q_{\nu}}{q^2 + m_{\pi}^2} \bar{u}(h_2) \frac{M}{g_r^{(0)}} O_{\mu\beta} q_{\beta} u(h_1) \quad (2.9)$$

Here $O_{\mu\nu}$ has the following expression

$$\begin{aligned} O_{\mu\nu} = & \gamma_5 \sigma_{\mu\zeta} k_{\zeta} \left(\frac{\partial \bar{V}_2^{(-)}}{\partial \nu} \right) \Big|_0 \frac{(h_1 + h_2)_{\nu}}{2M} + i \gamma_5 [(h_1 + h_2)_{\mu} k_{\nu} - \\ & - (h_1 + h_2)_{\nu} k_{\mu}] \bar{V}_2^{(0)}(0) + \epsilon_{\mu\nu\lambda\alpha} k_{\lambda} \gamma_{\alpha} \bar{V}_4^{(0)}(0) + \\ & + \gamma_5 (\gamma_{\mu} k_{\nu} - \delta_{\mu\nu}) \bar{V}_3^{(-)}(0) \end{aligned} \quad (2.10)$$

and the $\bar{V}$'s are the form factors appearing in the usual non-Born pion photoproduction amplitudes.

The standard expression for V and A will be used without second-class currents.

Some comments are in order, although the various terms can be easily visualized through the accompanying diagrams. In Figs. 1a, b, c, the full current is used at the weak vertex, i.e. with the explicit presence of the pion pole in the pseudo-scalar term. Figures 1d, e, f, are self-explanatory and, as well as the previous ones, had already been taken into account in Ref. 4. Figure 1g represents the structure part of the gauge term 1f linear in q and k, obtained by application of low-energy theorems, and is new. Figures 1h and i, which correspond to the Born and structure part of the virtual pion photoproduction are of the second order and higher, so that they should be consistently dropped. They have been retained because of the presence of the possibly anomalous pion propagator; it will however

be shown that in the physical process they can be safely neglected as well as the term of Fig. 1d.

Let us now turn to the evaluation of the new contributions as compared with the dominant ones of Figs. 1a and b. It is easy to realize that the terms in g'_A and $f_1^{V'}$ (where these form factors are dominated by the A_1 and ρ masses, respectively) are at most $O(m_\mu^2/M_\rho^2)$ with respect to the former, as well as $r(0)\delta_{\mu\nu}\gamma_5$, since $r(0) \approx 2MG'_A$. As far as the terms of Fig. 1g containing the photoproduction form factors are concerned, an immediate evaluation of them can be obtained, for example, from Chemtob and Rho¹⁴⁾:

$$\begin{aligned}\bar{V}_4^{(0)}(0) &= \frac{2 g_{\pi\pi\gamma} g_{\pi NN}}{M_\rho^3} & \bar{V}_3^{(-)}(0) &= \frac{4\pi}{3} \frac{\gamma_n - \gamma_p}{g_\pi} \frac{0.074}{m_\pi^3} \\ \bar{V}_2^{(0)}(0) &= \frac{2 g_{\pi\pi\gamma} g_{\pi NN}}{M_\rho^3} \frac{k_\nu}{2M}\end{aligned}$$

Therefore $\bar{V}_2^{(0)}$ and $\bar{V}_4^{(0)}$ are $O(m_\mu^2/M_\rho^2)$, whereas $\bar{V}_3^{(-)}$ as well as $\bar{V}_1^{(-)}/\partial\nu$ (estimated with the usual assumption of N^* dominance) are numerically small. This is probably not unexpected since these extra contributions are known to give a negligible contribution in this energy region to pion photoproduction, which is rather well reproduced by the Kroll-Rudermann term alone (automatically included in writing the standard form of the weak axial-vector current). Therefore Rood-Tolhoek's Hamiltonian can be *a posteriori* used on firm grounds from the point of view of elementary particles.

3. THE NUCLEAR HAMILTONIAN

The suggestion that in such a two-step process (weak and electromagnetic vertices) the usual one-body Hamiltonian might be altered through excitation of intermediate nuclear states was made long ago by Christillin and Rosa-Clot¹⁵⁾, whereas more recently the role of the isobar in the intermediate states was investigated by Ohta¹⁶⁾ in connection with nuclear renormalization of the axial-vector coupling constants.

We try here to give a heuristic discussion of all possible nuclear effects. It is first of all clear that the formal expression for the amplitude on a nucleus

$$\mu^- + (N, Z)_a \rightarrow \nu + \gamma + (N+1, Z-1)_b \quad (3.1)$$

from the initial ground state $|a\rangle$, to any allowed final state $|b\rangle$, is in principle a two-body object. As a matter of fact, the total amplitude T , by using standard reduction techniques, is

$$\begin{aligned} T \sim & \bar{u}(\nu) \gamma_\nu (1 + \gamma_5) S(\mu - k) \not{x} u(\mu) \int d^4 y e^{i(q-k) \cdot y} \langle b | V_\nu^-(y) + A_\nu^-(y) | a \rangle \\ & + \epsilon_\mu^* L_\nu \iint d^4 x d^4 y e^{iq \cdot x - ik \cdot y} \langle b | T \{ J_\mu^{em}(x), V_\nu^-(y) + A_\nu^-(y) \} | a \rangle + \\ & + \epsilon_\mu^* L_\nu S_{\mu\nu} \end{aligned} \quad (3.2)$$

where $S_{\mu\nu}$ stands for the seagull terms, $S(\mu - k)$ is the muon propagator, A^- , V^- , J^{em} are respectively the axial and vector weak current and the electromagnetic current, ϵ_μ is the photon polarization and $L_\nu = \bar{u}(\nu)(1 - \gamma_5)\gamma_\nu u(\mu)$. The second term can be transformed into

$$\begin{aligned} T_{\mu\nu} \sim & 2\pi \delta(E_f - E_i) \sum_n \iint d^3 x d^3 y \times \\ & \left\{ \frac{\langle b | J_\mu^{em}(\underline{x}, 0) e^{-i \underline{k} \cdot \underline{x}} | n \rangle \langle n | (V_\nu^-(\underline{y}, 0) + A_\nu^-(\underline{y}, 0)) e^{i \underline{q} \cdot \underline{y}} | a \rangle}{E_{na} - (m_\mu - \nu)} + \right. \\ & \left. + \frac{\langle b | (V_\nu^-(\underline{y}, 0) + A_\nu^-(\underline{y}, 0)) e^{i \underline{q} \cdot \underline{y}} | n \rangle \langle n | J_\mu^{em}(\underline{x}, 0) e^{-i \underline{k} \cdot \underline{x}} | a \rangle}{E_{na} + k} \right\} \end{aligned} \quad (3.3)$$

where n is any accessible intermediate state; in particular nuclear states, nuclear states + a pion, a nucleon-antinucleon pair ($N\bar{N}$) represented diagrammatically by Z graphs, and other particles.

Seagull terms depend on the form of the Hamiltonian¹⁷⁾ and hence of the currents. In other words if in Eq. (3.3) both positive and negative energy intermediate states are allowed (i.e. a $N\bar{N}$ pair) seagull terms have a given form,

determined by general principles, whereas if one reduces the currents non-relativistically first, "new" seagulls corresponding to the neglected negative energy states appear.

Let us now sketch, by considering the different intermediate states, how one recovers the previous one-body Hamiltonian and which are the corrections, neglecting for the moment exchange effects, i.e. using the impulse approximation at each vertex.

i) Z graph

In such a case $E_{na} = 2M$ and hence neglecting terms of the order of $(\bar{E}/2M)$, where $\bar{E}$ is the mean nuclear excitation energy ~ 20 MeV,

$$T_{N\bar{N}} \sim \frac{1}{2M} \sum_n \langle b | J^{\mu} | n, N\bar{N} \rangle \langle n, N\bar{N} | V + A | a \rangle + \text{cr.} \quad (3.4)$$

Now both the term in g_p , the space component of the weak current, and the time component of the axial-vector current connect large components of the Dirac spinors for antinucleon intermediate states. By properly contracting them with ϵ_{μ}^L we obtain to the leading order^{†)}

$$T_{N\bar{N}} \sim \frac{1}{2M} \sum_{i=1}^A \tau_i^- \left\{ (\underline{\sigma}_i \cdot \underline{\epsilon} + i \underline{\sigma}_i \cdot (\underline{\tau}_i \wedge \underline{\epsilon})) g_V + \underline{\sigma}_i \cdot \underline{\epsilon} (g_A + g_P) \right\} e^{-i(\underline{k} + \underline{\nu}) \cdot \underline{r}_i} \quad (3.5)$$

where the isospin geometrical factor $[(1 + \tau_1^3)/2] \tau_1^-$ has killed the contribution of the first piece of the T-ordered product. Equation (3.5) shows, as already known, that the dominating contribution of the pseudoscalar coupling constant and less relevant parts from g_V and g_A come from the above-mentioned Z graphs. As a consequence of this fact, it is straightforward to see that corrections to g_p (in the absence of exchange effects), because of the collective excitations, are $O(E/M)$ and hence negligible. It is also clear that i-j correlations are necessarily absent and that the presence of an intermediate pion is already included in the expression of g_p .

†) Here and in the following the trivial non-relativistic reduction of the lepton current $\sim (1 - \underline{\sigma} \cdot \underline{\hat{v}})$ will be omitted.

ii) Nuclear states

We now proceed to the evaluation of terms with intermediate nuclear states by performing the standard non-relativistic reduction of the currents.

