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EVOLUTION OF PARTON DENSITIES BEYOND LEADING ORDER :

THE NON-SINGLET CASE

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A B S T R A C T

We develop a technique, based explicitly on the factorization property of mass singularities, which allows to calculate the evolution of parton densities beyond leading order. We present the results for the evolution of hadronic structure functions as well as of parton fragmentation functions into hadrons. Within our scheme the predictions for a particular process are obtained by convoluting a universal parton density with a "short distance" cross-section specific to the process. As an application we calculate the QCD predictions for the Q^2 dependence of deep inelastic lepton-hadron scattering and of one particle inclusive e^+e^- annihilation cross-sections. Our results for electroproduction agree with those obtained with the operator product expansion technique. Physical quantities in scattering are related to the corresponding ones in annihilation by analytic continuation, whereas the Gribov-Lipatov relation is strongly violated.

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1. - INTRODUCTION

The possibility of quantum chromodynamics being the true theory for strong interactions is being tested in an increasing number of processes. Perturbative predictions for a large class of processes like the evolution of hadronic structure functions and fragmentation functions or the inclusive muon pair production and one hadron inclusive large angle scattering in hadron-hadron collisions are based on the factorization property of mass singularities. These manifest themselves as logarithms of the ratio of some large scale characterizing the process over some unknown effective quark and gluon mass and their presence spoils the meaning of a perturbative expansion in the "falling" coupling constant.

The factorization theorem allows to reabsorb such singularities into the definition of parton densities : predictions for any hard process made in terms of parton distributions experimentally determined in a given process and at a given energy scale will be free of mass singularities to any order of perturbation theory. It follows that a comparison between theory and experiment can be carried out beyond leading order. Apart from a general motivation of improving the severity of the tests, the analysis beyond leading order leads to a more precise determination of the effective scale Λ which governs the strength of the "falling" coupling constant.

For deep inelastic lepton-hadron scattering (DIS) the operator product expansion (OPE) ¹⁾ provides a systematic scheme for calculating, via the renormalization group, the logarithmic scaling violations to arbitrary accuracy in α .

It is well-known, however, that an expansion in terms of local operators is not relevant for the calculation of the pattern of the scaling violation in the inclusive quark $\rightarrow$ hadron fragmentation functions. On the other hand, the latter quantities are experimentally accessible with present e^+e^- machines in a range of energy scales unexplored by DIS experiments ; this pushes one to construct the diagrammatic scheme for a systematic calculation of higher order effects in QCD, applicable to any hard process. A technique of this type has been developed for fragmentation functions by Mueller ²⁾, where new kinds of operators, the "cut vertices", are introduced : as in OPE, ultra-violet divergences appearing in the radiative corrections are related to an external momentum scale dependence. Some preliminary calculations (φ^3 in six dimensions) within this scheme have been done recently ³⁾.

The method developed in this paper is based on the factorization properties of mass singularities in the axial gauge ; consequently, we get much more insight on the physical aspects of the perturbative QCD calculations than in the formal OPE-like methods. Our scheme incorporates many of the current ideas available on the factorization market, like the generalized ladder (2PI kernel) decomposition ⁴⁾, finiteness of the 2PI kernels in the axial gauge ⁵⁾, etc. The mass singularities are regularized by means of the dimensional regularization technique ^{6),7)} : in the massless theory in $n = 4 + \epsilon$ dimensions ($\epsilon > 0$) they appear as poles in ϵ . The dimensional method, combined with the minimal subtraction prescription of 't Hooft simplifies essentially the practical calculations as compared with more traditional schemes (like, e.g., the off-shell method) and, even more important, assures the gauge invariance of the relevant quantities in the intermediate steps of the calculation.

The most essential technical ingredient of our method is the use of the light-like gauge ($n_\mu A_\mu = 0$, $n^2 = 0$). It enables us to retain, in the higher order calculations, the probabilistic interpretation typical to leading log approximation, simplifies enormously the calculation as compared with the $n^2 \neq 0$ case, and provides a direct link with the OPE technique for electroproduction.

Within our scheme, the QCD predictions for a specific hard process are given, in x space, in the form of the convolution of some universal parton density $\Gamma(x, Q^2)$ with the "short-distance" cross-section $C(x, Q^2)$, characteristic for a given process.

The density $\Gamma(x, Q^2)$ depends only on whether the evolving parton has space-like momentum (like in DIS) or time-like momentum [like in single-particle inclusive e^+e^- annihilation (SIA)]. In the following, we refer to both kinematical configurations as spacelike and timelike ones, denoting the corresponding densities by Γ_S and Γ_T , respectively.

The Q^2 evolution of the densities Γ_S and Γ_T is governed by the probabilities $P_S(x, \alpha)$, $P_T(x, \alpha)$ which can be calculated perturbatively in the "falling" coupling constant ; to order α (leading logs) they coincide with the Parisi-Altarelli ⁸⁾ (PA) probability $P(x, \alpha) = (\alpha/2\pi) C_F ((1+x^2)/(1-x))_+$ both for spacelike and timelike densities.

As an application of the scheme proposed in the paper we present the results for next-to-leading corrections to non-singlet quantities in DIS and SIA, i.e., we calculate P_S and P_T to order α^2 as well as C_{DIS} and C_{SIA} to order α .

The x moments of $P_S(x, \alpha)$, as will be demonstrated in the paper, coincide to all orders with the anomalous dimensions of twist-two operators (calculated according to 't Hooft's scheme). As a check of our calculation, we have compared the x moments for the $O(\alpha^2)$ term in $P_S(x, Q^2)$ with the existing formulas for the non-singlet twist-two anomalous dimensions of the Wilson operators⁹⁾ at two loops.

We agree with these results.

The x representation of $P(x, \alpha)$ appears to be much simpler than the formulas for moments and admits a nice physical interpretation like the generalization to two loops of the PA probabilities. We also expect that the x space formulas can be more useful in practice for data samples suffering from uncertainties in the low x region, which complicate the experimental analysis of moments. In fact, predictions in x space for scaling violations at, say, x_0 do involve the knowledge of experimental structure functions for $x \geq x_0$ only.

The detailed description of the method is presented in Section 2. Section 3 contains a pedagogical application of the scheme to the leading log calculation, where we also collect some technical details concerning the light-like gauge. Next-to-leading corrections for spacelike densities are presented and discussed in Section 4. In Section 5 we demonstrate the equivalence of our method in the spacelike case with the OPE technique. In Section 6 we discuss some general relations between space- and timelike densities (like the analytic continuation rule and the Gribov-Lipatov relation) and we present the $O(\alpha^2)$ corrections to timelike probability $P_T(x, \alpha)$. The "short-distance" cross-sections for DIS and SIA are derived in Section 7. Some comments about various probabilistic interpretations of our results are given in Section 8, where we also present some numerical estimates of the relative strength of corrections for DIS and SIA.

Concluding remarks are given in Section 9.

2. - FACTORIZATION OF MASS SINGULARITIES : THE GENERAL FRAMEWORK

2.1. - History

The factorization properties of the perturbative expansion in QCD have been studied extensively during the last few years by means of various methods and techniques ¹⁰⁾⁻¹²⁾. As far as the general technique is concerned, the reader interested in the history of the factorization programme is referred to the Zimmermann-Symanzik proof of the OPE in perturbation theory ⁴⁾. The Mueller approach ^{2),4)} is the continuation of these ideas in the application for timelike processes, like SIA, etc. The factorization of the collinear logarithms $\ln Q^2/p^2$ is realized here by taking the limit $Q^2 \rightarrow \infty$. In this way one introduces some extra ultra-violet divergences into the theory, which in turn can be factorized by means of the powerful methods of renormalization theory.

An independent approach consists in taking the limit $p^2 \rightarrow 0$ that generates mass singularities ¹³⁾. It appears that the general factorization technique is very similar in both methods. In fact, it has been demonstrated recently ⁵⁾ that the Mueller method (originally formulated for φ^4 theory) can be applied in QCD to factor out the mass singularities to all orders in perturbation theory, assuming that one works in the axial gauge.

Our method should be considered as an application and a continuation of the ideas presented in the papers quoted above (to which we also refer for a more complete review of the relevant literature). We use the 2PI kernel expansion like in Ref. 4), we factor out the mass singularities like in Ref. 5) and finally we use the renormalization group methods like in Ref. 2).

On the other hand, our main goal is to construct a scheme which is as close as possible to parton model intuition and, simultaneously, which enables one to perform explicit higher order calculations with minimal effort. Therefore we work in the light-like gauge and use dimensional regularization of mass singularities which, in turn, forces us to reformulate many technical points in the constructions proposed in the papers quoted above. Therefore, in the following, we present the full scheme without further reference to the original papers.

2.2. - Generalized ladder expansion

The general strategy consists in a rearrangement of the perturbative expansion which makes explicit the factorization into a part which does not contain any mass singularity and another one which contains all (and only) mass singularities. For definiteness, we will present our construction for DIS as an example of spacelike processes. The universal parton densities $\Gamma_S(x)$ and $\Gamma_T(x)$, extracted in these reactions, can then be used in any other hard process if only the appropriate "short-distance" cross-section is calculated. In Sections 2.2-2.7, we present the method for the spacelike case. The timelike configuration can be treated in a similar way, with some modifications which are discussed in Section 2.8.

To fix our ideas, let us consider the matrix element squared for virtual (spacelike) photon-quark interaction, denoted by blob M in Fig. 1. In the axial gauge, the integrations giving rise to mass singularities are easily localizable once the expansion of M in 2PI kernels C_0 and K_0 (see Fig. 1) is performed. It has been proved ⁵⁾ that the 2PI amplitudes in the axial gauge are finite as long as the external legs are kept unintegrated ; consequently, all collinear singularities originate from the integrations over the momenta flowing in the lines connecting various kernels. (By definition, the kernels K_0 contain the full propagators of upper lines, and do not contain lower lines.)

As seen in Fig. 1, all kernels entering the generalized ladder expansion (GLE) depend on two momenta and carry two pairs of spinor (or vector) indices. In the following we will use the following compact notation :

- a) all indices and four-momentum dependences will be suppressed, i.e., we will write, e.g., C_0 for $C_{0\alpha\alpha'}^{\mu\nu}(q,p)$ in the first term on the right-hand side in Fig. 1, K_0 instead of $K_{\alpha\alpha'}^{\beta\beta'}(k,p)$ in the second term, etc. ;
- b) the product C of two kernels A, B will be denoted as $C = A \cdot B$ and is defined as

$$C_{\beta\beta'}^{\alpha\alpha'}(k_1, k_2) = \sum_{\gamma\gamma'} \int \frac{d^m \ell}{(2\pi)^m} A_{\gamma\gamma'}^{\alpha\alpha'}(k_1, \ell) B_{\beta\beta'}^{\gamma\gamma'}(\ell, k_2) \quad (2.1)$$

(see Fig. 2) ;

c) the contractions of the spinor indices will be denoted as follows :

$$\begin{aligned} [K A] & \text{ denotes } \sum_{\alpha\alpha'} (K)_{\alpha\alpha'} A_{\beta\beta'}^{\alpha\alpha'} \\ A[K] & \text{ denotes } \sum_{\beta\beta'} A_{\beta\beta'}^{\alpha\alpha'} (K)_{\beta\beta'} \end{aligned} \quad (2.2)$$

the contraction of the vector (photon) indices will be denoted by $[C_0^{(i)}]$ or $[M^{(i)}]$, $i=1,2,3$ and is defined as the projection on a particular structure function.

The GLE in Fig. 1 can be now written as

$$M = C_0(1 + K_0 + K_0^2 + \dots) = C_0 \frac{1}{1 - K_0} \equiv C_0 \Gamma_0 \quad (2.3)$$

In Eq. (2.3) the perturbative series for M has been rearranged in such a way that the factorizability starts to be visible : C_0 (as a 2PI kernel) is finite whereas Γ_0 contains all mass singularities. Both C_0 and Γ_0 are, however, still coupled by the $d^n k$ integration as well as by spinor indices.

2.3. - "Infinite momentum" frame

Both for technical purposes as well as to keep the parton interpretation of all formal operations, it is very convenient to use the following light-cone parametrization of the momenta ¹⁴⁾ :

$$\begin{aligned} P &= (P_0, \vec{P}_\perp, p_m) = (P, \vec{0}, P) \\ n &= \left(\frac{p_m}{2P}, \vec{0}, -\frac{p_m}{2P} \right) \\ k &= \left(xP + \frac{k_\perp^2}{4xP}, \vec{k}_\perp, xP - \frac{k_\perp^2}{4xP} \right) \end{aligned} \quad (2.4)$$

The covariant parametrization (2.4) looks very similar to the infinite momentum frame limit in old-fashioned perturbation theory, where the parton picture is relevant. In particular, the quantity $x = (kn/pn)$ can be interpreted as the IMF momentum fraction of momentum p carried by k . The light-like gauge gluon propagator has the form

$$\mathcal{D}_{\mu\nu}(l) = \frac{i}{l^2} d_{\mu\nu}(l); \quad d_{\mu\nu}(l) = -g_{\mu\nu} + \frac{l_\mu n_\nu + l_\nu n_\mu}{n l} \quad (2.5)$$

and during the calculation we encounter many terms like $(l n / k n)$, $(p n / l n)$, etc., which now have a simple parton interpretation. In particular, the well-known $1/l n$ singularities of the axial gauge can be interpreted as the soft singularity dx/x of the gluon spectrum.

Finally, the parametrization (2.4) is equivalent to the Sudakov ¹⁵⁾ decomposition :

$$k_\mu = x p_\mu + d n_\mu + k_{\perp\mu} = k_{\parallel\mu} + d n_\mu + k_{\perp\mu} \quad (2.6)$$

where now

$$d = \frac{k^2 + k_\perp^2}{2x}, \quad x = \frac{k n}{p n} \text{ and } k_\perp = (0, \vec{k}_\perp, 0)$$

The Jacobian in the IMF variables has the simple form

$$d^m k = \frac{dx}{2|x|} d^{m-2} \vec{k}_\perp d k^2 \quad (2.7)$$

2.4. - Projectors

The full factorization of the formula $M = C_0 \Gamma_0$ can now be realized by introducing a suitable projector $|P$ which :

- a) decouples C_0 and Γ_0 in spinor indices and
- b) extracts the singular part of $d^m k$ integrals (i.e., poles in ϵ) decoupling C_0 and Γ_0 in momentum space.

In order to specify the form of $|P$, let us denote its action in the spinor and momentum space by, respectively, P_n and P_ϵ , so that $|P = P_\epsilon \otimes P_n$.

The form of P_n follows from the general decomposition of $K_0(k,p)$ into invariant amplitudes :

$$K_0 \not{p} = A \not{k}_{||} + B \not{k}_{\perp} + C \not{\epsilon} \frac{k^2}{\mu} + D \not{\epsilon} \not{k}_{\perp} \not{k}_{||} \frac{1}{\mu} \quad (2.8)$$

Since the scalar amplitudes A, B, C, D are dimensionless and there is no other scale inside K_0 except for μ (the unit of mass in the dimensional regularization), the integral over k giving rise to the poles in ϵ is of the form

$$\int_0^{-Q^2} \frac{dk^2}{k^2} \int_0^{-(1-x)k^2} \frac{dk_{\perp}^2}{k^2} \left(\frac{k_{\perp}^2}{\mu^2} \right)^{\epsilon} \left[A \left(\frac{k^2}{\mu^2}, x, \epsilon \right) \not{k}_{||} + B \left(\frac{k^2}{\mu^2}, x, \epsilon \right) \not{k}_{\perp} + \dots \right] \quad (2.9)$$

The upper limit of the dk^2 integral is determined by the large momentum scale of the process and will be schematically denoted by $-Q^2$; as will be shown, its actual value is irrelevant. The upper limit on k_T^2

$$k_{\perp}^2 < (1-x)(-k^2) \quad (2.10)$$

is determined by the condition $(p-k)^2 > 0$ (all internal lines of K_0 in the $S=p-k$ channel are on-shell). Consequently, $k_T \sim \sqrt{-k^2}$ and all the terms proportional to k^2 or to k_T kill the collinear pole in ϵ . Therefore the mass singularity is entirely due to the first term in (2.8) which can be easily projected out since $n^2 = nk_T = 0$:

$$K_0 \not{p} = \not{\epsilon} \left[\frac{\not{\epsilon}}{4kn} K_0 \not{p} \right] + \text{finite part} \quad (2.11)$$

The operator P_n is defined as follows :

$$AP_n B = A \not{\epsilon} \left[\frac{\not{\epsilon}}{4kn} B \right] \quad (2.12)$$

i.e., it decouples A and B in the spinor indices, projecting out the singular part of B . The operator P_e sets $k^2 = 0$ in the $A \not{\epsilon}$ part and extracts the pole in the dk^2/k^2 integral from the $[\not{\epsilon} B / (4kn)]$ part.

E.g., the factorization of the mass singularity in $C_0 K_0$ term looks as follows :

$$F_i = \frac{1}{2} [C_0^{(i)} K_0] = \frac{1}{2} [C_0^{(i)} P K_0] + \frac{1}{2} [C_0^{(i)} (1-P) K_0] \quad (2.13)$$

where the second term is finite and

$$\begin{aligned} \frac{1}{2} [C_0^{(i)} P K_0] &= \frac{1}{2} [C_0^{(i)} P_\epsilon \otimes P_\pi K_0] = \frac{1}{2} [C_0^{(i)} k] P_\epsilon \left[\frac{\pi}{4k\eta} K_0 \right] = \\ &= \int \frac{dx}{x} \frac{1}{2} [C_0^{(i)} k]_{k^2=0} \cdot \tilde{\Gamma}(\frac{q^2}{\mu^2}, x, \frac{1}{\epsilon}) \end{aligned} \quad (2.14)$$

where

$$\tilde{\Gamma}(\frac{q^2}{\mu^2}, x, \frac{1}{\epsilon}) = x \cdot PP \left\{ \int \frac{d^m k}{(2\pi)^m} \delta(x - \frac{k\eta}{pn}) \left[\frac{\pi}{4k\eta} K_0 \right] \right\} \quad (2.15)$$

and PP denotes the pole part of the $d^m k$ integral. From kinematics $k = xp$ (i.e., $k_T = 0$) for $k^2 = 0$ so that $[C_0^{(i)} k]_{k^2=0}$ can be taken out from the $d^{m-2} k_T$ integral. Therefore, after the action of P, both kernels are coupled only by the x integral, which can be written in the form of a convolution

$$\frac{1}{2} [C_0^{(i)} P K_0] = \int_0^1 dy \tilde{C}_0^{(i)}(\frac{q^2}{\mu^2}, y) \int_0^1 dz \tilde{\Gamma}(\frac{q^2}{\mu^2}, z, \frac{1}{\epsilon}) \delta(x_B - yz) \quad (2.16)$$

where

$$\tilde{C}_0^{(i)}(\frac{q^2}{\mu^2}, y) = \frac{1}{2} [C_0^{(i)} k]_{k^2=0} \quad (y = \frac{-q^2}{2kq}) \quad (2.17)$$

and $x_B = (-q^2/2pq)$ is the Bjorken variable.

2.5. - Factorization

In previous sections we have collected all technicalities needed to carry out the factorization of mass singularities in GLE [e.g., (2.3)]. The procedure is iterative : first we factorize the singular part of the last kernel K_0 :

$$M = C_0 \left[1 + \sum_{i=1}^{\infty} K_0^{i-1} \mathbb{P} K_0 + \sum_{i=1}^{\infty} K_0^{i-1} (1-\mathbb{P}) K_0 \right] = C_0 \left[1 + \sum_{i=0}^{\infty} K_0^i (1-\mathbb{P}) K_0 \right] + M \mathbb{P} K_0 \quad (2.18)$$

which can be written as

$$M(1-\mathbb{P}K_0) = C_0 \left[1 + \sum_{i=0}^{\infty} K_0^i (1-\mathbb{P}) K_0 \right] \quad (2.19)$$

In the next step we factorize the singular part of $K_0(1-\mathbb{P})K_0$ on the right-hand-side of Eq. (2.19) which gives

$$M(1-\mathbb{P}K_0 - \mathbb{P}(K_0(1-\mathbb{P})K_0)) = C_0 \left[1 + (1-\mathbb{P})K_0 + \sum_{i=0}^{\infty} K_0^i (1-\mathbb{P})(K_0(1-\mathbb{P})K_0) \right] \quad (2.20)$$

etc. Finally, we end up with a series, which can be written in a compact form

$$M \left(1 - \mathbb{P} \left(K_0 \frac{1}{1 - (1-\mathbb{P})K_0} \right) \right) = C_0 \frac{1}{1 - (1-\mathbb{P})K_0} \quad (2.21)$$

The kernel $1/(1-(1-\mathbb{P})K_0)$ is defined by the series expansion in which $(1-\mathbb{P})$ acts on the full expression on the right :

$$\frac{1}{1 - (1-\mathbb{P})K_0} \equiv 1 + (1-\mathbb{P})K_0 + (1-\mathbb{P})(K_0(1-\mathbb{P})K_0) + \dots \quad (2.22)$$

Formula (2.21) can now be written in the convolutive form, like Eq. (2.16). By taking x_B moments, dividing both sides of Eq. (2.21) by the singular part $1 - \mathbb{P}K_0(1-(1-\mathbb{P})K_0)^{-1}$ and inverting back to the x space we can also write Eq. (2.22) in the form

$$M = \left(C_0 \frac{1}{1 - (1-\mathbb{P})K_0} \right) \left(\frac{1}{1 - \mathbb{P}K} \right) \quad (2.23)$$

where $K = K_0(1/(1-(1-\mathbb{P})K_0))$ and the kernel $1/(1-\mathbb{P}K)$ is defined by the series expansion in which $\mathbb{P}$ acts only on the adjacent K on the right :

$$\frac{1}{1 - \mathbb{P}K} \equiv 1 + \mathbb{P}K + (\mathbb{P}K)(\mathbb{P}K) + \dots \quad (2.24)$$

Finally let us note that formula (2.23) can be derived immediately by formal manipulations with the ladder expansions like in Ref. 5) :

$$\begin{aligned} \frac{1}{C_0} M_0 &= (1 - K_0)^{-1} = (1 - (1 - \mathbb{P}) K_0 - \mathbb{P} K_0)^{-1} = \{ [1 - \mathbb{P} K_0 (1 - (1 - \mathbb{P}) K_0)^{-1}] [1 - (1 - \mathbb{P}) K_0] \}^{-1} \\ &= (1 - (1 - \mathbb{P}) K_0)^{-1} (1 - \mathbb{P} K_0 (1 - (1 - \mathbb{P}) K_0)^{-1}) \end{aligned} \quad (2.25)$$

The rules (2.22) and (2.24) are, however, far from being obvious in this way of derivation, and an incorrect interpretation of series (2.22) and (2.24) leads to wrong results : e.g. $(\mathbb{P} K)(\mathbb{P} K)$ contains only a double pole in ϵ whereas $\mathbb{P}(K \mathbb{P} K)$ has both a single and a double pole.

The final step in constructing the quark structure function consists in performing the necessary contractions and multiplying M by Z_F - the residue of the pole of the full quark propagator. (In Section 3 we demonstrate that in our scheme Z_F has the form

$$Z_F = 1 + \frac{1}{\epsilon} Z_1(\alpha) + \frac{1}{\epsilon^2} Z_2(\alpha) + \dots \quad (2.26)$$

i.e., it contains only collinear poles.)

The final form of the result is the following :

$$F_i(\frac{Q^2}{\mu^2}, x_B, \alpha, \frac{1}{\epsilon}) = \int_0^1 dy C_i(\frac{Q^2}{\mu^2}, y, \alpha) \int_0^1 dz \Pi_S(z, \alpha, \frac{1}{\epsilon}) \delta(x_B - yz) \quad (2.27a)$$

where

$$F_i = \frac{1}{2} [M^{(i)} \not{x}] = \frac{1}{2} [C_0^{(i)} \frac{1}{1 - K_0} \not{x}] \quad (2.27b)$$

$$C_i = \frac{1}{2} [C_0^{(i)} \frac{1}{1 - (1 - \mathbb{P}) K_0} \not{x}]_{k^2=0} \quad (y = \frac{-q^2}{2kq}) \quad (2.27c)$$

$$\Pi_S(z, \alpha, \frac{1}{\epsilon}) = Z_F \left[\delta(1-z) + PP \int \frac{d^m k}{(2\pi)^m} \delta(z - \frac{km}{pn}) \not{x} \left[\frac{\not{x}}{4km} K \frac{1}{1 - \mathbb{P} K} \not{x} \right] \right] \quad (2.27d)$$

or, in terms of x_B moments :

$$F_i(Q^2/\mu^2, N, \alpha, 1/\epsilon) = C_i(Q^2/\mu^2, N, \alpha) \Gamma_S(N, \alpha, 1/\epsilon) \quad (2.28)$$

Here and in the following we use the simplified notation $f(N) = \int_0^1 dx x^{N-1} f(x)$.

The density $\Gamma_S(x, \alpha, 1/\epsilon)$ is normalized to the cross-section $d\sigma/dx$, i.e., if $C_i = \delta(1-y)$, the inclusive lepton cross-section is related to Γ_S as

$$\frac{d\sigma}{dx dQ^2} = \frac{d\sigma_0}{dQ^2} \Gamma_S(x, \alpha, 1/\epsilon) \quad (2.29)$$

where $d\sigma_0/dQ^2$ is the elastic lepton-parton cross-section. The definition of the structure functions F_i in terms of the tensor $W_{\mu\nu}$ is given in Section 7.