This part of the amplitude can be elegantly analysed, remembering that $\mu - \nu = k + E_{ba}$, by expanding the energy denominators of Eq. (3.3) in powers of (E/k) . We then have

$$T_{\text{nuc.}} \sim \sum_n \frac{\langle b | J^{\text{em}} | n \rangle \langle n | V + A^- | a \rangle}{-k} \left(1 + \frac{E_{nb}}{k} + \dots \right) + \sum_n \frac{\langle b | V + A^- | n \rangle \langle n | J^{\text{em}} | a \rangle}{k} \left(1 - \frac{E_{na}}{k} + \dots \right) \quad (3.6)$$

The first term is simply $(1/k)[V^- + A^-, J^{\text{em}}]$, and it is straightforward to estimate a typical contribution by taking, for example, the non-relativistic reduction of the convective part of the e.m. current and of the leading axial-vector current

$$\frac{1}{k} \left[\sum_i \frac{1+\gamma_i^3}{2} \frac{\mathbf{p}_i \cdot \boldsymbol{\varepsilon}}{M} e^{-i\mathbf{k} \cdot \mathbf{r}_i}, \sum_j g_A \boldsymbol{\tau}_j \cdot \boldsymbol{\sigma}_j e^{-i\boldsymbol{\nu} \cdot \mathbf{r}_j} \right] \quad (3.7)$$

$$\approx \sum_i g_A \boldsymbol{\tau}_i \cdot \frac{\boldsymbol{\nu} \cdot \boldsymbol{\varepsilon}}{Mk} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma} e^{-i(\boldsymbol{\nu} + \mathbf{k}) \cdot \mathbf{r}_i} + \sum_i g_A \boldsymbol{\tau}_i \cdot \frac{\mathbf{p}_i \cdot \boldsymbol{\varepsilon}}{Mk} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma} e^{-i(\boldsymbol{\nu} + \mathbf{k}) \cdot \mathbf{r}_i}$$

where use has been made of the relation

$$[Aa, Bb] = \frac{1}{2} [A, B] \{a, b\} + \frac{1}{2} \{A, B\} [a, b]$$

for operators A, B and a, b belonging to different subspaces. The meaning of the result obtained is that if we neglect intermediate and final-state excitations ($E_{na} = E_{nb} = 0$) only one-body operators appear.

In other words, although it is possible for a nucleus to absorb a muon somewhere and to emit the photon somewhere else, as soon as we treat it like an object without possible excitation, $i-j$ correlations are not allowed and the process occurs again on a single nucleon.

We want to reiterate here some remarks on the contribution of the different terms. The results of Rood and Tolhoek obtained by reducing non-relativistically a set of Feynman diagrams are reproduced from the muon-radiating diagram and from the T-ordered product mainly by inclusion of the $\bar{N}N$ pair intermediate states (g_p and other smaller contribution) and secondarily by taking into account intermediate nuclear states (1/k bremsstrahlung terms of little relevance at the end of the spectrum). Corrections to this result come from the higher-order expansion of Eq. (3.6). Those linear in the excitation energy are

$$\begin{aligned}
 & - \frac{1}{k^2} \left[\langle b | J^{\mu} | n \rangle \langle n | V^+ A^- | a \rangle E_{nb} + \langle b | V^+ A^- | n \rangle \langle n | J^{\mu} | a \rangle E_{na} \right] \\
 & = - \frac{1}{k^2} \langle b | [V^+ A^-, [H, J^{\mu}]] | a \rangle
 \end{aligned}
 \tag{3.8}$$

and, in line with the approximations of this section, i.e. currents in the impulse approximation and hence a nuclear Hamiltonian H without exchange terms, it is straightforward to prove that again only one-body operators appear in Eq. (3.8) (i.e. no coherence effects between $i - j$ nucleons appearing in the two currents). Indicatively a typical contribution is

$$\sum_i g_A \tau_i^- \frac{\sigma_i \cdot \underline{\tau}}{\underline{\tau} \cdot \underline{\tau}} \frac{\hbar \underline{\tau} \cdot \underline{\tau}}{Mk} \frac{\underline{q} \cdot \underline{k}}{Mk} e^{-i(\underline{k} + \underline{\nu}) \cdot \underline{r}_i}$$

therefore depressed with respect to the already small 1/k terms roughly by the naively expected E/k factor. In this connection it is worth recalling that in combining the electromagnetic and the weak current both isovector and isotensor contributions arise in the total amplitude (the isoscalar term being forbidden because of the charge selection rule); the former allow transition to the $\Delta T = 1$ states, the latter to the $\Delta T = 1, 2$ states. The results obtained, being one-body operators, give necessarily $\Delta T = 1$ contributions. It is only by relaxing the preceding condition (absence of exchange terms) that possible $i - j$ correlations arise. They can of course contribute to both isospin channels. $\Delta T = 1$ terms can easily be shown to be of the same order of magnitude as Eq. (3.8).

The contributions to the $\Delta T = 2$ transitions which do not interfere with the rest of the amplitude, can be best evaluated in a standard way considering their contribution to the rate. Moreover, isotensor excitation operators have a geometrical weight $T/(T + 2)$ and $2/(T + 2)$ in the $T + 1$ and $T + 2$ channels. The contributions to $\Delta T = 2$ transitions can therefore be straightforwardly proved to be totally negligible for medium-heavy nuclei^{†)}.

At this point let us make a little digression on the behaviour of the amplitude versus the photon energy.

Suppose in fact, we are interested in the process at very low photon energy. In this case we must keep the amplitude in its general form, Eq. (3.3), which describes the process as a sequence of ordinary^{††)} muon capture followed by γ emission and vice versa.

In particular, below nuclear resonances, taking for illustration the same currents as in the evaluation of Eq. (3.6), the first term of its expansion in powers of k/E is

$$T \sim \langle b | \left[\sum_i \frac{1+\tau_3}{2} \tau_i \cdot \epsilon, \sum_j g_A \tau_j \cdot \sigma_j \right] e^{-i \mathbf{z} \cdot \mathbf{h}} | a \rangle \quad (3.9)$$

$$= \langle b | \sum_i g_A \tau_i (\mathbf{h} \cdot \epsilon) (\sigma_i \cdot \sigma) e^{-i \mathbf{z} \cdot \mathbf{h}} | a \rangle$$

This shows quite clearly how the effective one-body Hamiltonian in use is appropriate in describing the process for high k 's only. At low energy it gets drastically modified and big corrections are to be expected.

iii) Exchange terms

It is well known that the nuclear interaction modifies the previously defined currents in the sense that simple additivity of the contribution of individual

†) In $T = 0$ nuclei isotensor operators contribute only to the $\Delta T = 2$ channel with weight 1. An explicit estimate had been made by the authors for ^{40}Ca . We thank H.P.C. Rood for pointing out a numerical factor which further reduces the predicted small contribution, bringing it into agreement with the discussion of this section.

††) This statement is only partially true because of the absence of the g_p contribution to the "weak" nuclear excitation.

nucleons (impulse approximation) no longer holds true. In order to study these effects we begin by considering the correction to the electromagnetic and weak vector currents, both of which we will assume exactly conserved to avoid unnecessary complications¹⁸⁾.

Along standard lines one has that the continuity equation

$$\vec{k} \cdot \vec{J}^a(\vec{k}) = -i [H, J_0^a(\vec{k})] \quad (3.10)$$

(where a stands for the isospin index) implies the presence of a term such that

$$\vec{k} \cdot \Delta \vec{J}^a(\vec{k}) = \sum_{i,j} (\tau_i^a - \tau_j^a) (e^{i\vec{k} \cdot \vec{r}_i} - e^{i\vec{k} \cdot \vec{r}_j}) V_{ij}^{\text{ex}} \quad (3.11)$$

where V_{ij}^{ex} is an exchange potential.

Such terms do appear in the currents of the time-ordered product, however their explicit calculation is not necessary thanks to the Siegert theorem, i.e. to the possibility of expressing the main contribution of the current's matrix elements through multipoles of the charge distribution $J_0^a(\vec{k})$.

However exchange contributions appear explicitly in the seagull term and it has been shown¹⁷⁾ that a unique feature of two-current nuclear processes is represented by the contact interaction of the external probes.

As a matter of fact if we analyse the amplitude given in Eq. (3.2) we easily verify that conservation of the vector currents does not automatically guarantee the gauge invariance of the full amplitude; on the contrary this condition gives rise to the following equations for the seagull terms (as far as vector-vector processes are concerned)

$$k_\mu q_\nu T^{\alpha\beta}_{\mu\nu} = i \varepsilon_{abc} k_\mu J_\mu^c(q+k) + k_\mu q_\nu S_{\mu\nu}^{\alpha\beta} \quad (3.12)$$

and using divergence conditions for the vector currents

$$\begin{aligned} \int \int d^4x d^4y \delta(x_0 - y_0) \langle a | [J_0^a(y), [H, J_0^b(x)]] | a \rangle e^{iqx} e^{-iky} = \\ = i \varepsilon_{abc} k_\mu J_\mu^c(q+k) + k_\mu q_\nu S_{\mu\nu}^{\alpha\beta} \end{aligned} \quad (3.13)$$

Developing the double commutator and expanding the result in powers of q and k , we find after straightforward but cumbersome calculations (with $\tilde{r}_{ij} = \tilde{r}_i - \tilde{r}_j$)

$$\begin{aligned} k_\mu q_\nu S_{\mu\nu}^{ab} = & \frac{k \cdot q}{M} \delta^{ab} \langle a | \sum_i e^{i(q-k) \cdot \tilde{r}_i} | a \rangle + \\ & + \sum_{i < j} \langle a | 4(\delta^{ab} - \frac{1}{3}(\tau_i^a \tau_j^b) \delta^{ab}) (\tilde{k} \cdot \tilde{r}_{ij}) (\tilde{q} \cdot \tilde{r}_{ij}) V_{ij}^{ex} \\ & + [\{\tau_i^a, \tau_j^b\} + \{\tau_j^a, \tau_i^b\} - \frac{4}{3}(\tau_i^a \tau_j^b) \delta^{ab}] (\tilde{k} \cdot \tilde{r}_{ij}) (\tilde{q} \cdot \tilde{r}_{ij}) V_{ij}^{ex} | a \rangle \end{aligned} \quad (3.14)$$

As expected from crossing properties the zero-order expansion of the amplitude in k, q , must be symmetric in isospin indices. In the case $a = 3$ and $b = -$ of RMC the only contribution is given by

$$k_\mu q_\nu S_{\mu\nu}^{3-} = \sum_{i < j} \langle a | (\{\tau_i^3, \tau_j^-\} + \{\tau_j^3, \tau_i^-\}) (\tilde{k} \cdot \tilde{r}_{ij}) (\tilde{q} \cdot \tilde{r}_{ij}) V_{ij}^{ex} | a \rangle \quad (3.15)$$

which is equivalent to a contribution to the amplitude of the form

$$T_{ex} \sim \langle a | \sum_{ij} \tau_{ij}^2 V_{ij}^{ex} | a \rangle$$

with the sum on the nucleons running only on proton-neutron couples. This term corresponds to the static limit of all possible diagrams in which the photon and the weak vertex can couple to the basic exchange potential diagram. This term has, apart from a different structure of the isospin operator, the same form as the usual dipole sum rule enhancement factor. We have therefore a contribution of order $(ZN/MA)\kappa$ (where $\kappa \simeq 1$), i.e. depressed by the usual factor $\mu/2M$ with respect to the muon radiating diagram.