2.6. - Renormalization group equation

Equation (2.21) has the structure of a typical relation between bare and renormalized quantities in perturbation theory :

$$M_B \cdot Z_B = M_R \quad (2.30)$$

where the (divergent) bare quantity $M_B = M$, multiplied by the corresponding (divergent) renormalization constant $Z_M = 1 - PK$ gives the (finite) renormalized result $M_R = C_0(1/(1-(1-P)K_0))$. In this sense the factorization of the mass singularities in the dimensional scheme is formally very similar to the ultra-violet technique of OPE and Mueller. On the other hand, Eq. (2.27) has a clear parton interpretation : $\Gamma(x, \alpha, 1/\epsilon)$ describes the density of partons in the parent quark, is independent on a particular hard process and contains all long distance (perturbative) effects, resulting in poles in ϵ , whereas $C_i(Q^2/\mu^2, x, \alpha)$ plays the rôle of a "short-distance" cross-section : indeed, as can be seen by looking at Eqs. (2.27) for F_i and C_i , the latter is very similar to the former but with the $1/\epsilon$ poles subtracted off, i.e., it describes the short-distance effects characteristic to a particular hard process. Thus, keeping the nice parton interpretation, we can simultaneously use the powerful methods of the renormalization group, typical for the ultra-violet techniques.

Let us formulate more precisely the meaning of the renormalization of collinear divergences. We start the calculation with $p^2 \neq 0$ in $m = 4 - \epsilon$ dimensions ($\epsilon > 0$) and subtract the ultra-violet poles. The collinear divergences are regulated by p^2 and the renormalization group gives only trivial information : $F_i(Q^2/\mu_1^2, \alpha(\mu_1^2)) = F_i(Q^2/\mu_2^2, \alpha(\mu_2^2))$.

We now continue analytically the result to $m = 4 + \epsilon$ ($\epsilon > 0$) dimensions and take $p^2 = 0$; this operation generates collinear poles in F_i . Since F_i is now a "bare" quantity and does not depend on μ , the μ^2 (and in consequence Q^2) dependence of G_i - the "renormalized" quantity - is determined by singularities of Γ^{-1} .

However, in order to use the renormalization group methods we have first to demonstrate that $\Gamma_S(x, \alpha, 1/\epsilon)$ is indeed independent of Q^2 (i.e., on the upper limit of the dk^2 integral), as anticipated in Eq. (2.27). In the ultra-violet method, the analogous property of Γ (independent of p^2/μ^2) is a direct consequence of the renormalizability of the theory, whereas in our case the Q^2/μ^2 independence of Γ is by no means obvious. In fact it is the most essential point of the whole factorization programme in the axial gauge, where the finiteness of the kernel K_0 proved in Ref. 5) enters in a crucial way. The proof goes as follows : as seen from (2.21) and (2.23), the operator P extracting the collinear poles acts on kernels $K_0, K_0(1-P)K_0, \dots, K_0[(1-P)K_0]^i$ which are finite in the limit $\epsilon \rightarrow 0$ before the last integral dk^2/k^2 is performed. The structure of Γ_S is therefore

$$\begin{aligned} \Gamma_S(Q^2/\mu^2, x, 1/\epsilon) &= \delta(1-x) + \frac{1}{\epsilon} \Gamma_1(Q^2/\mu^2, x) + \frac{1}{\epsilon^2} \Gamma_2(Q^2/\mu^2, x) + \dots = \\ &= \delta(1-x) + PP \int_0^{Q^2} \frac{dk^2}{k^2} \Phi(k^2/\mu^2, x, \epsilon) \end{aligned} \quad (2.31)$$

where

$$\lim_{\epsilon \rightarrow 0} \Phi(k^2/\mu^2, x, \epsilon) < \infty \quad (2.32)$$

Differentiating both sides of Eq. (2.31) over Q^2 , using (2.32) and comparing coefficients of the same powers of $1/\epsilon$ on both sides we find $(\partial/\partial Q^2)\Gamma_1 = (\partial/\partial Q^2)\Gamma_2 = \dots = 0$, which implies $(\partial/\partial Q^2)\Gamma = 0$. Q.E.D.

The rest of this section follows a well-known pattern of arguments ⁹⁾.
Let us denote by $\mathcal{D}$ the total derivative over μ :

$$\mathcal{D} = \mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g, \epsilon) \frac{\partial}{\partial g} \quad (2.33)$$

where

$$\beta(g, \epsilon) = \beta(g) + \frac{1}{2} g \epsilon \quad (2.34)$$

is the Peterman-Stueckelberg-Gell-Mann-Low ^{16), 17)} β function in $m = 4 + \epsilon$ dimensions. Since the left-hand-side of Eq. (2.28) is independent of μ (as a physical quantity in m dimensions) we get

$$\mathcal{D} \ln F_i = 0 = \mathcal{D} \ln C_i + \mathcal{D} \ln \Gamma \quad (2.35)$$

i.e.,

$$[\mathcal{D} - \gamma(N, \alpha)] C_i(\frac{Q^2}{\mu^2}, N, \alpha) = 0 \quad (2.36)$$

where $\gamma(N, \alpha)$ is defined as

$$\gamma(N, \alpha) = -\mathcal{D} \ln \Gamma = -\beta(g, \epsilon) \frac{\partial}{\partial g} \ln \Gamma(N, \alpha, \frac{1}{\epsilon}) \quad (2.37)$$

and, from Eq. (2.36), it does not contain poles in ϵ (since both $\mathcal{D}$ and C_i do not). Using the expansion

$$\Gamma(N, \alpha, \frac{1}{\epsilon}) = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} \Gamma_k(N, \alpha) \quad (2.38)$$

we get from (2.34) and (2.36)

$$\gamma(N, \alpha) = -\alpha \frac{\partial}{\partial \alpha} \Gamma_1(N, \alpha) \quad (2.39)$$

For a further comparison with the OPE results, let us parametrize Γ_1 as

$$\Gamma_1(N, \alpha) = - \sum_{k=1}^{\infty} \left(\frac{\alpha}{4\pi} \right)^k \frac{1}{k} \gamma_{k-1}(N) \quad (2.40)$$

so that

$$\gamma(N, \alpha) = \sum_{k=1}^{\infty} \left(\frac{\alpha}{4\pi} \right)^k \gamma_{k-1}(N) = \left(\frac{\alpha}{4\pi} \right) \gamma_0(N) + \left(\frac{\alpha}{4\pi} \right)^2 \gamma_1(N) + \dots \quad (2.41)$$

[It will be shown in Section 5 that $\gamma(N, \alpha)$ coincides with the anomalous dimension of the twist-two non-singlet Wilson operator, calculated in the 't Hooft scheme.]

Equation (2.37) can be integrated and we get

$$\Gamma(N, \alpha, \frac{1}{\epsilon}) = \exp \left[- \int_0^{\alpha} \frac{d\lambda}{\lambda} \frac{\gamma(N, \lambda)}{2\bar{\beta}(\lambda) + \epsilon} \right] \quad (2.42)$$

where

$$\bar{\beta}(\alpha) = \frac{1}{g} \beta(g) = - \left[\left(\frac{\alpha}{4\pi} \right) \beta_0 + \left(\frac{\alpha}{4\pi} \right)^2 \beta_1 + \dots \right] \quad (2.43)$$

Equation (2.28) can be therefore written as

$$F_i(Q^2/\mu^2, N, \alpha, \frac{1}{\epsilon}) = C_i(Q^2/\mu^2, N, \alpha) \exp \left[- \int_0^{\alpha} \frac{d\lambda}{\lambda} \frac{\gamma(N, \lambda)}{2\bar{\beta}(\lambda) + \epsilon} \right] \quad (2.44)$$

or, since the left-hand-side does not depend on μ ,

$$F_i(Q^2/\mu^2, N, \alpha, \frac{1}{\epsilon}) = C_i(1, N, \alpha(Q^2)) \exp \left[- \int_0^{\alpha(Q^2)} \frac{d\lambda}{\lambda} \frac{\gamma(N, \lambda)}{2\bar{\beta}(\lambda) + \epsilon} \right] \quad (2.45)$$

where $\alpha(Q^2) \equiv \bar{\alpha}(\ln Q^2/\mu^2; \alpha)$ is the running coupling constant.

2.7. - Bare and dressed parton densities

In order to construct the hadronic structure functions, partonic quantities [Eq. (2.45)] should be convoluted with the "bare" density of quarks inside the hadron $q_{B,H}(x, \alpha, 1/\epsilon)$. According to the GLE (2.3), $q_{B,H}$ is given by

$$q_{B,H}(x, \alpha, \frac{1}{\epsilon}) = x \int \frac{d^m p}{(2\pi)^m} \delta(x - \frac{p_0}{p_H}) \left[\frac{\gamma}{4p_H} H \not{p}_H \right] \quad (2.45)$$

where H is the two-particle irreducible hadron $\rightarrow$ quark kernel (see Fig. 3). For our purposes we need only one property of H , i.e., that it is soft in the following sense :

$$H(\vec{p}, x) = \frac{1}{p^2} \int \frac{d^{m-2} p_\perp}{2 (2\pi)^m} \left[\frac{\gamma}{4p_H} H \not{p}_H \right] \lesssim \frac{C}{|p^2|^\delta} ; \left(\begin{array}{l} |p|^2 \rightarrow \infty \\ \delta > 0 \end{array} \right) \quad (2.47)$$

This assumption is consistent with all phenomenological evidence concerning the internal structure of hadrons like asymptotic behaviour of the electromagnetic form factors of hadrons, wide angle elastic scattering, etc. Assuming softness of the hadronic blob we can extend the p^2 integral to infinity :

$$q_{B,H}(x, \alpha, \frac{1}{\epsilon}) = \int \frac{d^m p^2}{p^2} H(\vec{p}, x) = \int \frac{d^m p^2}{p^2} H(\vec{p}, x) + O\left[\left(\frac{\mu^2}{Q^2}\right)^\delta\right] \quad (2.48)$$

so that the "bare" density contains only power-like corrections to the Q^2 dependence of the structure functions (higher twists in the Wilson expansion). On the other hand, the lower limit of the dp^2/p^2 integral will generate the mass singularities which, according to the LNK theorem ¹⁸⁾, must cancel exactly the divergences of $\Gamma(N, \alpha, 1/\epsilon)$. Therefore the "dressed" density

$$q_H(N, Q^2) = \exp\left[-\int_0^{\alpha(Q^2)} \frac{d\lambda}{\lambda} \frac{\gamma(N, \lambda)}{2\beta(\lambda) + \epsilon}\right] q_{B,H}(N, \alpha, \frac{1}{\epsilon}) \quad (2.49)$$

is free of mass singularities and evolves with Q^2 according to

$$Q^2 \frac{\partial}{\partial Q^2} q_H(N, Q^2) = -\frac{1}{2} \gamma(N, \alpha(Q^2)) q_H(N, Q^2) \quad (2.50)$$

or, in the integrated form

$$q_H(N, Q^2) = E(N; Q^2, Q_0^2) q_H(N, Q_0^2) \quad (2.51)$$

where

$$E(N; Q^2, Q_0^2) = \exp \left(- \int_{\alpha(Q_0^2)}^{\alpha(Q^2)} \frac{\gamma(N, \lambda)}{2\beta(\lambda)} d\lambda \right) \quad (2.52)$$

Let us define the probability $P_{qq}(x, \alpha)$ as follows :

$$\int_0^1 dx x^{N-1} P_{qq}(x, \alpha) = -\frac{1}{2} \gamma(N, \alpha) \quad (2.53)$$

The formula (2.50) can be now written in the x space :

$$Q^2 \frac{\partial}{\partial Q^2} q_H(x, Q^2) = \int_x^1 \frac{dz}{z} P_{qq}\left(\frac{x}{z}, \alpha(Q^2)\right) q_H(z, Q^2) \quad (2.54)$$

i.e., it has the form of the "master equation" of Altarelli and Parisi.

Expanding P_{qq} in powers of α

$$P_{qq}(x, \alpha) = \left(\frac{\alpha}{2\pi}\right) P_{qq}^{(0)}(x) + \left(\frac{\alpha}{2\pi}\right)^2 P_{qq}^{(1)}(x) + \dots \quad (2.55)$$

we see from (2.54) and (2.55) that $P_{qq}^{(0)}(x)$ must coincide with the Parisi-Altarelli (PA) probability :

$$P_{qq}^{(0)}(x) = c_F \left(\frac{1+x^2}{1-x} \right) \quad (2.56)$$

Therefore, $P_{qq}(x, \alpha)$ can be considered as a generalization to all orders in α of the PA probability (see Section 8 for a more detailed discussion of this point).