It is perhaps interesting at this point to contrast this result with the one obtained in Compton scattering, at similar energies. There the equivalent term, imposed by gauge invariance, was found to be of the same order as the dominating one due to incoherent scattering¹⁷⁾. The reason for the depression of this exchange contribution in RMC is the scale $1/\mu$ introduced by the radiation from the muon. Thus the CVC hypothesis allows the determination of exchange terms for e.m. weak vector current correlations. As for the axial part, the situation is not as

favourable. In addition to the PCAC hypothesis, one has to have the extra knowledge of the coupling of the pion to the nucleus. This last problem has been discussed in detail for space-like pions, as for instance in ordinary muon capture^{10,19)}. In that case the nuclear medium produces a strong quenching of the pseudoscalar coupling constant. Ohta¹⁶⁾ has suggested that such a quenching should also affect the g_p coefficient of RMC. This has been criticized^{†)} on the grounds that the main contribution to the terms in g_p arises from the graph of Fig. 1b, for which the pion is time-like ($\underline{q} \approx 0$, $q_0 \approx m_\mu$). This process can be thought of as S-wave radiative pion capture, to which a weak vertex is attached, about which the following semiquantitative considerations can be made. The effect of the nuclear medium on the propagator of the virtual pion can be described by an optical potential similar to the one relevant for the physical pion. In the $\underline{q} = 0$ limit only its local part matters, modifying the propagator from its free value $1/(m_\pi^2 - m_\mu^2)$ to $1/(m_\pi^2 - m_\mu^2 + 2m_\pi V_{opt})$. It is therefore easy to see, using standard values of V_{opt} ²⁰⁾, that the effects of the medium are not supposed to cause large effects. Moreover, the strong Kroll-Rudermann term $\underline{\sigma} \cdot \underline{\epsilon}$, dominant for radiative pion capture is not expected to be strongly renormalized by the medium either. However, the possibility of nuclear anomalies cannot be ruled out. We hence take the common attitude of using the Hamiltonian we have derived within the impulse approximation with nucleonic values of the coupling constants, in order to have a firm theoretical basis for the analysis of future experiments. Their results would thus provide interesting information on possible nuclear renormalization effects.

4. ORDINARY AND RADIATIVE CAPTURE IN THE CLOSURE APPROXIMATION

In order to get a semiquantitative feeling of the order of magnitude of RMC along the nuclear table and to discuss the consequences of the closure approximation, the results for ordinary capture rates will first be recalled using this approximation, leaving the more exact treatment to later on.

†) M. Ericson, private communication.

The total capture rate can be written, with standard approximations, as

$$\Lambda_0 = \frac{G^2}{2\pi} |\phi_\mu^2|_{av} \int \frac{d\bar{\nu}}{4\pi} \sum_b \nu_b^2 | \langle b | O^- | a \rangle |^2 \quad (4.1)$$

The evaluation of the matrix element in the Fermi gas model yields, in the energy weighted form,

$$\int \frac{d\bar{\nu}}{4\pi} \sum_b \nu_b^2 | \langle b | O^- | a \rangle |^2 = \frac{\bar{\nu}^4 Z}{2M^* \bar{E}'} \frac{1}{1-\alpha} \left\{ 1 - \frac{3}{8} x \gamma_- - \right. \\ \left. - \frac{3}{16} \frac{1}{x} \gamma_- \gamma_+ + \frac{1}{128} \frac{1}{x^3} \gamma_-^3 \right\} \quad (4.2)$$

where M^* is the effective mass, $\alpha = (N - Z)/A$, $\gamma_\pm = (1 + \alpha)^{2/3} \pm (1 - \alpha)^{2/3}$, $x = \bar{\nu}/2k_F$, k_F is the Fermi momentum taken to be 1.36 fm^{-1} , $\bar{E}'$ the excitation energy of the capturing nucleus, $\bar{\nu} = \nu_{av}$.

Although completely equivalent to the non-energy weighted form, Eq. (4.2) has the advantage of being less dependent on correlations. In particular, it can be expanded with a good approximation to the first order in α , yielding

$$\frac{\bar{\nu}^4 Z}{2M^* \bar{E}'} \left[1 - \left(\frac{k_F}{\bar{\nu}} - 1 + \frac{\bar{\nu}}{4k_F} \right) \alpha \right] \simeq \frac{\bar{\nu}^4 Z}{2M^* \bar{E}'} (1 - \delta \alpha) \quad (4.3)$$

where δ , which is expressed in terms of the parameter of the model k_F and of the neutrino momentum, is rather constant with the value $\delta \simeq 2.5$.

Therefore the final expression for the rate is

$$\Lambda_0 = \frac{G^2}{2\pi} |\phi_\mu^2|_{av} \frac{\bar{\nu}^4 Z}{2M^* \bar{E}'} (1 - \delta \alpha) \quad (4.4)$$

$\bar{E}'$ and $\bar{\nu}$ are as usual related by $\bar{E}' + \bar{\nu} = \nu_\mu \equiv m_\mu - \epsilon + E_C$, ϵ and E_C standing, respectively, for the muon binding energy in the 1S orbit and for the Coulomb energy shift between capturing and final nucleus.

To justify the use of the Fermi gas model, some comments are in order. It is well known that total rates have been successfully fitted, assuming a constant $\bar{\nu}$,

by the so-called Primakoff plot²¹⁾ which has been shown by Bell and Løvseth²²⁾ to correspond to a truncation to the first order in α of the Fermi gas calculation.

The full expression was on the contrary in disagreement with experiment. Subsequently, through a combined use of energy weighted and non-energy weighted sum rules, Christillin, Dellafiore and Rosa-Clot⁹⁾ were able to predict a decrease in the average momentum $\bar{v}$, contrary to the previous assumption, reproducing at the same time the rates without any truncation. In this connection let us stress that M^* is the only phenomenological parameter of the theory [see Eq. (4.4)] which intervenes also in the prediction of $\bar{v} = \bar{v}(\alpha, M^*)$.

Therefore, the Fermi gas model calculations can be considered as a self-consistent improvement^{†)} of the Primakoff plot, with additional support for the prediction of a decreasing $\bar{v}$, as seen from neutron multiplicities²⁴⁾.

We now consider RMC, with the same technique, taking into account only the contribution of the free muon radiating diagram. Within this approximation, the spectrum has the well-known form

$$N(k)dk = \frac{e^2}{2\pi^2} |\Phi_\mu|_{av}^2 \frac{1}{m_\mu^2} \int \frac{d\hat{k}}{4\pi} \int \frac{d\hat{v}}{4\pi} (g_v^2 + 3g_A^2) \sum_b |\langle b | O_r^- | a \rangle|^2 k(k_{b2}-k)^2 dk \quad (4.5)$$

Then in so far as the Hamiltonian and hence the operator

$$O_r^- = \sum_i \tau_i^- e^{-i(\vec{v}+\vec{k})\cdot\vec{r}_i} = \sum_i \tau_i^- e^{-i\vec{S}_i\cdot\vec{r}_i}$$

of RMC has been shown to be a one-body operator, the matrix element we are interested in is exactly the same as in Eq. (4.2). It is then easy to get

$$N(k)dk = \frac{e^2}{2\pi^2} |\Phi_\mu|_{av}^2 \frac{k_\mu^4}{m_\mu^2} \int \frac{d\hat{k}}{4\pi} \frac{S^2 Z}{2M^* E'} (1-\delta\alpha) (g_v^2 + 3g_A^2) X(1-X)^2 dX \quad (4.6)$$

†) This in spite of more general criticisms questioning the applicability at all of such a model²³⁾, because of bad convergence properties.

where a closure value k_m has been introduced for the photon momentum

$$x = \frac{k}{k_m}, \quad \delta' \approx \frac{k_F}{k_m \sqrt{(1-x)^2 + x^2 + 2x(1-x)\gamma}} - 1$$

in close similarity to Eq. (4.3),

$$S^2 = k_m^2 [(1-x)^2 + x^2 + 2x(1-x)\gamma], \quad \gamma = \frac{\hat{v} \cdot \hat{k}}{v_\mu}$$

with the usual energy-conservation relation $E' + v + k = v_\mu$.

Using now the closure approximation in its strictest sense, i.e. assuming the excitation energy to be the same for the two operators O^- and O_r^- and hence k_m equal to v_{av} , one obtains a ratio to ordinary capture of

$$R(x)dx \approx \frac{e^2}{\pi} \frac{k_m^2}{m_\mu^2} (1-x)(1-2x+2x^2)x(1-x)^2 dx \quad (4.7)$$

This order of magnitude estimate predicts a decrease of the ratio along the table because of the factor $1 - \alpha$ and of the decrease of v_{av} ⁹⁾ as a function of α . As a matter of fact, with typical $v_{av} \sim 90$ MeV for $\alpha = 0$ and $v_{av} \sim 80$ MeV for $\alpha = 0.2$, one has an integrated relative spectrum for $k \geq 60$ MeV of 9×10^{-6} and 3×10^{-6} , respectively.

The first number, remembering that the canonical g_p value roughly doubles the rate in this region, may be compared with the experimental numbers of $R(k > 57 \text{ MeV}) \approx 20 \times 10^{-6}$ for ^{40}Ca . The second one might serve with the same provision as an indication for lead. However, the closure approximation has to be questioned here, especially for heavy nuclei.

As a matter of fact a slightly different choice $k_m = 75$ MeV reduces the previous estimate considerably more than naively expected by simple power counting because of the smaller fraction of the available spectrum

$$\int_0^1 (1-2x+2x^2)x(1-x)^2 dx$$

which corresponds to the *fixed* measured part of the spectrum $k \geq 60$ MeV, and brings down the above number to $\approx 1 \times 10^{-6}$. It is worth stressing that the introduction

of k_m might be of some significance for total RMC rates only, with the same rôle played by v_{av} in ordinary muon capture.

For this reason the more sophisticated method based on the calculation of the response function in the Fermi gas model will be used in the following.

5. MUON CAPTURE: NUMERICAL RESULTS AND THE EFFECTIVE MASS

As already stated, the only free parameter in our model is given by the effective mass, which we use to simulate exchange effects. Its value is fixed by a fit to ordinary total capture rates, which we will report in this section, the improvement upon a previous treatment⁹⁾ consisting in the inclusion of recoil terms.