We can also express the densities $\Gamma(x, \alpha, 1/\epsilon)$ and $E(x, Q_0^2, Q^2)$ in terms of $P(x, \alpha)$ as follows :

$$\Gamma(x, \alpha, \frac{1}{\epsilon}) = 1 + \frac{2}{\epsilon} \left[\left(\frac{\alpha}{2\pi} \right) P_{qq}^0(x) + \frac{1}{2} \left(\frac{\alpha}{2\pi} \right)^2 P_{qq}^1(x) + \dots \right] + O\left(\frac{1}{\epsilon^2}\right) \quad (2.57)$$

$$E(N; Q^2, Q^2) = \exp \left(\int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} P_{qq}(N, \alpha(k^2)) \right) \quad (2.58)$$

The final formula for the deep-inelastic structure function measured in lepton (ℓ) - hadron (H) scattering reads

$$F_i^{\ell H} = \sum_q C_i^{\ell q} \left(\frac{Q^2}{\mu^2}, N, \alpha \right) q_{H,i}(N, Q^2) \quad (2.53)$$

where the sum runs over all flavours. In fact, in order to construct the non-singlet quantity one has to form appropriate linear combinations of $F_i^{\ell H}$ for different currents or for different targets. This operation is always implicit throughout the paper.

2.8. - Modifications for the timelike densities

The factorization procedure for timelike densities is very similar to the spacelike case discussed up to now. The generalized ladder expansion for single-particle inclusive annihilation as well as relevant notation are given in Fig. 4. We now work in the frame where p and n are parametrized as before, whereas

$$k = \left(\frac{1}{x} P + \frac{k_1^2 + k_2^2}{4 \frac{1}{x} P}, \vec{k}_1, \frac{1}{x} P - \frac{k_1^2 + k_2^2}{4 \frac{1}{x} P} \right) \quad 0 \leq x \leq 1$$

If we adopt the "mirror" convention with respect to the spacelike case, i.e., the products of kernels are written from below, all formulas are formally identical as in the spacelike construction [up to Eq. (2.26)]. A small modification arises if we want to work out the correct normalization of the single-particle inclusive cross-section. In order to do that we have to multiply M by the flux factor and by the phase-space volume $(d^m p / (2\pi)^m) \cdot 2\pi \delta^+(p^2)$. However, on the right-hand side of Eq. (2.23) we would like rather to associate the phase space volume $(d^m k / (2\pi)^m)$ with C in order to retain the probabilistic interpretation of C as a "short-distance"

cross-section [if we proceed in complete analogy with the spacelike version, the original $d^m k / (2\pi)^m$ integral from the line k joining C and Γ will be included in the definition of Γ_T , in a similar way as in Eq. (2.27)]. In the collinear configuration, forced by the projector $\mathbb{P}$ we have

$$\frac{d^m p}{(2\pi)^m} 2\pi \delta^+(p^2) = x^{2+\epsilon} \frac{d^m k}{(2\pi)^m} 2\pi \delta^+(k^2) \quad (2.54)$$

i.e., the phase space volume can be shifted formally to the C blob, leaving an extra factor $x^{2+\epsilon}$ in the Γ part. However, since the x^ϵ term does not commute with the projector $\mathbb{P}$, the shift must be performed before the factorization procedure, i.e., at the level of Eq. (2.3). In order to preserve the ladder structure, the factor x^ϵ must be distributed (in the sense of a convolution) among all kernels K_0 , i.e., each kernel $K_0(k_1, k_2)$ gets its own factor $(k_{2n}/k_{1n})^\epsilon$, which will be denoted as follows :

$$K_0(k_1, k_2) \left(\frac{k_{2n}}{k_{1n}} \right)^\epsilon \equiv \tilde{K}_0(k_1, k_2) ; \quad \tilde{K} = \tilde{K}_0 \frac{1}{1 - (\mathbb{P} \tilde{K}_0)} \quad (2.55)$$

The final result for SIA structure functions is the following :

$$F_i(Q^2/\mu^2, x_A, \alpha, \frac{1}{\epsilon}) = \int dy C_i(Q^2/\mu^2, y, \alpha, \frac{1}{\epsilon}) \Gamma_T(z, \alpha, \frac{1}{\epsilon}) \delta(x_A - yz) \quad (2.56)$$

where

$$x_A = \frac{2pq}{q^2}$$

$$C_i = \frac{1}{2} \left[C_0^{(i)} \frac{1}{1 - (\mathbb{P} \tilde{K}_0)} \not{k} \right] \quad \left(y = \frac{2kq}{q^2} \right)$$

and

$$\Gamma_T(z, \alpha, \frac{1}{\epsilon}) = \tilde{\chi}_F \left\{ \delta(1-z) + z^2 \mathbb{P} \int \frac{d^m k}{(2\pi)^m} \delta(z - \frac{p\eta}{k\eta}) \left[\frac{\pi}{4k\eta} \tilde{K} \frac{1}{1 - \mathbb{P} \tilde{K}} \not{k} \right] \right\}$$

The density Γ_T in Eq. (2.56) is normalized to the single-particle inclusive cross-section $d\sigma/dx_A d\Omega$, i.e., if $C_i = \delta(1-y)$, then

$$\frac{d\sigma}{dx_A d\Omega} = \frac{d\sigma_0}{d\Omega} \Gamma_T(x_A, \alpha, \frac{1}{\epsilon}) \quad (2.57)$$

where $d\sigma_0/d\Omega$ is the elastic cross-section $e^+e^- \rightarrow q\bar{q}$.

The remaining part of the scheme presented for the spacelike case is identical for timelike densities (i.e., RG equation, hadronic part, etc.).

3. - A PEDAGOGICAL EXAMPLE : REDERIVATION OF THE LEADING LOG RESULT

As a simple illustration of the method we calculate here the $O(\alpha)$ contribution to $\gamma(x, \alpha)$ [Eq. (2.41)] which is equivalent to the re-summation of all leading logarithms (i.e., all terms of type α/ϵ in the dimensional regularization) of $\Gamma_S(x, \alpha, 1/\epsilon)$. To order α , Γ_S is given by

$$\Gamma_S(x, \alpha, \frac{1}{\epsilon}) = \mathcal{Z}_F \left[1 + PP \left\{ x \int \frac{d^m k}{(2\pi)^m} \delta(x - \frac{k_n}{p_n}) \left[\frac{\alpha}{4k_n} K_{op} \right] \right\} \right] \quad (3.1)$$

The diagram contributing to K_o and the one for Z_F are presented in Fig. 5. More explicitly, we have

$$\left[\frac{\alpha}{4k_n} K_{op} \right] = \frac{g^2}{\mu \epsilon} \times T_F \left[\frac{\alpha}{4k_n} k_\mu \gamma^\mu p \gamma^\nu k_\nu \right] \frac{1}{(k^2)^2} d_{\mu\nu}(\ell) 2\pi \delta^+(\ell^2) (2\pi)^m \delta(p - k - \ell) \frac{d^m \ell}{(2\pi)^m}$$

Performing trivial integrations over $d^m \ell$ and over $d^{m-2} \vec{k}_T$ and summing over colour, we get :

$$\begin{aligned} \Gamma_S(x, \alpha, \frac{1}{\epsilon}) &= \mathcal{Z}_F \left[1 + PP \left\{ \frac{\alpha}{2\pi} C_F \int_0^1 \frac{d(-k^2)}{(-k^2)} \left(\frac{-k^2}{4\pi\mu^2} \right)^{\epsilon/2} \left[\frac{1+x^2}{1-x} + \frac{\epsilon}{2} (1-x) \right] \right\} \right] \\ &= \mathcal{Z}_F \left[1 + \frac{\alpha}{2\pi} C_F \frac{2}{\epsilon} \frac{1+x^2}{1-x} \right] \quad \text{where} \quad C_F = \frac{N-1}{2N} = \frac{4}{3} \end{aligned} \quad (3.2)$$

In order to calculate Z_F , defined as the residue of the pole in the full quark propagator

$$G(p) \sim \mathcal{Z}_F \frac{i \not{p}}{p^2} \quad \text{for } p^2 \rightarrow 0 \quad (3.3)$$

let us consider the general structure of the quark self-energy Σ , related to G as

$$G(p) = \frac{i}{\not{p} - \Sigma} \quad (3.4)$$

The general decomposition of Σ into invariant amplitudes has the form

$$\Sigma = A(p^2/\mu^2, \epsilon) \not{p} + B(p^2/\mu^2, \epsilon) \not{n} \frac{p^2}{2pn} \quad (3.5)$$

After simple algebra we get for G

$$G(p) = \frac{1}{1-\sigma} \left[\frac{i \not{p}}{p^2} - \frac{B}{1-A} \not{n} \frac{p^2}{2pn} \right] \quad (3.6)$$

where $\sigma = A+B$, so that

$$\chi_F = \frac{1}{1-\sigma} \quad (3.7)$$

The diagram contributing to σ at order α is shown in Fig. 5. In order to isolate ultra-violet divergences, we calculate the loop integral in $m=4-\eta$ dimensions ($\eta > 0$). The Feynmann parameter method gives the result

$$\begin{aligned} \Sigma = & - \left(\frac{\alpha}{2\pi} \right) \Gamma\left(\frac{\eta}{2}\right) \int_0^1 dx \left[\frac{-4\pi\mu^2}{x(1-x)p^2} \right]^{\eta/2} \left\{ \left(1 - \frac{\eta}{2}\right)(1-x)\not{p} + \frac{2x}{1-x} \not{n} \frac{p^2}{2pn} \right\} \\ & + \left(\frac{\alpha}{2\pi} \right) \frac{2}{\eta} \int_0^1 dx \left[(1-x)\not{p} + \frac{2x}{1-x} \not{n} \frac{p^2}{2pn} \right] \end{aligned} \quad (3.8)$$

The second term in the above expression contains the subtraction of the ultra-violet pole. Here, the well-known peculiarity of the light-like gauge ¹⁹⁾ is visible : in order to subtract the pole, one has to introduce the gauge dependent counter terms [like $\bar{\psi} \not{n} (\partial^\mu n_\mu)^{-1} \psi$ in our case]. Effects of this type come from integrals like

$$\gamma = \int \frac{d^m l}{(2\pi)^m} \frac{1}{l^2 (p-l)^2 (2pn)} \quad (3.9)$$

where the UV divergent integrand is multiplied by the axial gauge denominator $1/\ell n$. A naïve power counting suggests that γ is finite, as is indeed the case as long as $n^2 \neq 0$. On the other hand the integral then contains the term $\ell n^2(n^2 p^2/(pn)^2)$ so that the limit $n^2 \rightarrow 0$ does not exist. If we put $n^2 = 0$ from the beginning, the power counting breaks down and the integral develops a spurious ultra-violet pole, as seen in Eq. (3.8).

The above-mentioned peculiarity of the light-like gauge was the origin of doubts raised by some people whether this particular gauge can be accepted at all for practical calculations. Our prescription to overcome this difficulty is very simple : subtract all the UV poles, normal and spurious ones. The discussion of the two limits ($n^2 \rightarrow 0$, $\eta \rightarrow 0$ and vice versa) presented above seems to indicate that the spurious pole in the $n^2 = 0$ gauge replaces in a sense the term $\ell n^2(n^2 p^2/(pn)^2)$ in the $n^2 \neq 0$ gauge. The terms of this type, in turn, must finally cancel if we calculate a gauge invariant quantity. Therefore, in all physical applications the local subtraction of spurious poles in the $n^2 = 0$ gauge should be equivalent to the global cancellation of $\ell n^2(n^2 p^2/(pn)^2)$ terms in the $n^2 \neq 0$ gauge.

The sceptical reader may ask why we subtract the spurious poles if the argument for doing so is that they will cancel anyway in the final result for the physical quantity. The answer is that it does not matter which version we choose (subtract or let them cancel at the end) as long as we calculate the full cross-section. On the other hand, in the factorization scheme discussed in the paper the cancellation would have to take place independently in C and Γ , in order to preserve the finiteness of 2PI kernels. In fact, something opposite happens. In the $O(\alpha)$ calculation of the full deep inelastic structure function of the quark, the $\ell n^2(n^2 p^2/(pn)^2)$ terms (and, consequently, also the spurious poles in the light-like gauge) from Z_F cancel with those from the photon vertex. It means that the factorizability in the light-like gauge would be broken without local subtractions of spurious poles. (On the other hand, once we do subtract them, the scheme works : our spacelike two-loop anomalous dimensions coincide with the OPE result and, hopefully, it will be a challenge for field theory experts to provide a more formal support for our "phenomenological" rules.) The final step in the calculation of Z_F is the extraction of the collinear poles. Formula (3.8), being an analytic function in η , can now be continued to $m = 4 + \epsilon$ dimensions. In the resulting expression [i.e., in Eq. (3.8) with $\eta = -\epsilon$] we can take the limit $p^2 \rightarrow 0$ and we obtain

$$\sigma = - \left(\frac{\alpha}{2\pi} \right) C_F \frac{2}{\epsilon} \int_0^1 dx \frac{1+x^2}{1-x} \quad (3.10)$$

Thus the full result is equal to the subtracted ultra-violet pole with opposite sign. The above property of Z_F is valid to any order in α : since the m dimensional phase space for $m=4+\epsilon$ has the form $(p^2/\mu^2)^{\lambda\epsilon}$ where $\lambda > 0$, after taking the limit $p^2 \rightarrow 0$ everything vanishes except the subtracted ultra-violet poles. Consequently, to any order in α , Z_F contains only collinear poles, i.e., it is of the form of Eq. (2.26). Combining (3.2) and (3.10), we get

$$\begin{aligned} \Gamma_S(x, \alpha, \frac{1}{\epsilon}) &= 1 + \left(\frac{\alpha}{2\pi} \right) C_F \frac{2}{\epsilon} \left[\frac{1+x^2}{1-x} - \delta(1-x) \int_0^1 dy \frac{1+y^2}{1-y} \right] \equiv \\ &\equiv 1 + \left(\frac{\alpha}{2\pi} \right) \frac{2}{\epsilon} C_F \left(\frac{1+x^2}{1-x} \right)_+ \end{aligned} \quad (3.11)$$

i.e., the well-known result of Ref. 7).

In the above calculation we have also encountered another type of singularity typical of an axial gauge': the integral (3.10) is in fact divergent at $x=1$. According to the IMF parton picture, the singularity at $x=1$ [i.e., for $(\ell n/pn)=0$, where ℓ is the momentum of the gluon] plays the rôle of the infra-red divergence of the gluon spectrum. As seen from Eq. (3.11), the precise nature of this singularity is irrelevant at the one-loop level since it finally cancels once the real and the virtual contributions are taken into account.

In higher orders, however, where the number of independent integrations increases, it is much more difficult to keep the infra-red divergences in the unintegrated form up to the end of the calculation. We have also found examples where the formal manipulations with the convolutions of integrands like in (3.10) lead to different results depending on particular substitutions, changes of integration variables, etc. Therefore we have used the principal value prescription for regularizing the $1/\ell n$ divergences ¹⁹⁾

$$\frac{1}{2n} \rightarrow \frac{1}{2} \left(\frac{1}{ln + i\delta(pn)} + \frac{1}{ln - i\delta(pn)} \right) = \frac{ln}{(ln)^2 + \delta^2(pn)^2} \quad (3.12)$$

where the pn scalar product is added in order to preserve the homogeneity in n (p is the momentum of the incoming parton, fixed once forever). When the pole appears inside the integration range, the result is finite, but as seen from the example (3.10), also end point singularities may occur. For those we have to keep $\delta \neq 0$ as long as $\log \delta$ divergences will be cancelled. In practical calculations it is convenient to isolate the δ divergences in the form of universal integrals, like

$$I_0 = \int_0^1 du \frac{u}{u^2 + \delta^2}, \quad I_1 = \int_0^1 du \frac{u}{u^2 + \delta^2} \ln u, \dots \quad (3.13)$$

so that the following simple rules are valid

$$\begin{aligned} \frac{1}{1-x} &= \frac{1-x}{(1-x)^2 + \delta^2} = I_0 \delta(1-x) + \left(\frac{1}{1-x} \right)_+ \\ \frac{1}{1-x} \ln(1-x) &= \frac{1-x}{(1-x)^2 + \delta^2} \ln(1-x) = I_1 \delta(1-x) + \left(\frac{\ln(1-x)}{1-x} \right)_+ \end{aligned} \quad (3.14)$$

For example, Eq. (3.10) can now be written as

$$\mathcal{Z}_F = 1 + \left(\frac{\alpha}{2\pi} \right) C_F \frac{1}{\epsilon} [-4I_0 + 3] \quad (3.15)$$

The price paid by introducing the principal value regulator is the lack of rescaling invariance

$$d_{\mu\nu}(\lambda l) \neq d_{\mu\nu}(l) \quad (3.16)$$

which was obeyed by the unregularized propagator. Consequently, the renormalization constants depend, in general, on the IMF fractions k_{in}/k_{jn} . For example, if we calculate $Z_F(x) \equiv Z_F$ for the particle with momentum k , so that $kn/pn = x \neq 1$, we get

$$\begin{aligned} Z_F(x) &= 1 - \left(\frac{\alpha}{2\pi}\right) C_F \frac{2}{\epsilon} \int_0^1 dz (1+z^2) \cdot \text{PV} \left[\frac{1}{x(1-z)} \right] x \\ &= 1 + \left(\frac{\alpha}{2\pi}\right) C_F \frac{1}{\epsilon} [-4I_0 - 4\ln|x| + 3] \end{aligned} \quad (3.17)$$

At the end of this Section we present the $O(\alpha)$ formulas for all renormalization constants relevant for the non-singlet calculation. The corresponding diagrams are presented in Fig. 6 and the results are as follows :

$$\begin{aligned} Z_F(x) &= 1 + \left(\frac{\alpha}{2\pi}\right) C_F \frac{1}{\epsilon} [-4I_0 - 4\ln|x| + 3] \\ Z_G(x) &= 1 + \left(\frac{\alpha}{2\pi}\right) \frac{1}{2} C_G \frac{1}{\epsilon} [-8I_0 - 8\ln|x| + \frac{22}{3}] \\ Z_{N_F}(x) &= 1 + \left(\frac{\alpha}{2\pi}\right) \frac{1}{\epsilon} N_F T_F \left(-\frac{4}{3}\right) \\ Z_F^{(V)}(x_1, x_2, x_3) &= 1 + \left(\frac{\alpha}{2\pi}\right) \frac{1}{\epsilon} \left(C_F - \frac{1}{2} C_G\right) [4I_0 + 2\ln|x_1| + 2\ln|x_2| - 3] \\ Z_G^{(V)}(x_1, x_2, x_3) &= 1 + \left(\frac{\alpha}{2\pi}\right) \frac{1}{\epsilon} \left(\frac{1}{2} C_G\right) [8I_0 + 2\ln|x_1| + 2\ln|x_2| + 4\ln|x_3| - 3] \end{aligned} \quad \text{where} \quad \begin{cases} C_F = \frac{N^2 - 1}{2N} = \frac{4}{3} \\ C_G = N = 3 \\ T_F = \frac{1}{2} \end{cases} \quad (3.18)$$

Let us summarize the discussion of all the peculiarities of the light-like gauge with a simple demonstration of how all suspicious objects like spurious ultra-violet poles, $\ln x$ terms, and I_0 and I_1 divergences, cancel if we form a gauge invariant quantity.

For any physical quantity, all the renormalization constants can be collected in vertices in the form of the bare coupling

$$\begin{aligned} g_0(x_1, x_2, x_3) &= g \mu^{-\epsilon/2} [Z_F^{(V)}(x_1, x_2, x_3) + Z_G^{(V)}(x_1, x_2, x_3)] \sqrt{Z_F(x_1) Z_F(x_2) [Z_G(x_3) + Z_{N_F}(x_3)]} \\ &= g \mu^{-\epsilon/2} \left(\frac{\alpha}{2\pi}\right) \left(-\frac{1}{\epsilon}\right) \left[-\frac{11}{3} C_G + \frac{2}{3} N_F\right] \end{aligned} \quad (3.19)$$

i.e., all the divergences cancel and we get the correct one-loop value for the β function.

4. - SPACELIKE PARTON DENSITIES : NEXT-TO-LEADING CORRECTIONS

4.1. - q $\bar{q}$ mixing

In terms of the probability $P_{qq}(x, \alpha)$ introduced in Section 2.7, the calculation of the next-to-leading corrections to the parton densities is equivalent to the derivation of $P_{qq}^{(1)}(x)$ in Eq. (2.55). For this reason, we have to calculate the coefficient of the single pole in $\Gamma_S(x, \alpha, 1/\epsilon)$ to order α^2 [Eq. (2.57)]. Expanding the formula (2.27d) up to terms quadratic in K_0 we have

$$\Gamma = \frac{\bar{\chi}_F}{1 - PK} = \bar{\chi}_F [1 + PK_0 + P(K_0^2) - P(K_0 PK_0)] \quad (4.1)$$

The diagrams contributing to various terms of the right-hand side in Eq. (4.1) are given in Fig. 7. As seen in the Figure, there is a new qualitative phenomenon appearing at order α^2 : the $q\bar{q}$ mixing. Diagrams B to I describe the process $q \rightarrow q + X$ whereas diagram A corresponds to the $q \rightarrow \bar{q} + (qq)$ transition. Therefore, beyond leading log level, "non-singlet" is no longer equivalent to "valence" : diagram A will result in some net number of antiquarks in the parent quark.

As a consequence, we have to consider two different densities : $\Gamma_{qq}(x, \alpha, 1/\epsilon)$ and $\Gamma_{q\bar{q}}(x, \alpha, 1/\epsilon)$ and to solve a mixing problem of the form :

$$\begin{aligned} F_i^{lq} &= C_i^{lq} \Gamma_{qq} + C_i^{l\bar{q}} \Gamma_{q\bar{q}} \\ F_i^{l\bar{q}} &= C_i^{lq} \Gamma_{q\bar{q}} + C_i^{l\bar{q}} \Gamma_{q\bar{q}} \end{aligned} \quad (4.2)$$

[Formulas like (4.2), where the arguments are suppressed, can be interpreted both in the N space (as products) or in the x space (as convolutions).] Since $\Gamma_{qq} = \Gamma_{\bar{q}\bar{q}}$, $\Gamma_{q\bar{q}} = \Gamma_{\bar{q}q}$, the equations (4.2) can be decoupled immediately :

$$F_i^{l(\pm)} = 2C_i^{l(\pm)} \Gamma_{(\pm)} \quad (4.3)$$

where

$$\begin{aligned} F_i^{\ell(\pm)} &\equiv F_i^{\ell q} \pm F_i^{\ell \bar{q}} \\ C_i^{\ell(\pm)} &\equiv \frac{1}{2} (C_i^{\ell q} \pm C_i^{\ell \bar{q}}) \\ \Gamma(\pm) &= \Gamma_{qq} \pm \Gamma_{q\bar{q}} \end{aligned} \quad (4.4)$$

The hadronic structure functions can now be written as

$$\begin{aligned} F_i^{\ell H} &= \sum_q (C_i^{\ell(+)} \Gamma_{(+)} q_{B,H}^{(+)} + C_i^{\ell(-)} \Gamma_{(-)} q_{B,H}^{(-)}) = \\ &= \sum_q (C_i^{\ell(+)} q_H^{(+)} + C_i^{\ell(-)} q_H^{(-)}) \end{aligned} \quad (4.5)$$

where

$$q^{(\pm)} \equiv q \pm \bar{q} \quad (4.6)$$

The corresponding probabilities will be denoted as follows :

$$\begin{aligned} \Gamma_{qq}(x, \alpha, \frac{1}{\epsilon}) &= \delta(1-x) + \frac{2}{\epsilon} \left\{ \left(\frac{\alpha}{2\pi} \right) P_{qq}^{(0)}(x) + \frac{1}{2} \left(\frac{\alpha}{2\pi} \right)^2 P_{qq}^{(1)}(x) + \dots \right\} + O\left(\frac{1}{\epsilon^2}\right) \\ \Gamma_{q\bar{q}}(x, \alpha, \frac{1}{\epsilon}) &= \frac{2}{\epsilon} \left\{ \frac{1}{2} \left(\frac{\alpha}{2\pi} \right)^2 P_{q\bar{q}}^{(1)}(x) + \dots \right\} + O\left(\frac{1}{\epsilon^2}\right) \end{aligned} \quad (4.7)$$

$$\begin{aligned} P_{qq}(\alpha, x) &= \left(\frac{\alpha}{2\pi} \right) P_{qq}^{(0)}(x) + \left(\frac{\alpha}{2\pi} \right)^2 P_{qq}^{(1)}(x) + O(\alpha^3) \\ P_{q\bar{q}}(\alpha, x) &= \left(\frac{\alpha}{2\pi} \right)^2 P_{q\bar{q}}^{(1)}(x) + O(\alpha^3) \\ P_{(\pm)}(\alpha, x) &= P_{qq}(\alpha, x) \pm P_{q\bar{q}}(\alpha, x) \end{aligned} \quad (4.8)$$

The hadronic densities

$$q_H^{(\pm)}(x, Q^2) = q_H(x, Q^2) \pm \bar{q}_H(x, Q^2)$$

evolve with Q^2 as

$$Q^2 \frac{\partial}{\partial Q^2} q_H^{(\pm)}(x, Q^2) = \int_x^1 \frac{dz}{z} P_{(\pm)}\left(\frac{x}{z}, \alpha(Q^2)\right) q_H^{(\pm)}(z, Q^2) \quad (4.9)$$