The total muon capture rate is

$$\Lambda_\mu = \Lambda_0 + \Lambda' \quad (5.1)$$

where the exact expression for Λ_0 , given in the closure approximation in the previous section, is reported in Appendix B. Λ' can be estimated beyond closure along the same lines as $E\nu$ times the excitation spectrum. The result is, however, well reproduced by the analogous expression in closure²⁵⁾

$$\Lambda' = \frac{g_V^2 + (G_A - H_A)g_A}{2\pi} \left| \phi_\mu^2 \right|_{av} \frac{\bar{\nu}^3 Z}{M} \frac{1}{1-\alpha} \left\{ 1 + \frac{1}{128\chi^3} - \frac{3}{8}\chi - \frac{3}{16}\chi^2 \right\} \quad (5.2)$$

where the ratio of these coupling constants to those appearing in ordinary capture

$$\frac{g_V^2 + (G_A - H_A)g_A}{g_V^2 + 3G_A^2 + H_A^2 - 2G_A H_A} \simeq 0.35$$

and $\Lambda' \simeq 0.15 \Lambda_0$ ^{†)}.

†) The difference between the quoted values and those of Ref. 25, is due to the fact that there the p/M terms were actually considered to be p/M^* terms. Although the treatment of exchange effects in H_{eff} is by no means definite, it seemed to us more correct to keep here to a more conventional viewpoint. Therefore results according to Eq. (5.2) do but slightly differ (because of ν variation) from those previously calculated in Ref. 26.

The fit of the previous expression to experimental rates yields a χ^2 of 2600 and $M^* = 0.5$ M. For illustration we report instead of the usual plot the numerical results in Table 1, using the experimental values of Eckhause et al.²⁷⁾.

Apart from nuclei around $\alpha \sim 0.07$, showing structure effects, the well-known over-all agreement is apparent. Therefore the above-mentioned value of M^* will be used in the following. As concerns recoil terms in RMC, the procedure used in Ref. 25 cannot be straightforwardly adapted, because of more complicated kinematical conditions. Radiative spectra will therefore be given without these terms. We know that in the $N = Z$ case, the combination of recoil coupling constants is $g_V^2 + g_A^2$, i.e. essentially the one appearing in ordinary capture, Eq. (5.2). The same thing should probably happen for $N \neq Z$ nuclei as well. If this is true, consideration of the above coupling constant ratio to the main term

$$\simeq (g_V^2 + g_A^2) / (g_V^2 + 3g_A^2 + 2g_A g_p \eta + 2g_p^2 \eta^2)$$

shows that the contribution of recoil terms may be expected to be smaller in RMC than in ordinary capture.

In the absence of a quantitative estimate, this represents a theoretical uncertainty of our calculation. Its order of magnitude is however $\leq 10\%$, i.e. well below the sensitivity of the spectrum versus g_p .

6. THE PHOTON SPECTRUM FOR RADIATIVE CAPTURE

The photon spectrum for RMC can be expressed, when neglecting velocity-dependent terms, as

$$N_\lambda(k)dk = \frac{e^2}{2\pi^2} |\Phi_\mu^2|_{av} \frac{1}{m_\mu^2} \int \frac{d\hat{k}}{4\pi} \int \frac{d\nu}{4\pi} G_{r,\lambda}^2 k\nu^2 \sum_b \sum_a |\langle b | \sigma_r^- | a \rangle|^2 dk \quad (6.1a)$$

where λ refers to the photon circular polarization $\pm$, Σ 's have an obvious meaning, $|\Phi_\mu^2|_{av}$ is the muon wave function averaged over the nuclear volume and $G_{r,\lambda}^2 = A_\lambda(k,y) \pm \hat{k} \cdot \hat{\zeta} B_\lambda(k,y)$, which contains the relevant coupling constants, is given in Appendix A.

Equation (6.1) can be alternatively rewritten as

$$N(k)dk = \frac{e^2}{2\pi^2} |\phi_\mu|^2 \frac{1}{2v_\mu^2} \int \frac{d\hat{k}}{4\pi} \int \frac{d\hat{v}}{4\pi} A(k, \gamma) k v^2 \sum_b \sum_a |\langle b | \hat{O}_\gamma | a \rangle|^2 dk \quad (6.1b)$$

It is apparent, as is well known, that all relevant dynamical quantities pertaining to the problem are related to the response function

$$R(E', s) = \sum_b \sum_a |\langle b | \sum_i \tau_i e^{-i \sum_j t_j} | a \rangle| \delta(E_b - E') \quad (6.2)$$

where $s = v + k$, in terms of which everything can be restated.

For a better understanding, let us discuss the general features of the problem using the kinematical constraints in the E', s plane, before turning to the solution of Eqs. (6.2) and (6.1b).

$$\text{Energy momentum conservation} \quad E' = v_\mu - k - v = v_\mu - k - \sqrt{s^2 + k^2 - 2kst}$$

gives us for the limiting cases, $t = \cos \theta_{sk} = \pm 1$ for $k \geq v_\mu/2$, the straight lines $E' = v_\mu - s$, $E' = v_\mu - 2k + s$ in the upper E' plane.

The allowed domain for the reaction is therefore the hatched area of Fig. 2, inside which dynamics of the Fermi gas model will place further constraints (for a detailed and analytical discussion see Appendix B), the most obvious of them being the fact that for $\alpha > 0$ nuclei, the excitation energy E' be definite positive, i.e. $E' = (n_F^2 - p_F^2)/2M^*$ necessary to push a proton above the filled-up neutron states.

We are hence in a region of intermediate excitation energy, bounded by photo-reactions ($E' = s$) and ordinary muon capture ($E' = v_\mu - s$, s representing always the appropriate momentum transfer), with a favoured momentum transfer lying preferentially in the high s part of the allowed region. Therefore the combined suppression of the very low excitation energy region and the simultaneous request of a

relatively high momentum transfer, give a reasonable support to our choice of the nuclear model in the cases we are treating. Explicit calculations quantifying the preponderance of the domain lying closer to the muon capture line, supporting the above viewpoint, are given in Appendix B.

As far as the actual calculations are concerned, the excitation spectrum of the Fermi gas model with the coupling constants $A_\lambda(k, y)$ has been integrated numerically. Results for the photon spectrum $N(k) = N_+(k) + N_-(k)$ are given in Table 2 for a sample of medium-heavy nuclei as a function of k . For orientation purposes, coupling constants appear in its expression to the leading order in g_V , g_A , and g_P as $\sim g_V^2 + 3g_A^2 + 2g_A g_P \eta + 2(g_P \eta)^2$.

For each nucleus, absolute predictions of the photon spectrum in $\text{MeV}^{-1} \text{ sec}^{-1}$ are reported for the various values of the pseudoscalar coupling constant (expressed as Δ times the standard Goldberger-Treiman value), as well as the corresponding values of ordinary total capture rates in 10^5 sec^{-1} . In the last column the average values of E' , defined in a standard way as a weighted mean over the spectrum, are given for each photon momentum for the case $\Delta = 1$, results for the other values being very similar. At the bottom the average excitation energy $\bar{E}'$ for ordinary capture is also reported. It is evident that there occurs a sizeable decrease of E' with increasing photon energy as intuitively expected. It is also worth stressing its smaller value with respect to the average excitation energy in ordinary capture.

This represents a quantitative self-commenting illustration of the inadequacy, anticipated many times, of the closure approximation both as regards a common average excitation value for all k 's and its identification with the corresponding value in muon capture.

Photon spectra are seen not to vary drastically, either in shape or in magnitude along the nuclear table^{†)}, therefore putting all nuclei on an equal footing

†) A small qualitative difference between our results and Rood and Tolhoek's, concerning the relative position of the $\Delta = -1$ and $\Delta = 0$ spectra can be easily explained, as due to our more exact treatment. See also the comments in next paragraph and in Appendix A.

from the point of view of experimental feasibility. The sensitivity to the various values of g_p is essentially the same as in $N = Z$ nuclei. Heavy nuclei are therefore expected to represent a crucial source of information about the nuclear value of the pseudoscalar coupling constant.

7. PHOTON POLARIZATION AND MUON SPIN
PHOTON ANGULAR CORRELATION

The circular polarization of the photons emitted in radiative muon capture is given by

$$P(k) = \frac{N_+(k) - N_-(k)}{N_+(k) + N_-(k)} \quad (7.1)$$

Another observable of interest (because of the absence of final-state interactions) is the muon photon angular correlation. The possibility of its observation stems from the fact that the initially polarized μ in cascading down to the $1s$ orbit still retains a sizeable percentage of residual polarization, typically of the order of 15%. Its expression is

$$\Gamma(k) = \frac{B_+(k) + B_-(k)}{A_+(k) + A_-(k)}$$

where

$$B(k) = \int \frac{d\Omega}{4\pi} B(k, \Omega) \quad A(k) = \int \frac{d\Omega}{4\pi} A(k, \Omega) \quad (7.2)$$

Both quantities are easily obtained from Appendix A. For orientation purposes we can however give the following approximate expression to the leading order in g_V , g_A and g_p , which displays, in a rough way, the dependence both on the magnitude and on the sign of g_p

$$P(k) = \Gamma(k) \simeq \frac{g_V^2 + 3g_A^2 + 2g_A g_p \eta}{g_V^2 + 3g_A^2 + 2g_A g_p \eta + 2g_p^2 \eta^2} \quad (7.3)$$

Finally we give in Table 3 both quantities, for the nuclei considered in the previous paragraph as well as the average value of $\theta_{\nu k}$ at the different energies. This average is taken over the full expression of the differential rate, i.e. spectrum and coupling constants. It is evident that $\theta_{\nu k}$ is peaked at 90° , but with a smoother behaviour superimposed.

With increasing k , i.e. decreasing E' , $\theta_{\nu k}$ goes from smaller to larger values, i.e. the higher the photon momentum the more the photon and neutrino tend to be antiparallel.

This causes some structure which is not present in the oversimplified Eq. (7.3) or in Rood and Tolhoek's results either. This is especially evident, for instance, in the case of a magnified g_p contribution, i.e. $\Delta = 1, 2$ in the form of a ripple in $\Gamma(k)$. $\Gamma(k)$ and $P(k)$ are indeed, in a first approximation, model-independent. This depends, however, on the use of a linearized expression in y for the differential rate, which allows one to discard the linear terms for numerical reasons, and on the closure approximation, which washes out the y dependence in g_p and permits factorization of the excitation spectrum so that Γ and P are essentially given by Eq. (7.3). When both these assumptions are relaxed our results obtain. The previous quantities do not show a drastic variation for the different nuclei and in each case, apart from the obvious case of $\Delta = 0$, which yields $P(k) \approx \Gamma(k) \approx 1$, the predicted values for $\Delta = 1$ are always definitely bigger than for $\Delta = -1, 2$. The different behaviour in k in these two cases is also accompanied by markedly different values of the radiative spectrum, so that maximum information should come of course from a combined measurement of these quantities. To illustrate the previous statement $P(k)$ and $\Gamma(k)$ and the spectrum have been plotted for ^{208}Pb in Fig. 3.