It will be convenient in the following to consider independently the contributions from Z_F and from "real" diagrams where at least one internal line is cut. For this reason we introduce the following notation (where q stands for any of the subscripts $qq, +, -$) :

$$\Gamma_q(x, \alpha, \frac{1}{\epsilon}) = Z_F \hat{\Gamma}_q(x, \alpha, \frac{1}{\epsilon}) \quad (4.10)$$

where

$$\begin{aligned} Z_F &= 1 + \frac{2}{\epsilon} \left\{ \left(\frac{\alpha}{2\pi} \right) \zeta^{(0)} + \frac{1}{2} \left(\frac{\alpha}{2\pi} \right)^2 \zeta^{(1)} + \dots \right\} + O\left(\frac{1}{\epsilon^2}\right) \\ \hat{\Gamma}_q(x, \alpha, \frac{1}{\epsilon}) &= \delta(1-x) + \frac{2}{\epsilon} \left\{ \left(\frac{\alpha}{2\pi} \right) \hat{P}_q^{(0)}(x) + \frac{1}{2} \left(\frac{\alpha}{2\pi} \right)^2 \hat{P}_q^{(1)}(x) + \dots \right\} + O\left(\frac{1}{\epsilon^2}\right) \end{aligned} \quad (4.11)$$

so that

$$P_q(x, \alpha) = \hat{P}_q(x, \alpha) + \zeta(\alpha) \delta(1-x) \quad (4.12)$$

where

$$\zeta(\alpha) = \left(\frac{\alpha}{2\pi} \right) \zeta^{(0)} + \left(\frac{\alpha}{2\pi} \right)^2 \zeta^{(1)} + \dots \quad (4.13)$$

For $\Gamma_{q\bar{q}}$ we have

$$\begin{aligned} \Gamma_{q\bar{q}}(x, \alpha, \frac{1}{\epsilon}) &= Z_F \hat{\Gamma}_{q\bar{q}}(x, \alpha, \frac{1}{\epsilon}) \\ \hat{\Gamma}_{q\bar{q}}(x, \alpha, \frac{1}{\epsilon}) &= \frac{2}{\epsilon} \left\{ \left(\frac{\alpha}{2\pi} \right)^2 \frac{1}{2} \hat{P}_{q\bar{q}}^{(1)}(x) + \dots \right\} + O\left(\frac{1}{\epsilon^2}\right) \end{aligned} \quad (4.14)$$

so that

$$\hat{P}_{q\bar{q}}(x) = P_{q\bar{q}}(x) \quad (4.15)$$

The $O(\alpha^2)$ contribution to $\Gamma_{q\bar{q}}$ is given by diagram A of Fig. 7. There is, however, one very important subtlety here : in order to retain the probabilistic interpretation of $C_i^{\bar{q}}$ as a "short-distance" cross-section, we have to modify the action of the P projector on the antiquark line. The projector P_n inserted between G and K acts as follows

$$C P_n K = c \cancel{k} \left[\frac{\cancel{k}}{4kn} K \right] \quad (4.16)$$

so that $\cancel{k}$ imitates the sum over the polarizations of the incoming quark in the "short-distance" cross-section. But for the incoming antiquark with momentum k we need $-\cancel{k}$ in order to get the correct sign of $C_i^{\bar{q}}$. Therefore, we write Eq. (4.16) as follows

$$C P_n K = c (-\cancel{k}) \left[\frac{(-\cancel{k})}{4kn} K \right] \quad (4.17)$$

Now $C_i^{\bar{q}}$ can be calculated according to the usual Feynman rules but the spinor indices of the antiquark in $\Gamma_{q\bar{q}}$ should be contracted with $-\cancel{k}$, i.e., the diagram A enters $\Gamma_{q\bar{q}}$ with opposite sign.

4.2. - Fermion number conservation

At the leading log level we have

$$\int_0^1 dx P_{q\bar{q}}^{(0)}(x) = 0 \quad (4.18)$$

and, consequently $E(1; Q^2, Q_0^2) \equiv 1$, i.e., the first moment of the non-singlet quark density is conserved (does not evolve with Q^2). Since the physical origin of Eq. (4.18) is the fermion number conservation, we expect that a similar relation is valid also in higher orders in α . As seen from the calculation of $P_{q\bar{q}}^{(0)}(x)$ in Section 3, Eq. (4.18) is a consequence of the relation

$$\frac{1}{Z_F} = \int_0^1 dx \hat{\Gamma}_{qq}(x) \quad (4.19)$$

between virtual and real diagrams. The generalization of this formula to higher orders can be guessed by parton model arguments. According to the relation

$$|P_{\text{DRESSED}}\rangle = \sqrt{Z_F} \{ |P_{\text{BARE}}\rangle + \dots \} \quad (4.20)$$

between bare and dressed partons in old-fashion perturbation theory, the left-hand side of Eq. (4.19) can be interpreted as the number of dressed valence partons in the bare one. The right-hand side admits the same interpretation since the density $\hat{\Gamma}$ is normalized to one incoming bare quark (the Z_F factor is lacking), whereas the quark "interacting with γ " is dressed (i.e., it contains the full propagator). At higher orders, however, $\hat{\Gamma}_{qq}$ contains also some contamination of quarks from $q\bar{q}$ pairs (diagram C of Fig. 7) which is equal to $\int_0^1 \Gamma_{q\bar{q}} dx$; this should be subtracted to give the following relation

$$\frac{1}{Z_F} = \int_0^1 dx [\hat{\Gamma}_{qq}(x) - \hat{\Gamma}_{q\bar{q}}(x)] \quad (4.21)$$

In the following, we will demonstrate more formally that Eq. (4.21) is valid to order α^2 . The proof is based on the Ward identity: first we relate the cut diagrams $\hat{\Gamma}$ to the corresponding uncut objects using dispersion relations and then, via the Ward identity, we relate the uncut vertex to Z_F .

We denote by $\Gamma_\mu(p_1, p_2)$ the proper fermion vertex with vector coupling γ_μ for zero momentum transfer ($p_1 = p_2 = p$, $p^2 \neq 0$) and we contract it with n^μ and $\not{p}$:

$$\frac{1}{4} n^\mu \Gamma_\mu(p, p) \not{p} = \int_{-\infty}^{\infty} dx \tilde{\Gamma}(x) \quad (4.22)$$

where

$$\tilde{\Gamma}(x) = \delta(1-x) + \int \frac{d^m k}{(2\pi)^m} \delta(x - \frac{k_n}{p_n}) [\gamma T_{\beta\beta'}] \quad (4.23)$$

and $T_{\beta\beta'}^{\alpha\alpha'}(k, p)$ is the fully connected four-point function. Equation (4.23) can be written as

$$\tilde{\Gamma}(x) = \delta(1-x) + \int_{-\infty}^{\infty} \frac{dk^2}{(k^2 + i\epsilon)^2} \int_0^{\infty} dk_{\perp}^2 T(s, u; k^2) \quad (4.24)$$

where

$$\begin{aligned} s &= (p-k)^2 = (1-x)(p^2 - \frac{k^2}{x}) - \frac{k_{\perp}^2}{x} \\ u &= (p+k)^2 = (1+x)(p^2 + \frac{k^2}{x}) + \frac{k_{\perp}^2}{x} \end{aligned} \quad (4.25)$$

Let us write the forward dispersion relation for $T(s, u; k^2)$:

$$T(s, u; k^2) = \frac{1}{2\pi i} \left[\int_0^{\infty} \frac{ds'}{s' - s - i\epsilon} \text{Disc}_s T(s', u'; k^2) + \int_0^{\infty} \frac{du'}{u' - u - i\epsilon} \text{Disc}_u T(s', u'; k^2) \right] \quad (4.26)$$

where $s' + u' = s + u = 2p^2$. (For simplicity, we neglect subtractions which, as can be immediately verified, do not change the proof.) The integral over k^2 in Eq. (4.24) can be performed by closing the contour around the singularities of the integrand in the complex k^2 plane. Since the singularities in k^2 of the explicitly written propagators, as well as those hidden in the discontinuities $\text{Disc}_s T$, $\text{Disc}_u T$, are located below the real axis, the non-zero contribution to the integral can be obtained only if the dispersive denominators in Eq. (4.26) develop the poles in the upper half plane. As seen from (4.25), it provides the conditions : $0 < x < 1$ for the s channel discontinuity and $-1 < x < 0$ for the u channel discontinuity. Closing the contours in the upper k^2 half-plane, i.e., replacing the dispersive denominators by the corresponding δ functions and performing ds' , du' integrals, we get

$$\frac{1}{4} \gamma_{\mu} \Gamma_{\mu}(p, p) \gamma = 1 + \int_0^1 dx \Gamma_s(x) + \int_{-1}^0 dx \Gamma_u(x) \quad (4.27)$$

where

$$\Gamma_{(c)}(x) = \int \frac{d^m k}{(2\pi)^m} \delta(x - \frac{k\eta}{pm}) [\chi \text{Disc}_c T \not{p}] \quad , \quad c=s,u \quad (4.28)$$

Now, let us perform, on both sides of Eq. (4.27), the operation discussed in Section 3 : we continue the result to $m=4+\epsilon$ dimensions and take $p^2=0$. Now, we observe that there is a one-to-one correspondence between s channel discontinuities in $\Gamma_{(s)}(x)$ and the diagrams contributing to $\hat{\Gamma}_{qq}(x)$ so that

$$1 + \int_0^1 dx \Gamma_{(s)}(x) = \int_0^1 dx \hat{\Gamma}_{qq}(x) \quad (4.29)$$

Diagram B (uncut) has both s and u channel discontinuities ; the latter coincides with $\hat{\Gamma}_{q\bar{q}}(x)$ up to signs : the sign of k is reversed and an over-all (-1) is missing, which was introduced in Section 4.1 in order to get the correct sign for $C_1^{\bar{q}}$. Therefore

$$\Gamma_{(u)}(x) = - \hat{\Gamma}_{q\bar{q}}(-x) \Theta[x(1+x)]$$

so that

$$\frac{1}{4} \eta^\mu \Gamma_\mu(p,p) \not{p} = \int_0^1 dx \left[\hat{\Gamma}_{qq}(x) - \hat{\Gamma}_{q\bar{q}}(x) \right] \quad (4.30)$$

In order to relate the left-hand side of Eq. (4.9) to Z_F , let us consider the limit

$$\frac{1}{4} \eta^\mu \Gamma_\mu(p,p) \not{p} = \frac{1}{4} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left[q^\mu \Gamma_\mu(p, p+q) \not{p} \right]_{q=\lambda n} \quad (4.31)$$

Now, for the right-hand side in Eq. (4.31) we can use the Ward identity :

$$\begin{aligned} \frac{1}{4} \eta^\mu \Gamma_\mu(p,p) \not{p} &= 1 - \frac{1}{4} \lim_{\lambda \rightarrow 0} \text{Tr} [\Sigma(p+q) \not{p} - \Sigma(p) \not{p}] = \\ &= 1 - \frac{1}{4} \text{Tr} \left[\eta^\mu \frac{\partial \Sigma}{\partial p^\mu} \not{p} \right] \end{aligned} \quad (4.32)$$

The scalar amplitudes A and B [see Eq. (3.5)] depend on p^2 like $(4\pi\mu^2/p^2)^{c\eta}$ where $c > 0$ and $m = 4 - \eta$ ($\eta > 0$). Since, after making the derivative $\partial/\partial p_\mu$, we continue the result to $m = 4 + \epsilon$ ($\epsilon > 0$) and put $p^2 = 0$, all the derivatives $(\partial/\partial p_\mu)A$, $(\partial/\partial p_\mu)B$ give a zero contribution, i.e., only the derivatives over explicit $\not{p}$ and p^2 in Eq. (3.5) survive and we get

$$\frac{1}{4} \eta^\mu \Gamma_\mu(p, p) \not{p} = 1 - A - B = \frac{1}{\mathcal{Z}_F} \quad (4.33)$$

Combining (4.33) with (4.30) we obtain the promised relation (4.21).

The practical advantage of this relation is obvious : we can omit the calculation of complicated two-loop virtual diagrams contributing to Z_F . The full result for $P_{(\pm)}(x, \alpha)$ can be entirely expressed by the real diagrams $\hat{P}_{(\pm)}(x, \alpha)$ in the following way :

$$\begin{aligned} \zeta(\alpha) &= - \int_0^1 dx \hat{P}_{(-)}(x, \alpha) \\ P_{(+)}(x, \alpha) &= \hat{P}_{(+)}(x, \alpha) - \delta(1-x) \int_0^1 dz \hat{P}_{(-)}(z, \alpha) \\ P_{(-)}(x, \alpha) &= \hat{P}_{(-)}(x, \alpha) - \delta(1-x) \int_0^1 dz \hat{P}_{(-)}(z, \alpha) \end{aligned} \quad (4.34)$$

or, using + distributions $([f(x)]_+ = f(x) - \delta(1-x) \int_0^1 dz f(z))$

$$\begin{aligned} P_{(+)}(x, \alpha) &= [\hat{P}_{(+)}(x, \alpha)]_+ + 2\delta(1-x) \int_0^1 dz P_{q\bar{q}}(z, \alpha) \\ P_{(-)}(x, \alpha) &= [\hat{P}_{(-)}(x, \alpha)]_+ \end{aligned} \quad (4.35)$$

As a consequence of Eqs. (4.35), the density $q_H^{(-)}(x, Q^2)$ is conserved :

$$Q^2 \frac{\partial}{\partial Q^2} \int_0^1 dx q_H^{(-)}(x, Q^2) = 0 \quad (4.36)$$

whereas $q_H^{(+)}$ evolves with Q^2 :

$$Q^2 \frac{\partial}{\partial Q^2} \int_0^1 dz q_H^{(+)}(z, Q^2) = 2 \int_0^1 dz P_{q\bar{q}}(z, \alpha(Q^2)) \int_0^1 dx q_H^{(+)}(x, Q^2) \quad (4.37)$$

We can also write the formulas for the evolution of quarks and antiquarks :

$$\begin{pmatrix} q_H(N, Q^2) \\ \bar{q}_H(N, Q^2) \end{pmatrix} = \begin{pmatrix} E_{qq} & E_{q\bar{q}} \\ E_{q\bar{q}} & E_{qq} \end{pmatrix} \begin{pmatrix} q_H(N, Q_0^2) \\ \bar{q}_H(N, Q_0^2) \end{pmatrix} \quad (4.38)$$

where

$$\begin{aligned} E_{qq}(N; Q^2, Q_0^2) &= \cosh \left(\int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} P_{qq}(N, \alpha(k^2)) \right) \exp \left(\int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} P_{q\bar{q}}(N, \alpha(k^2)) \right) \\ E_{q\bar{q}}(N; Q^2, Q_0^2) &= \sinh \left(\int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} P_{q\bar{q}}(N, \alpha(k^2)) \right) \exp \left(\int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} P_{qq}(N, \alpha(k^2)) \right) \end{aligned} \quad (4.39)$$

E.g., if we start with one quark at $x=1$ for $Q^2=Q_0^2$, the relative abundance of quarks and antiquarks evolves with Q^2 as follows ^{*)} :

$$\frac{\bar{q}_H(N, Q^2)}{q_H(N, Q^2)} = \tanh \left(\int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} P_{q\bar{q}}(N, \alpha(k^2)) \right) \quad (4.40)$$

As seen from Eq. (4.37), the Q^2 evolution of $q_H^{(\pm)}$ starts at order α^2 ; in the leading log approximation both $q_H^{(\pm)}$ are conserved. Consequently, since also $C_i^{\ell(\pm)}$ are then equal to their naive parton model values, we have in the leading log limit the set of well-known sum rules for non-singlet structure functions.

With the next-to-leading effects included, most of the parton model sum rules are violated. Denoting the $O(\alpha)$ correction to $C_i^{\ell(\pm)}$ as

$$C_i^{\ell(\pm)}(1, N; \alpha(Q^2)) = C_{i, PM}^{\ell(\pm)} \left[1 + \frac{\alpha(Q^2)}{2\pi} B_i(N) \right] \quad (4.41)$$

^{*)} For a discussion of this point within OPE, see Ref. 20).

where $C_{i,PM}^{l(\pm)}$ are the parton model values, we find (see Section 7) that $B_1(1) \neq 0$, $B_3(1) \neq 0$, so that, e.g., the Gross-Llewellyn Smith sum rule is broken by the corrections to the "short distance" cross-section. On the other hand, $B_2(1) = 0$ so that the sum rules for $F_2(Q^2/\mu^2, 1)$ could be valid beyond the leading log approximation. Since, however, $q_H^{(+)}$ starts to evolve with Q^2 the sum rule

$$Q^2 \frac{d}{dQ^2} \int_0^1 \frac{dx}{x} F_2(Q^2/\mu^2, x) = 0 \quad (4.42)$$

will be satisfied only if

$$C_{i,PM}^{l(H)} = C_{i,PM}^{lq} + C_{i,PM}^{l\bar{q}} = 0 \quad (4.43)$$

i.e., if only the $q_H^{(-)}$ part in Eq. (4.5) contributes.

For obvious reasons, the relation (4.43) cannot be satisfied by any non-singlet quantity measured by an electromagnetic current, where actually only the $q_H^{(+)}$ part contributes: $C_{i,PM}^{e(-)} = 0$. On the other hand, for the weak charged current we have

$$C_{2,PM}^{v-\bar{v},q} = -C_{2,PM}^{v-\bar{v},\bar{q}} \quad (4.44)$$

Consequently, e.g., $\int_0^1 1/x (F_2^{eP} - F_2^{eN})$ evolves with Q^2 , whereas $\int_0^1 (dx/x) [F_2^{vP} - F_2^{v\bar{P}}]$ is conserved (Adler sum rule).

The physical interpretation of this effect is very transparent: because of the $q\bar{q}$ mixing we get a number of $q\bar{q}$ pairs even if we form the non-singlet quantity. The sum rule for the number of fermions will therefore, be satisfied only if the current (i.e., $C_i^{l(\pm)}$) is able to distinguish the fermion number of quarks from the fermion number of antiquarks, i.e., if (4.43) holds.

4.3. - Results

The calculation of diagrams in Fig. 7, except for the increasing technical complexity, does not contain qualitatively new elements as compared with the leading log exercise (Section 3). In particular, we did not

encounter any new peculiarity of the light-like gauge except for those discussed in Section 3 : all spurious ultra-violet poles come from self-energy type integrals like in Eq. (3.9), all infra-red (δ) divergences can be collected in the form of I_0 and I_1 integrals which cancel among themselves after adding up all diagrams. Finally, we have explicitly verified diagram by diagram that the 2PI kernel is finite before the last integral dk^2/k^2 is performed. We illustrate how it is realized in practice on the example of diagram B of Fig. 7. The contribution to Γ_{qq} coming from this diagram can be written as

$$\Gamma_{qq}(x, \mu, \frac{1}{\epsilon}) = x(C_F^2 - \frac{1}{2}C_F C_G) \frac{g^4}{\mu^{2\epsilon}} PP \left\{ \int \frac{d^4 k}{(2\pi)^4} \delta(x - \frac{k\eta}{p\eta}) \frac{1}{(k^2)^2} \frac{W_2(k^2, k_1^2, k_2^2)}{k_1^2 k_2^2} d\Phi_2(l_1, l_2) \right\} \quad (4.45)$$

where $d\Phi_2(l_1, l_2)$ is the two-body phase space of on-shell gluons and

$$W_2 = \text{Tr} \left[\frac{\gamma}{4kn} \gamma_\delta \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta \gamma_\rho \gamma_\sigma \gamma_\tau \gamma_\lambda \gamma_\kappa \right] d_{\alpha\beta}(l_1) d_{\mu\nu}(l_2) \quad (4.46)$$

is a quadratic form in the variables k^2, k_1^2, k_2^2 with the coefficients depending on the IMF fractions

$$z_i = \frac{l_i \eta}{p\eta}, x_i = \frac{k_i \eta}{p\eta}, x = \frac{k\eta}{p\eta}$$

since

$$\vec{k}_{11} = -\vec{l}_{11}; \vec{k}_{12} = -\vec{l}_{12}, k_1^2 = -\frac{k_{11}^2}{z_1}, k_2^2 = -\frac{k_{12}^2}{z_2}$$

the subintegral $d\Phi_2$ has two symmetric regions of logarithmic collinear divergences :

$$\frac{dk_{11}^2}{k_{11}^2} \left(\frac{k_{11}^2}{4\pi\mu^2} \right)^{\epsilon/2} \quad \text{for } k_{11}^2 \ll k_1^2, k_{12}^2 \sim k_1^2$$

and another one with 1 and 2 exchanged.

From the popular helicity arguments for the gluon propagator in axial gauge, we expect that collinear divergences will be killed by $\sqrt{-k_1^2}, \sqrt{-k_2^2}$ factors from the trace W_2 . The question is how one can get square root factors from the expression which is a quadratic form in k^2, k_1^2, k_2^2 . In the IMF variables we get an elegant answer : replacing k^2 everywhere in W_2 by the transverse scalar product $\vec{k}_{T1} \vec{k}_{T2}$ by means of the relation

$$k^2 = x_1 k_2^2 + x_2 k_1^2 - 2\vec{k}_{11} \vec{k}_{12}$$

we obtain

$$\frac{W_2(k_1^2, k_2^2, k_3^2)}{k_1^2 k_2^2} = A \frac{\vec{k}_1 \cdot \vec{k}_2}{k_1^2} + B \frac{\vec{k}_1 \cdot \vec{k}_2}{k_2^2} + C \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_1^2 k_2^2} + D \quad (4.47)$$

(A, B, C, D being functions of x_1, x_2, x); the potentially divergent integrals are indeed of the form $dk_{T1}/\sqrt{k_{T1}^2}$, $dk_{T2}/\sqrt{k_{T2}^2}$, i.e., the poles in subintegrals are killed.

The final result is of the form

$$\Gamma_{qq}(x, \alpha, \frac{1}{\epsilon}) = (C_F^2 - \frac{1}{2} C_F C_G) \left(\frac{\alpha}{2\pi} \right)^2 \int_0^Q \frac{dk^2}{k^2} \left(\frac{-k^2}{4\pi\mu^2} \right)^\epsilon \mathcal{P}(x, \epsilon) \quad (4.48)$$

so that diagram B has only a single pole coming from the last integral. For diagrams containing ultra-violet poles or internal collinear poles (ladder) we encounter also terms of the type

$$\Gamma(x, \alpha, \frac{1}{\epsilon}) = \int_0^Q \frac{dk^2}{k^2} \left(\frac{-k^2}{4\pi\mu^2} \right)^{\epsilon/2} \left[\frac{1}{\epsilon} \left(\frac{-k^2}{4\pi\mu^2} \right)^{\epsilon/2} - \frac{1}{\epsilon} \right] \mathcal{P}(x, \epsilon) \quad (4.49)$$

so that the final result contains both single and double poles. Since the pole is always subtracted, the integrand is finite when $\epsilon \rightarrow 0$ and $\Gamma(x, \alpha, 1/\epsilon)$ does not depend on Q^2/μ^2 , as discussed in Section 2.6.

[Usual ultra-violet poles are subtracted by counter terms, spurious ultra-violet poles by hand, and the collinear pole of the ladder K_0^2 is subtracted by the term $-K_0 P K_0$ in Eq. (4.1).]

The final result for $\hat{P}(x, \alpha)$ up to order α^2 is the following :

$$\hat{P}_{qq}(x, \alpha) = \left(\frac{\alpha}{2\pi} \right) C_F \left(\frac{1+x^2}{1-x} \right) + \left(\frac{\alpha}{2\pi} \right)^2 \left[C_F^2 P_F(x) + \frac{1}{2} C_F C_G P_G(x) + C_F N_F T_F P_{\pi_F}(x) \right] \quad (4.50)$$

$$\hat{P}_{q\bar{q}}(x, \alpha) = \left(\frac{\alpha}{2\pi} \right)^2 \left(C_F^2 - \frac{1}{2} C_F C_G \right) P_A(x) \quad (4.51)$$

where

$$P_F(x) = -2 \frac{1+x^2}{1-x} \ln x \ln(1-x) - \left(\frac{3}{1-x} + 2x \right) \ln x - \frac{1}{2} (1+x) \ln^2 x - 5(1-x) \quad (4.52)$$

$$P_G(x) = \frac{1+x^2}{1-x} \left[\ln^2 x + \frac{11}{3} \ln x + \frac{67}{9} - \frac{\pi^2}{3} \right] + 2(1+x) \ln x + \frac{40}{3}(1-x) \quad (4.53)$$

$$P_{NF}(x) = \frac{2}{3} \left[\frac{1+x^2}{1-x} \left(-\ln x - \frac{5}{3} \right) - 2(1-x) \right] \quad (4.54)$$

$$P_A(x) = 2 \frac{1+x^2}{1+x} \int_{\frac{x}{1+x}}^{\frac{1}{1+x}} \frac{dz}{z} \ln \left(\frac{1-z}{z} \right) + 2(1+x) \ln x + 4(1-x) \quad (4.55)$$

The contributions of particular diagrams to $\Gamma_{qq}(x, \alpha, 1/\epsilon)$ are given in Table 1. Each of the quantities P_F, P_G, P_{NF} in Eq. (4.50) is a linear combination of functions displayed in the first vertical column, with the coefficients given in the Table. The entry of the Table, multiplied by the corresponding function of x from the first column, and by the appropriate colour factor from the first line, gives the correctly normalized contribution to $\hat{P}_{qq}^{(1)}(x)$; when further multiplied by $(\alpha/2\pi)^2$ and by $1/\epsilon$ or $1/\epsilon^2$, it gives the contribution to $\Gamma_{qq}(x, \alpha, 1/\epsilon)$.