In conclusion, a complete treatment has been given of the problem of photon polarization and of muon photon angular correlation, with explicit inclusion of the full muon propagator and of the excitation spectrum. Reported results are to a reasonable extent model-independent.

8. CONCLUSIONS

We have studied in this paper radiative muon capture by medium-heavy nuclei. To reach quantitative conclusions a number of problems have been overcome. First the non-relativistic reduction of the free proton amplitude obtained through low-energy theorems has been checked to essentially reproduce the effective Hamiltonian previously used in $N = Z$ nuclear calculations.

The rôle of nuclear intermediate states in the two-step process actually happening in nuclei has been investigated and shown not to influence sizeably, as regards vector-vector correlations, the one-body Hamiltonian valid for energetic photons. When using standard values for the axial-vector coupling constants, it has been confirmed that the high-energy part of the spectrum is essentially determined with a roughly equal weight by the radiation from the muon and by the g_p term due to photon emission by the proton. Previous results on the muon propagation in the finite-size nuclear Coulomb field, not greatly at variance with the free case, have been used to treat correctly the muon radiating diagram.

Finally the Fermi gas model has been used in the nuclear calculation. No closure approximation has been made by properly integrating its response function, in the energy momentum transfer plane. The only free parameter in the theory being the effective mass, which is determined through a fit to ordinary rates, we regard the excitation spectrum we use as a reasonable representative of the "real" situation.

The main difference of our results with respect to the closure approximation consists in a decreasing excitation energy for increasing k 's, or in turn in an increasing "equivalent" average maximum k_m in the closure language.

Radiative spectra do remain essentially constant as a function of A , therefore making heavy nuclei equally favourable targets, and spectra do exhibit the same sensitivity to g_p as in the $N = Z$ case.

Photon polarization and muon-spin photon angular correlation, which represent a complementary source of information, have been calculated as well. Their slightly lesser sensitivity to g_p is compensated by a greater model independence. Predictions for all of the previous quantities have been given as a function of different g_p values.

In summary, the present work constitutes, in view of its thorough analysis, a firm theoretical basis for the analysis of future experiments.

With reasonably accurate measurements spanning the periodic table, the determination of the value of the pseudoscalar coupling constant in nuclei seems therefore now quite possible.

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Table 1

Systematics of total muon capture rates. Experimental data from Ref. 27
Theoretical predictions: Fermi gas model, present article.

Nucleus	α	$\Lambda_{\text{exp}}/10^6 \text{ sec}$	$(\Lambda_{\text{th}} - \Lambda_{\text{exp}})/\Lambda_{\text{exp}}$ (%)
$_{22}\text{Ti}$	0.082	2.60 ± 0.04	-10.5
$_{23}\text{V}$	0.098	3.19 ± 0.06	-22.7
$_{24}\text{Cr}$	0.078	3.29 ± 0.06	-7.3
$_{25}\text{Mn}$	0.091	3.91 ± 0.07	-18.2
$_{26}\text{Fe}$	0.070	4.41 ± 0.03	-9.1
$_{27}\text{Co}$	0.085	4.96 ± 0.04	-17.0
$_{28}\text{Ni}$	0.047	5.92 ± 0.07	-5.7
$_{29}\text{Cu}$	0.088	5.67 ± 0.03	-13.5
$_{29}^{63}\text{Cu}$	0.079	5.72 ± 0.05	-9.9
$_{30}\text{Zn}$	0.084	5.74 ± 0.04	-3.8
$_{31}\text{Ga}$	0.111	5.68 ± 0.06	-8.3
$_{32}\text{Ge}$	0.119	5.53 ± 0.06	-0.9
$_{33}\text{As}$	0.120	6.06 ± 0.06	-0.7
$_{34}\text{Se}$	0.140	5.69 ± 0.05	+1.9
$_{37}\text{Rb}$	0.135	6.89 ± 0.14	-4.3
$_{38}\text{Sr}$	0.133	7.25 ± 0.14	-1.9
$_{38}^{88}\text{Sr}$	0.136	6.61 ± 0.27	+6.4
$_{39}\text{Y}$	0.123	7.89 ± 0.11	+2.3
$_{40}\text{Zr}$	0.124	8.59 ± 0.07	-0.9
$_{41}\text{Nb}$	0.118	10.40 ± 0.14	-9.7
$_{42}\text{Mo}$	0.124	9.22 ± 0.06	+3.9
$_{45}\text{Rh}$	0.126	10.01 ± 0.07	+8.8
$_{46}\text{Pd}$	0.136	10.00 ± 0.07	+6.4
$_{47}\text{Ag}$	0.129	10.88 ± 0.09	+7.4
$_{48}\text{Cd}$	0.148	10.62 ± 0.10	-0.0
$_{49}\text{In}$	0.147	11.37 ± 0.13	-3.0

Table 1 (cont.)

Nucleus	α	$\Lambda_{\text{exp}}/10^6 \text{ sec}$	$(\Lambda_{\text{th}} - \Lambda_{\text{exp}})/\Lambda_{\text{exp}}$ (%)
$_{50}\text{Sn}$	0.157	10.68 ± 0.12	+6.5
$_{51}\text{Sb}$	0.162	10.49 ± 0.14	-1.0
$_{52}\text{Te}$	0.186	9.06 ± 0.11	-2.6
$_{53}\text{I}$	0.165	11.20 ± 0.11	-3.3
$_{55}\text{Cs}$	0.173	10.98 ± 0.25	-1.2
$_{56}\text{Ba}$	0.183	10.18 ± 0.10	+1.1
$_{57}\text{La}$	0.178	10.71 ± 0.10	+2.3
$_{58}\text{Ce}$	0.171	11.11 ± 0.11	+4.5
$_{59}\text{Pr}$	0.163	13.45 ± 0.13	-3.3
$_{60}\text{Nd}$	0.167	12.32 ± 0.14	+4.4
$_{62}\text{Sm}$	0.174	12.22 ± 0.17	+6.1
$_{64}\text{Gd}$	0.186	12.09 ± 0.16	-0.2
$_{65}\text{Tb}$	0.182	12.73 ± 0.13	-1.0
$_{67}\text{Ho}$	0.188	12.95 ± 0.13	-4.3
$_{72}\text{Hf}$	0.192	13.03 ± 0.21	+0.1
$_{73}\text{Ta}$	0.193	12.86 ± 0.13	+2.3
$_{74}\text{W}$	0.195	13.13 ± 0.20	+1.3
$_{79}\text{Au}$	0.198	13.39 ± 0.11	+6.3
$_{80}\text{Hg}$	0.200	12.74 ± 0.26	+12.0
$_{81}\text{Tl}$	0.207	13.78 ± 0.18	+1.3
$_{82}\text{Pb}$	0.208	13.02 ± 0.11	+4.8
$_{83}\text{Bi}$	0.205	13.26 ± 0.10	+4.9
$_{90}\text{Th}$	0.224	13.10 ± 0.11	-7.9
$_{92}\text{U}$	0.227	11.0 ± 0.5	+11.8
$^{235}_{92}\text{U}$	0.217	14.7 ± 0.9	-6.6

Table 2

Predicted radiative spectrum $N(k)$ of RMC in the Fermi gas model for various nuclei and various values of g_p (see text). E' represents the average excitation energy at each value of k , whereas the average excitation energy $\bar{E}'$ in ordinary muon capture is shown at the bottom.

	k (MeV)	N(k) sec·MeV				E' (k) (MeV)
		$\Delta = -1$	$\Delta = 0$	$\Delta = 1$	$\Delta = 2$	
${}_{42}\text{Mo}$ $\alpha = 0.124$	60	10.9	10.5	15.4	25.4	25.3
	65	8.4	8.2	12.8	22.3	25.2
	70	5.2	5.1	8.5	15.6	24.9
	75	2.6	2.4	4.3	8.1	23.1
	80	1.1	1.0	1.8	3.5	21.0
	85	0.3	0.3	0.6	1.2	19.1
	90	0.1	0.1	0.1	0.2	17.3
	$\Lambda_\mu/10^5 \text{ sec}$	125	109	97	88	31
${}_{50}\text{Sn}$ $\alpha = 0.157$	60	10.1	9.5	13.9	23.4	27.3
	65	7.7	7.4	11.7	20.7	27.0
	70	4.7	4.5	7.7	14.4	26.9
	75	2.1	2.0	3.5	6.8	25.0
	80	0.8	0.7	1.3	2.5	22.9
	85	0.2	0.2	0.3	0.7	20.4
	90	0.0	0.0	0.0	0.1	17.2
	$\Lambda_\mu/10^5 \text{ sec}$	135	118	105	96	32
${}_{64}\text{Gd}$ $\alpha = 0.186$	60	9.8	9.0	13.2	22.5	29.0
	65	7.4	6.9	11.0	19.8	28.8
	70	4.4	4.1	7.3	13.8	28.6
	75	1.8	1.6	3.0	5.9	26.8
	80	0.5	0.5	0.9	1.8	25.1
	85	0.1	0.1	0.2	0.3	23.2
	90	0.0	0.0	0.0	0.0	12.6
	$\Lambda_\mu/10^5 \text{ sec}$	154	135	120	109	33

Table 2 (cont.)

	k (MeV)	N(k) sec·MeV				E' (k) (MeV)
		$\Delta = -1$	$\Delta = 0$	$\Delta = 1$	$\Delta = 2$	
${}_{74}\text{W}$ $\alpha = 0.195$	60	10.1	9.2	13.5	23.3	29.5
	65	7.6	7.0	11.3	20.5	29.2
	70	4.5	4.2	7.5	14.3	28.9
	75	1.8	1.6	3.0	5.9	27.2
	80	0.5	0.4	0.8	1.7	24.9
	85	0.1	0.1	0.1	0.3	22.6
	90	0.0	0.0	0.0	0.0	12.3
	$\Lambda_{\mu}/10^5 \text{ sec}$	172	150	133	122	33
${}_{82}\text{Pb}$ $\alpha = 0.208$	60	9.3	8.4	12.6	21.7	30.2
	65	6.9	6.4	10.3	18.9	29.9
	70	4.0	3.7	6.7	12.9	29.9
	75	1.4	1.3	2.4	4.8	28.1
	80	0.3	0.3	0.6	1.2	25.9
	85	0.0	0.0	0.1	0.1	23.8
	90	0.0	0.0	0.0	0.0	0.0
	$\Lambda_{\mu}/10^5 \text{ sec}$	177	155	138	126	34
${}_{92}\text{U}$ $\alpha = 0.227$	60	7.1	6.3	9.4	16.7	31.7
	65	5.3	4.7	7.7	14.3	31.2
	70	2.9	2.7	4.9	9.6	30.9
	75	0.9	0.8	1.5	3.2	29.1
	80	0.2	0.1	0.3	0.6	26.7
	85	0.0	0.0	0.0	0.0	17.2
	90	0.0	0.0	0.0	0.0	0.0
	$\Lambda_{\mu}/10^5 \text{ sec}$	158	138	123	112	35

Table 3

Predictions of the photon circular polarization $P(k)$ and of the muon spin photon angular correlation $\Gamma(k)$ as a function of k for various nuclei and various values of g_p , in the Fermi gas model. $\langle\theta_{\nu k}\rangle$ stands for the average value at each k of the angle between photon and neutrino.

k (MeV)	P(k) for $\Delta =$				$\Gamma(k)$ for $\Delta =$				$\langle\theta_{\nu k}\rangle$ (°)
	-1	0	1	2	-1	0	1	2	
$_{42}\text{Mo}$									
60	0.61	0.96	0.86	0.62	0.69	1	0.81	0.52	88
65	0.53	0.96	0.85	0.61	0.67	1	0.81	0.55	96
70	0.45	0.95	0.84	0.59	0.64	1	0.81	0.55	98
75	0.41	0.95	0.82	0.57	0.60	1	0.79	0.53	97
80	0.37	0.95	0.80	0.54	0.56	1	0.77	0.50	96
85	0.33	0.95	0.78	0.51	0.52	1	0.74	0.47	95
90	0.28	0.95	0.76	0.50	0.48	1	0.72	0.44	95
$_{50}\text{Sn}$									
60	0.60	0.96	0.85	0.60	0.67	1	0.78	0.47	84
65	0.52	0.96	0.84	0.59	0.64	1	0.79	0.51	92
70	0.42	0.95	0.83	0.57	0.61	1	0.79	0.53	97
75	0.38	0.95	0.81	0.55	0.58	1	0.77	0.51	96
80	0.34	0.95	0.79	0.52	0.54	1	0.75	0.48	96
85	0.29	0.95	0.77	0.50	0.49	1	0.73	0.45	95
90	0.25	0.95	0.75	0.44	0.44	1	0.70	0.40	94
$_{64}\text{Gd}$									
60	0.61	0.97	0.84	0.58	0.65	1	0.76	0.43	80
65	0.51	0.96	0.83	0.58	0.62	1	0.77	0.48	89
70	0.41	0.95	0.82	0.56	0.59	1	0.78	0.52	96
75	0.36	0.95	0.80	0.54	0.55	1	0.76	0.50	96
80	0.31	0.95	0.78	0.51	0.51	1	0.74	0.46	95
85	0.27	0.95	0.74	0.48	0.47	1	0.70	0.43	95
90	0.22	0.95	0.73	0.45	0.41	1	0.68	0.40	93

Table 3 (cont.)

k (MeV)	P(k) for $\Delta =$				$\Gamma(k)$ for $\Delta =$				$\langle \theta_{vk} \rangle$ (°)
	-1	0	1	2	-1	0	1	2	
^{74}W									
60	0.61	0.97	0.84	0.57	0.64	1	0.75	0.42	80
65	0.51	0.96	0.82	0.57	0.61	1	0.76	0.46	88
70	0.39	0.95	0.81	0.55	0.58	1	0.77	0.51	96
75	0.34	0.95	0.79	0.53	0.54	1	0.75	0.49	96
80	0.30	0.95	0.77	0.50	0.50	1	0.73	0.45	95
85	0.25	0.95	0.75	0.47	0.45	1	0.70	0.43	95
90	0.21	0.95	0.73	0.45	0.40	1	0.67	0.39	93
^{82}Pb									
60	0.60	0.97	0.83	0.57	0.63	1	0.74	0.39	77
65	0.50	0.96	0.82	0.56	0.60	1	0.75	0.44	87
70	0.38	0.95	0.81	0.55	0.57	1	0.77	0.50	96
75	0.33	0.95	0.79	0.52	0.53	1	0.75	0.48	96
80	0.29	0.95	0.77	0.50	0.48	1	0.73	0.45	95
85	0.23	0.95	0.75	0.47	0.43	1	0.70	0.41	94
90	0.00	0.00	0.00	0.00	0.00	0	0.00	0.00	
^{92}U									
60	0.59	0.97	0.81	0.55	0.61	1	0.72	0.35	74
65	0.50	0.96	0.81	0.54	0.57	1	0.73	0.41	85
70	0.36	0.95	0.80	0.54	0.55	1	0.76	0.49	96
75	0.31	0.95	0.78	0.51	0.51	1	0.73	0.46	96
80	0.26	0.95	0.76	0.48	0.46	1	0.71	0.43	95
85	0.22	0.95	0.74	0.46	0.42	1	0.68	0.40	94
90	0.00	0.00	0.00	0.00	0.00	0	0.00	0.00	

APPENDIX A

Our starting point for the dependence of RMC on the various coupling constants is essentially Rood and Tolhoek's result, i.e.

$$\begin{aligned}
 H_{eff} = & \frac{1}{\sqrt{2} m_\mu} (1 - \hat{\underline{s}} \cdot \hat{\underline{v}}) \sum_{i=1}^A \left[\left\{ g_1(\underline{\sigma}_i \cdot \underline{\underline{s}}) + g_5(\underline{\sigma}_i \cdot \hat{\underline{k}})(\underline{v} \cdot \underline{\underline{s}})/2M + g_6 \frac{\underline{v} \cdot \underline{\underline{s}}}{2M} \right\} \tau_i^- \right. \\
 & + \left\{ g_2 \underline{\underline{s}} + g_3 i(\underline{\underline{s}} \wedge \underline{\underline{v}}) + g_4(\underline{\sigma}_i \cdot \underline{\underline{s}}) \underline{\underline{s}}/2M + g_7 i(\underline{\underline{s}} \wedge \underline{\underline{s}})/2M + \right. \\
 & + \left. g_8(\underline{\sigma}_i \cdot \underline{\underline{s}}/2M) \underline{\underline{s}} + g_9(\underline{v} \cdot \underline{\underline{s}}/2M) \underline{\underline{v}} + g_{10}(\hat{\underline{v}} \cdot \underline{\underline{s}}) \underline{\underline{s}}/2M \right\} \cdot \underline{\sigma}_i \tau_i^- + \\
 & + \left. \left\{ -g_v \lambda_+ (\underline{\sigma}_i \cdot \underline{\underline{v}})(\underline{\sigma}_i \cdot \underline{\underline{k}})/M - g_A \lambda_+ (\underline{\sigma}_i \cdot \underline{\underline{v}})(\underline{\sigma}_i \cdot \underline{\underline{k}})/M + \eta g_P^N(\underline{\sigma}_i \cdot \hat{\underline{k}})(\underline{\underline{k}} \cdot \underline{\underline{s}})/M \right\} \tau_i^- \right] \delta^3(\underline{r} - \underline{r}_i)
 \end{aligned} \quad (A.1)$$

where $\underline{\underline{s}} = \underline{\underline{k}} + \underline{\underline{v}}$, $\hat{\underline{v}}$ and $\hat{\underline{k}}$ are the unit vectors along $\underline{\underline{v}}$ and $\underline{\underline{k}}$ and

$$\begin{aligned}
 g_1 &= -g_v \lambda_+ (1 + \underline{\underline{s}} \cdot \hat{\underline{k}}/2M) - g_v \eta + g_A \lambda \eta \mu - g_M \lambda \eta \mu (\underline{v} \cdot \hat{\underline{k}}/2M) \\
 g_2 &= -g_A \lambda_+ - g_P^N \eta + g_v \lambda \eta \mu - g_A \eta \\
 g_3 &= -g_A \lambda_+ + g_M \eta + g_v \eta - g_A \lambda \eta \\
 g_4 &= g_A \lambda_+ + g_P^L \lambda_- \\
 g_5 &= g_v \lambda_+ + g_M \lambda \eta \mu \\
 g_6 &= g_v \lambda_+ + g_P^N \lambda \eta \mu \\
 g_7 &= g_v \lambda_+ + g_M \lambda_+ + g_P^N \lambda \eta \\
 g_8 &= -g_v \lambda_+ - g_M \lambda_+ \\
 g_9 &= g_v \lambda_+ + g_M \lambda_+ \\
 g_{10} &= g_P^N \eta \ 4M \nu / (m_\pi^2 + (\mu - \nu - k)^2)
 \end{aligned} \quad (A.2)$$

The standard notations are used

$$\eta = m_\mu/2M \quad \lambda_\pm = \frac{1}{2}(1 \pm \lambda) \quad \mu = 1 + \mu_p - \mu_n = 4.7$$

$$g_P^N = \frac{\Delta \ 2M m_\mu g_A}{m_\pi^2 + (\mu - \nu)^2} \quad g_P^L = \frac{\Delta \ 2M m_\mu g_A}{m_\pi^2 + (\mu - \nu - k)^2}$$

Δ is a parameter assuming the values -1, 0, 1, 2 to allow the simulation of violations of the Goldberger-Treiman relation ($\Delta = 1$) and the numerical values $g_V = 0.98$, $g_A = -1.23$, $g_M = 3.7$ in units of $10^{-5}/M^2$ will be used. We further stress that in Eq. (A.1) we have neglected terms of the order of $(1/k)$ which are important only for low-energy photons. Moreover the terms containing λ_+ in the previous expression, easily shown to come from the radiation by the muon, will be multiplied by the factor $\sqrt{G_{\ell\ell}}$ depending on the nucleus and to a very good extent constant in k , as discussed in Ref. 11, to account for non-locality effects and attraction by the nuclear Coulomb field.

Such an effective Hamiltonian, summing over lepton spin and final nuclear states, in the SU(4) limit, gives a contribution to the differential rate of the form

$$G_{r,\lambda}^2 = A_\lambda(k, \gamma) + (\vec{k} \cdot \vec{\zeta}) B_\lambda(k, \gamma) \quad (A.3)$$

where $\vec{\zeta}$ is the residual muon polarization, $y = \cos \theta_{\nu k}$ and

$$\begin{aligned} A_\lambda(k, \gamma) = & g_1^2(1+\lambda\gamma) + g_2^2 + g_3^2(2+\lambda\gamma) - 2g_2g_3\lambda\gamma + \\ & + [g_2g_4(-1+\gamma^2)(1-x) - g_3g_4(1-x+2\lambda x+2\lambda(1-x)\gamma+2x\gamma \\ & + (1-x)\gamma^2) - 2g_2g_7\lambda(x+(1-x)\gamma) + g_3g_7(1-x+2x\gamma+ \\ & + (1-x)\gamma^2) - 2g_2g_8(1-x+x\gamma) + 2g_3g_8\lambda(x+(1-x)\gamma) \\ & + g_2g_9(-1+\gamma^2)(1-x) + 2g_3g_9(1-\gamma^2)(1-x) + g_2g_{10}(1-\gamma^2)(1-x) \\ & + g_3g_{10}\lambda x(1-\gamma^2) - g_1g_5\lambda(1-\gamma^2)(1-x) + g_1g_6(-1+\gamma^2)(1-x)] k + \\ & + [g_4^2(1+\lambda\gamma)(1-2x(1-x)(1-\gamma)) + g_4g_{10}(-1+\gamma^2)(1-2x(1-x)(1-\gamma)) \\ & + g_{10}^2(1-\gamma^2)(\frac{1}{2} - x(1-x)(1-\gamma))] k^2 \end{aligned} \quad (A.4)$$

$$\begin{aligned}
 B_2(k, y) = & g_1^2(\lambda + y) - g_2^2 y + g_3^2 \lambda + 2g_2 g_3 \lambda + \\
 & + [g_2 g_4 \lambda (-1 + y^2)(1-x) - g_3 g_4 (2x + \lambda - \lambda x + 2(1-x)y + 2\lambda xy \\
 & + \lambda(1-x)y^2) + 2g_2 g_7 \lambda (x + (1-x)y) - g_3 g_7 (-\lambda + 2x + \\
 & + 2\lambda xy - 2xy + \lambda(1-x)y^2) + 2g_2 g_8 (x + y(1-x)) - \\
 & - 2g_3 g_8 \lambda(1-x) + g_2 g_9 \lambda(1-y^2)(1-x) - g_3 g_9 (1-y^2)(1-x) + \\
 & + g_3 g_{10} (\lambda - \lambda x + x + (1-x)y - \lambda(1-x)y^2 - xy^2 - (1-x)y^3 - \\
 & - g_1 g_5 (1-y^2)(1-x) - g_1 g_6 \lambda(1-y^2)(1-x)] \kappa + \\
 & + [g_4^2 (\lambda + y)(1 - 2x(1-x)(1-y)) - g_4 g_{10} \lambda(1-y^2)(1 - 2x(1-x)(1-y)) \\
 & - g_{10}^2 y(1-y^2)(\frac{1}{2} - x(1-x)(1-y))] \kappa^2
 \end{aligned}$$

where $x = k/(\nu_\mu - E')$ and $\kappa = (\nu_\mu - E')/2M$.

Minor modifications with respect to Rood and Tolhoek come from our standard definition of g_p (see above) and from not using the closure approximation.

As a matter of fact, the pseudoscalar term denominator will be correctly expressed as $m_\pi^2 + (\mu - \nu)^2 = m_\pi^2 - m_\mu^2 + 2m_\mu(m_\mu - E' - k)$ with E' function of k and y , as given by our model, instead of approximating $\nu_\mu - E' = \nu_\mu - \bar{E}' = k_m$.

APPENDIX B

To gain a better insight into the contribution of different regions in the energy-momentum plane in the Fermi gas model, the exact analytical solution for the spectrum will be worked out in this Appendix, assuming coupling constants independent of k .

As is well known, our nuclear states are Fermi spheres of neutrons and protons, filled up to n_F and p_F , respectively. The excitation operator

$$O_r^- = \sum_{i=1}^A \tau_i^- e^{-i(\underline{\nu} + \underline{k}) \cdot \underline{r}_i}$$

transforms a proton into a neutron and displaces the proton Fermi sphere by $\underline{\nu} + \underline{k} = \underline{s}$ in momentum space (see Fig. 4).

The RMC can then be simply expressed in terms of the integral:

$$\begin{aligned} I(k) &= \int \frac{d\hat{k}}{4\pi} \int \frac{d^3\nu}{4\pi} \sum_b \sum_a |K_b| |O_r^-| a \rangle|^2 \delta^4(p_f - p_i) = \\ &= \int \frac{d\hat{k}}{4\pi} \int \frac{d^3\nu}{4\pi} \int d^3n (1 - \theta(n_F - n)) \int d^3p \theta(p_F - p) \delta^3(\underline{p} - \underline{n} - \underline{s}) \delta(E_f - E_i) = \\ &= \int \frac{d\hat{k}}{4\pi} \int \frac{d^3\nu}{4\pi} \int d^3n (1 - \theta(n_F - n)) \theta(p_F - |\underline{n} + \underline{s}|) \delta(E' + \nu + k - \epsilon) \end{aligned} \quad (B.1)$$

where the θ functions explicitly take into account the effect of the Pauli principle.

As far as the energy is concerned, the usual choice $E' = (\underline{n}^2 - \underline{p}^2)/2M$ will be made, i.e. purely kinetic neglecting the muon momentum. Exchange effects will be simulated by an effective mass M^* so that with the definition $n_z = -\underline{n} \cdot \hat{s}$ the energy reads $E' = (-s^2 + 2sn_z)/2M^*$.

Moreover, in the transition the number of protons changes from the parent to the daughter nucleus so that the Coulomb shift must be properly taken into account, as can be seen for instance from Fig. 5, where it is shown that corresponding isobaric states are lowered in the final nucleus by $\sim E_c$ (we have always properly taken into account the neutron proton mass difference, even without explicit indication). That is to say that the neglect of the Coulomb part in the

above calculation would give us approximately the $T + 1$ levels of the same nucleus, those excited by electron scattering, which we denote by E' . Their relation to the actual excitation energy E of the daughter nucleus is of course $E' = E + E_c$ and $E = E_{b,b_{gr}} + \Delta M$.

Energy conservation then reads at the spectrum end point ($\nu = 0$), which corresponds to ordinary muon capture,

$$k_M = m_\mu - \epsilon + E = (m_\mu - \epsilon + E_c) - E'.$$

As already remarked $m_\mu - \epsilon + E_c \equiv \nu_\mu$ is fairly constant around the value ≈ 113 MeV for $A > 40$. The energy variable E' we use represents a physical quantity, i.e. the energy of the $T + 1$ channel of the parent nucleus; we remark that a slightly decreasing k_M for heavy nuclei corresponds to an increase of E' , which is consistent with an excitation energy of the daughter nucleus $E_{b,b_{gr}}$ ranging between 15 and 20 MeV.

We now turn to Eq. (B.1) which reads with $\tilde{n}_\perp = \tilde{n} - \tilde{n}_z \hat{s}$

$$I(k) = \int \frac{d\hat{k}}{4\pi} \int \frac{d^3\nu}{4\pi} \int d\eta_z \delta(E' + k + \nu - \nu_\mu) \int d^2\eta_\perp (1 - \Theta(\eta_z - \eta) \Theta(\nu_F - |\eta_\perp + s|)) \quad (B.2)$$

with two distinct regions contributing to the last integral.

It is immediately realized that with a suitable change of variables the evaluation of the integral greatly simplifies.

Namely, for fixed k , $d^3\nu = d^3s$ so that the angular integration $d\hat{k}$ will yield 4π and, in terms of the variable $t = \hat{s} \cdot \hat{k}$,

$$\int d^3s = 2\pi \int s^2 ds \int dt.$$

Physically, for every k , a whole spectrum of values for the neutrino energy occurs, but the particle being undetected no information is lost with the above

transformation. Therefore by reexpressing the neutrino momentum in terms of the new variables and using E' as a function of s and n_z we obtain

$$I(k) = \frac{1}{2} \int S^2 ds \int dt \int dE' \frac{M^*}{s} \delta(E' + k + \sqrt{s^2 + k^2 - 2kst} - \nu_\mu) \cdot$$

$$\cdot \left\{ S_1 \Theta\left(\frac{M^* E'}{s} + \frac{s}{2} - \eta_F\right) \Theta\left(-\frac{M^* E'}{s} + \frac{s}{2} + \eta_F\right) + \right.$$

$$\left. + S_2 \Theta\left(E' - \frac{\eta_F^2 - k_F^2}{2M^*}\right) \Theta\left(-\frac{M^* E'}{s} - \frac{s}{2} + \eta_F\right) \right\} \quad (B.3)$$

with

$$S_1 = \pi \left(k_F^2 - \left(\frac{M^* E'}{s} - \frac{s}{2} \right)^2 \right)$$

$$S_2 = \pi \left(k_F^2 - \eta_F^2 + 2M^* E' \right) \quad (B.4)$$

The integration over dE' immediately gives $E' = \nu_\mu - k - \sqrt{s^2 + k^2 - 2kst}$ and the Θ functions then fix at given k the constraints in t, s

$$\nu_\mu - \frac{s^2}{2M^*} - \frac{k_F s}{M^*} < k + \sqrt{s^2 + k^2 - 2kst} < \nu_\mu + \frac{s^2}{2M^*} - \frac{\eta_F s}{M^*} \quad (B.5a)$$

$$\nu_\mu + \frac{s^2}{2M^*} - \frac{\eta_F s}{M^*} < k + \sqrt{s^2 + k^2 - 2kst} < \nu_\mu - \frac{\eta_F^2 - k_F^2}{2M^*}$$

Remembering that $k + \sqrt{s^2 + k^2 - 2kst} = k + v = \nu_\mu - E'$ we can visualize the above double inequalities as specifying, respectively, regions S_1 and S_2 of Fig. 6a in the E', s plane. At given k , the one-parameter family of curves $E' = E'(s, t) = \nu_\mu - k - \sqrt{s^2 + k^2 - 2kst}$ are sketched in Fig. 6b. As $t \rightarrow 1$ the curve $E'(s, t)$ tends to $E' = \nu_\mu - s$ for $s > k$ and to $E' = \nu_\mu - 2k + s$ for $s < k$; as $t \rightarrow -1$ it tends to $E' = \nu_\mu - 2k - s$. Superposition of Fig. 6a and 6b tells us, at fixed k , what are the limits in the multiple integration $\int ds/dt$; at given s , t must be such that $E'(s, t)$ lies either in region S_1 or in S_2 ; in the former case the integrand is the function S_1 , in the latter the function S_2 .

As k increases, whereas Fig. 6a remains the same, Fig. 6b changes; the lines $E' = \nu_\mu - 2k + s$ and $E' = \nu_\mu - 2k - s$ shift downward so that the overlap between Fig. 6a and 6b tends, loosely speaking, to decrease. One intermediate case is,

for example, shown in Fig. 7. In the case of Fig. 7 we shall also need the k -dependent values c_0, c_1, c_2 , in addition to the fixed interception points s_0, s_1, s_2, s_3 . As for the allowed t -values, we introduce at given s the limiting values t_2, t_1, t_0 , obtained by intercepting $E'(s, t)$ respectively with the two parabolas curving up and down, and with the straight line of Fig. 6a obeying the equations

$$\begin{aligned} E' &= \frac{s^2}{2M^*} + \frac{h_F s}{M^*} & \text{II} \\ E' &= -\frac{s^2}{2M^*} + \frac{u_F s}{M^*} & \text{I} \\ E' &= \frac{u_F^2 - h_F^2}{2M^*} \end{aligned} \quad (\text{B.5b})$$

Very straightforward calculation gives

$$\begin{aligned} s_0 &= u_F - h_F \\ s_1 &= -(h_F + M^*) + \sqrt{(h_F + M^*)^2 + 2M^* \nu_\mu} \\ s_2 &= u_F + M^* - \sqrt{(u_F + M^*)^2 - 2M^* \nu_\mu} \\ s_3 &= \nu_\mu - (u_F^2 - h_F^2)/2M^* \end{aligned} \quad (\text{B.6a})$$

$$\begin{aligned} c_0 &= 2k - \nu_\mu + (u_F^2 - h_F^2)/2M^* \\ c_1 &= u_F - M^* + \sqrt{(u_F - M^*)^2 - 2M^*(\nu_\mu - 2k)} \\ c_2 &= -(h_F - M^*) + \sqrt{(h_F - M^*)^2 + 2M^*(\nu_\mu - 2k)} \end{aligned} \quad (\text{B.6b})$$

$$\begin{aligned} t_0 &= (s^2 + k^2 - (\nu_\mu - (u_F^2 - h_F^2)/2M^* - k)^2)/2ks \\ t_1 &= (s^2 + k^2 - (\nu_\mu + \frac{s^2}{2M^*} - u_F s/M^* - k)^2)/2ks \\ t_2 &= (s^2 + k^2 - (\nu_\mu - \frac{s^2}{2M^*} - \frac{h_F s}{M^*} - k)^2)/2ks \end{aligned} \quad (\text{B.6c})$$

Before presenting the final result some comments are in order. Our previous figures showing the allowed region as a function of k in the E', s plane are actually a family of curves depending on p_F , n_F and hence on $\alpha = (N - Z)/A$. In this regard, it is immediately realized, for instance, that the lowest allowed E' is zero for $N = Z$ nuclei ($n_F = p_F$) and gradually rises according to $E' = (n_F^2 - p_F^2)/2M^*$, the same shift applying to the upper limit $E' = v_\mu - s_1$. Of course, since we must integrate S_1 and S_2 over the support, our integrals would in reality correspond to a tridimensional plot.

As an illustration we show in Fig. 8 the excitation spectrum for the integration path $E' = v_\mu - s$, corresponding to ordinary muon capture⁹⁾ for $s = v$ and to the photon spectrum end point when $s = k_M$.

As already commented upon in the case of ordinary muon capture, rates have been calculated⁹⁾ in the closure approximation, along the line $v = \text{const}$ instead of the kinematically correct path $E' = v_\mu - v$. The average v was then determined through energy conservation and the explicit calculation of energy-weighted and non-energy-weighted sum rules. Here the same comment applies with one basic difference, namely that for each k a different average energy should be used. In particular it is easy to recognize from the previous figures that for high k 's the average excitation energy must necessarily decrease.

The present procedure therefore automatically satisfies in the most exact way the above requirements.

Coming back to $I(k)$ it is straightforward to obtain (for $k \geq v_\mu/2 \approx 55$ MeV)

$$\begin{aligned}
 I(k) = \frac{M^*}{2} \Big\{ & \int_{s_0}^{s_1} s ds (A(t_2) - A(t_1)) + \int_{s_1}^{s_2} s ds (A(1) - A(t_1)) + \\
 & + \int_{s_2}^{s_3} s ds (B(t_1) - B(t_0)) + \int_{s_3}^{s_4} s ds (B(1) - B(t_0)) \Big\} \quad \frac{v_\mu}{2} < k < k_0 \\
 & + \frac{M^*}{2} \Big\{ \int_{s_1}^{s_2} s ds (A(1) - A(t_1)) + \int_{s_2}^{s_1} s ds (A(t_2) - A(t_1)) +
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{s_1}^{s_2} s ds (A(1) - A(t_1)) \Big\} & k_0 < k < s_2 \\
 & + \frac{M^*}{2} \left\{ \int_{c_0}^{c_1} s ds (B(1) - B(t_0)) + \int_{c_1}^{s_2} s ds (B(t_1) - B(t_0)) + \right. \\
 & \quad \left. + \int_{s_2}^{s_3} s ds (B(1) - B(t_0)) \right\} & k_0 < k < s_2 \quad (B.7) \\
 & + \frac{M^*}{2} \int_{s_1}^{s_2} s ds (A(1) - A(t_1)) & s_1 < k < s_2 \\
 & + \frac{M^*}{2} \int_{c_0}^{s_3} s ds (B(1) - B(t_0)) & s_2 < k < s_3
 \end{aligned}$$

where $k_0 = \frac{1}{2} \left(\nu_\mu - \frac{\eta_F^2 - k_F^2}{2M^*} + \eta_F - k_F \right)$

$$\begin{aligned}
 A(t) = \int dt \, \mathcal{S}_1 = \pi \Big\{ & \left[k_F^2 - \left(M^*(\nu_\mu - k)/s - \frac{s}{2} \right)^2 - M^{*2}(s^2 + k^2)/s \right] t^{(B.8)} \\
 & + M^{*2} k t^2 / s - 2M^*/(3ks^2) \cdot (M^*(\nu_\mu - k)/s - \frac{s}{2})(s^2 + k^2 - 2kst)^{3/2} \Big\}
 \end{aligned}$$

and

$$B(t) = \int dt \, \mathcal{S}_2 = \pi \left\{ \left[k_F^2 - \eta_F^2 + 2M^*(\nu_\mu - k) \right] t + \frac{2M^*}{3ks} (s^2 + k^2 - 2kst)^{3/2} \right\}^{(B.9)}$$

Analogous integrals in powers of y can be analytically performed expressing $y = \hat{v} \cdot \hat{k}$ in terms of the new variables as $(-k + st)/(k^2 + s^2 - 2kst)^{1/2}$.

Since all our results for the spectra are normalized to the total ordinary capture rate Λ_0 , in the sense that a χ^2 to the experimental data is used to determine M^* , it is straightforward to reduce the previous expression to the non-radiative case. In that case ($\int dt = 2$, $s \rightarrow \nu$), we obtain

$$\Lambda_0 = \frac{G^2}{2\pi} |\phi_\mu^2|_{qv} \int \frac{d\hat{\nu}}{4\pi} \sum_b \nu^2 |\langle b | \Sigma_i \gamma_i e^{-i\nu \cdot \hat{x}_i} | a \rangle|^2 = \quad (B.10)$$

$$\begin{aligned} &= \frac{G^2}{2\pi} |\phi_\mu^2|_{qv} M^* \left[\int_{\nu_1}^{\nu_2} S_1 \nu d\nu + \int_{\nu_2}^{\nu_3} S_2 \nu d\nu \right] = \\ &= \frac{G^2}{2\pi} |\phi_\mu^2|_{qv} \pi M^* \left\{ \left[-C^2 \ln x + 2CM^*x + (k_F^2 + C - M^{*2}) \frac{x^2}{2} - M^* \frac{x^3}{3} - \frac{x^4}{16} \right] \right|_{x=\nu_1}^{x=\nu_2} \\ &\quad + \left[(k_F^2 - \eta_F^2 + 2C) \frac{x^2}{2} - 2M^* \frac{x^3}{3} \right] \right|_{x=\nu_2}^{x=\nu_3} \end{aligned}$$

with the previous definition of symbols, $C = \nu_\mu M^*$, and

$$\begin{aligned} \nu_3 &= \nu_\mu - \frac{\eta_F^2 - k_F^2}{2M^*} \\ \nu_2 &= \frac{2\nu_\mu M^*}{M^* + \eta_F + \sqrt{(M^* + \eta_F)^2 - 2\nu_\mu M^*}} \\ \nu_1 &= \frac{2\nu_\mu M^*}{M^* + k_F + \sqrt{(M^* + k_F)^2 + 2\nu_\mu M^*}} \end{aligned} \quad (B.11)$$

The corresponding formula in Ref. 9 was unfortunately marred by some misprints.

In the numerical comparison it is useful to remember that $Z \sim \frac{4}{3}\pi p_F^3$.

Figure captions

- Fig. 1 : Diagrams contributing to radiative muon capture.
- Fig. 2 : Allowed region for RMC in the energy momentum transfer plane E',s because of kinematical constraints, for $N > Z$ nuclei. For details see text.
- Fig. 3 : For ^{208}Pb and for various values of g_p
a) Radiative spectrum $N(k)$ in $\text{MeV}^{-1} \text{ sec}^{-1}$.
b) Circular polarization $P(k)$.
c) Muon-spin photon angular correlation $\Gamma(k)$.
- Fig. 4 : Dynamics of the Fermi gas model.
- Fig. 5 : Simplified energy levels scheme for capturing and final nucleus.
- Fig. 6 : a) Allowed regions S_1 and S_2 for RMC in the Fermi gas model.
b) Behaviour of $E'(s,t)$ for given k .
- Fig. 7 : Allowed region for RMC for given k and relevant quantities pertaining to the calculation of the spectrum.
- Fig. 8 : Tridimensional excitation spectrum for ordinary muon capture in the E',s plane.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6a

Fig. 6b

Fig. 7

Fig. 8