5. - COMPARISON WITH THE OPE RESULTS

In the present section, we will compare our results for the probabilities $P(x, \alpha)$ with the corresponding two-loop results for the anomalous dimensions derived by the OPE method ⁹⁾.

In general, since the two schemes are based on completely different techniques one could expect that a comparison is possible only at the level of final physical quantities: it is well known that the intermediate results in higher order calculations depend on the particular method of factorization. However, as will be demonstrated in this Section, our scheme and the OPE method in the minimal subtraction scheme are identical: there is a one-to-one correspondence between our probabilities $P(x, \alpha)$ and the anomalous dimensions of the Wilson operators, as well as between the coefficient functions and the "short-distance" cross-sections.

The non-singlet anomalous dimension $\gamma(N, \alpha)$ is determined by the coefficient of the simple ultra-violet pole in the renormalization constant of the Wilson operator

$$O_{\mu_1 \dots \mu_N}^N(x) = \{ \bar{\psi}(x) \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_N} \psi(x) \}_{S,T} \quad (5.1)$$

where S, T denote the symmetric and traceless parts. The twist-two contribution can be extracted by considering the two-point (e.g., quark) Green functions with O^N insertion

$$\frac{1}{4} \langle p | O_{\mu_1 \dots \mu_N}^N(x) | p \rangle = O^N(\alpha, \epsilon) p^{\mu_1} \dots p^{\mu_N} - \text{traces} \quad (5.2)$$

and by taking only the first term written explicitly in Eq. (5.2). $O^N(\alpha, \epsilon)$ denotes the renormalized matrix element, related to the bare one in $m = 4 - \epsilon$ dimensions ($\epsilon > 0$) as follows

$$O^N(\alpha, \epsilon, p^2/\mu^2) = Z_N(\alpha, \frac{1}{\epsilon}) O_B^N[\alpha \cdot (\frac{p^2}{\mu^2})^\epsilon, \epsilon] \quad (5.3)$$

The collinear divergences are regularized by taking $p^2 \neq 0$. In Eq. (5.2) we contract the spinor indices by $\not{p}$ and the factor $1/4$ normalizes O^N to unity in the free field limit. The O^N part can be projected out by contracting the Lorentz indices in Eq. (5.2) with an arbitrary light-like vector Δ_μ ($\Delta^2 = 0$) :

$$O^N(\alpha, \epsilon) = \frac{1}{4} \langle p | \{ \bar{\psi}(x) \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_N} \psi(x) \}_{S,T} | p \rangle \frac{\Delta_{\mu_1} \dots \Delta_{\mu_N}}{(\Delta p)^N} \quad (5.4)$$

Since $O_{\mu_1 \dots \mu_N}^N$ is a gauge invariant operator, its pole part Z_N can be calculated in any gauge. Let us therefore consider Eq. (5.4) in the light-like gauge and take $\Delta = n$. Since now $n^\mu A_\mu = 0$, the covariant derivatives are reduced to usual ones, i.e., there are no effective vertices with gluon insertions whereas for the quark-quark vertex we have the simple rule $\not{n}(kn)^{N-1}$ where k is the momentum entering the vertex with O^N insertion. Equation (5.4) can be rewritten as follows

$$O^N(\alpha, \epsilon) = \int_{-1}^1 dx x^{N-1} O(x, \alpha, \epsilon) \quad (5.5)$$

where

$$O(x, \alpha, \epsilon) = Z_F \left[\delta(1-x) + x \int \frac{d^m k}{(2\pi)^m} \left[\frac{\not{x}}{4kn} T(p, k) \not{x} \right] \right] \quad (5.6)$$

and $T(p, k)$ is the fully connected four-point function. Thus, as seen from Eq. (5.6), the effective vertex for the O^N insertion in the light-like gauge $n = \Delta$ coincides with the $\not{x}/4kn$ vertex in our method.

We now proceed to the final step for the proof of the identity between the two schemes. Let us consider the left-hand side of Eq. (5.3) and continue analytically the quantity $O^N(\alpha, \epsilon, p^2/\mu^2)$ to $m = 4 + \epsilon$ ($\epsilon > 0$) dimensions. By looking at Eq. (5.3), one sees that now a $p^2 \rightarrow 0$ limit does exist and is equivalent to an $\alpha \rightarrow 0$ limit inside O_B^N . Then, O^N is simply reduced to $Z_N(\alpha, 1/\epsilon)$. In order to relate the right-hand side of Eq. (5.5) to the moments of $\Gamma^S(x, \alpha, 1/\epsilon)$, we repeat literally the chain of arguments used in Section 4.2 to relate Z_F to cut diagrams: we write the dispersion relation for $[(\not{x}/4kn)T(p, k)]$ and perform the k^2 integral by closing the contour around the dispersive poles. Finally we make the analytic continuation to $m = 4 + \epsilon$ dimensions and take the limit $p^2 \rightarrow 0$

$$Z_N(\alpha, \frac{1}{\epsilon}) = \int_{-1}^1 dx x^{N-1} \left[\Gamma_{qq}(x, \alpha, \frac{1}{\epsilon}) \theta(x) - \Gamma_{q\bar{q}}(-x, \alpha, \frac{1}{\epsilon}) \theta(-x) \right] \quad (5.7)$$

Comparing the coefficients of single poles on both sides of Eq. (5.7) we get

$$\begin{aligned} -\frac{1}{2} \gamma(N, \alpha) &= \int_{-1}^1 dx x^{N-1} \left[P_{qq}(x, \alpha) \theta(x) - P_{q\bar{q}}(-x, \alpha) \theta(-x) \right] \\ &= P_{qq}(N, \alpha) + (-1)^N P_{q\bar{q}}(N, \alpha) \end{aligned} \quad (5.8)$$

where $\gamma(N, \alpha)$ is the anomalous dimension of the operator (5.1) in the minimal subtraction scheme and we use the convention $f(N) = \int_0^1 dx x^{N-1} f(x)$.

Equation (5.8) illustrates the origin of the $(-1)^N$ factor in the formulas for $\gamma(N, \alpha)$: in the OPE method one calculate x moments of the total discontinuity of the forward Compton amplitude and therefore both the s channel ($0 < x < 1$) and the u channel ($-1 < x < 0$) contribute. All the diagrams with crossed fermion lines, like the one giving rise to $\Gamma_{q\bar{q}}$ in

the parton calculation can be replaced by the diagrams with crossed external bosons (see Fig. 8) : the s channel discontinuity of the diagram A_s is equivalent to the u channel discontinuity of the diagram A_u in Fig. 8. Consequently, in the language of the OPE method, antiquarks are described as quarks with a negative value of x .

Let us formulate it more precisely. Denoting the structure functions as $(\rho_1, \rho_2, \rho_3) = (F_1, 1/x F_2, F_3)$, we have the crossing relations

$$\rho_i^{\ell\bar{q}}(-x) = -\tau_i \rho_i^{\ell q}(x) \quad (5.9)$$

where $\tau_1 = \tau_2 = -\tau_3 = 1$ and the CPT relations

$$\rho_i^{\ell\bar{q}}(x) = \tau_i \rho_i^{\bar{\ell}q}(x) \quad (5.10)$$

Combining (5.9) and (5.10) we get

$$\rho_i^{\ell\bar{q}}(x) = -\rho_i^{\ell q}(-x) \quad (5.11)$$

Denoting by $\rho_i^{\ell}(x)$ the full discontinuity of the Compton amplitude for quark, we can write

$$\rho_i^{\ell}(x) = \rho_i^{\ell q}(x) \theta(x) - \rho_i^{\ell\bar{q}}(x) \theta(-x) \quad (5.12)$$

The same happens for the bare parton density [Eq. (2.46)] : the full discontinuity $f_{B,H}^{\ell}(x)$ of the hadronic blob can be written as

$$f_{B,H}^{\ell}(x) = q_{B,H}(x) \theta(x) - \bar{q}_{B,H}(-x) \theta(-x) \quad (5.13)$$

The result for the discontinuity $F_i^{\ell H}(x)$ obtained in the OPE method can be written in x space as follows :

$$F_i^{\ell H}(x) = \int_{-1}^1 dx_1 dx_2 dx_3 C_i^{\ell}(x_1) \Gamma_{\text{OPE}}(x_2) f_{B,H}^{\ell}(x_3) \delta(x - x_1 x_2 x_3) \quad (5.14)$$

where

$$\begin{aligned}\Pi_{OPE} &\equiv \Pi_{qq}(x)\theta(x) - \Pi_{q\bar{q}}(-x)\theta(-x) \\ c_i^{\ell}(x) &\equiv c_i^{\ell q}(x)\theta(x) - c_i^{\ell \bar{q}}(-x)\theta(-x)\end{aligned}\quad (5.15)$$

From Eqs. (5.13)-(5.15) it follows that all functions entering the formula (5.14) are of the form

$$\psi(z) = \psi_q(z)\theta(z) - \psi_{\bar{q}}(-z)\theta(-z) \quad (5.16)$$

For the $(-1,1)$ moments of ψ we have

$$\int_{-1}^1 dx x^{N-1} \psi(x) = e_N \psi_{(+)}(N) + o_N \psi_{(-)}(N) \quad (5.17)$$

where

$$\psi_{(\pm)}(N) = \int_0^1 dx x^{N-1} [\psi_q(x) \pm \psi_{\bar{q}}(x)] \quad (5.18)$$

and

$$e_N = \frac{1}{2} [1 + (-1)^N] \quad ; \quad o_N = \frac{1}{2} [1 - (-1)^N] \quad (5.19)$$

so that the even (e_N) or odd (o_N) moments in the interval $(-1,1)$ are equal to usual $(0,1)$ moments for $(\pm)$ combinations of q and $\bar{q}$. Using the formula (5.18) we get from (5.14) :

$$\begin{aligned}\int_{-1}^1 dx x^{N-1} \rho_i^{\ell}(x) &= [e_N c_i^{\ell(+)} + o_N c_i^{\ell(-)}] [e_N \Pi_{(+)} + o_N \Pi_{(-)}] \\ &\quad \cdot [e_N \rho_{B,H}^{(+)} + o_N \rho_{B,H}^{(-)}] = \\ &= e_N c_i^{\ell(+)} \Pi_{(+)} q_{B,H}^{(+)} + o_N c_i^{\ell(-)} \Pi_{(-)} q_{B,H}^{(-)} = \\ &= e_N c_i^{\ell(+)} q_H^{(+)} + o_N c_i^{\ell(-)} q_H^{(-)}\end{aligned}\quad (5.20)$$

In order to express the left-hand side of Eq. (5.20) by the $(0,1)$ moments, we have to form linear combinations of $\rho_i^{(\eta)H}$ with a definite crossing parity, like

$$\rho_i^{(\eta)H} = \rho_i^{eH} + \eta \rho_i^{\bar{e}H} \quad (5.21)$$

where $\eta = \pm 1$. Using the crossing relations (5.9) we get

$$\rho_i^{(\eta)H}(-x) = -\tau_i \eta \rho_i^{(\eta)H}(x) \quad (5.22)$$

so that Eq. (5.20) can be written as follows :

$$[1 + \tau_i \eta (-1)^N] \rho_i^{(\eta)H}(N) = e_N c_i^{(\eta)(+)} q_H^{(+)} + o_N c_i^{(\eta)(-)} q_H^{(-)} \quad (5.23)$$

which is equivalent to

$$\rho_i^{(\eta)H}(N) = e_N \theta(\tau_i \eta) c_i^{(\eta)(+)} q_H^{(+)} + o_N \theta(-\tau_i \eta) c_i^{(\eta)(-)} q_H^{(-)} \quad (5.24)$$

Equation (5.24) expresses the OPE result in terms of the parton model quantities used in our scheme. On the other hand, the corresponding result for $\rho_i^{(\eta)H}$ in our scheme, following from (4.5) and (5.10), reads

$$\rho_i^{(\eta)H}(N) = \theta(\tau_i \eta) c_i^{(\eta)(+)} q_H^{(+)} + \theta(-\tau_i \eta) c_i^{(\eta)(-)} q_H^{(-)} \quad (5.25)$$

As seen from Eq. (5.24), the OPE method provides only the values of even or odd moments depending on the crossing parity of the particular combination $\rho_i^{(\eta)H}$ of the structure functions. For example, we get from (5.24) only even values of N for

$$\begin{aligned} & \int_0^1 dx x^{N-1} [F_3^{\nu p}(x) - F_3^{\bar{\nu} p}(x)] \\ & \int_0^1 dx x^{N-1} [F_1^{ep}(x) - F_1^{eN}(x)] \\ & \int_0^1 dx x^{N-2} [F_2^{ep}(x) - F_2^{eN}(x)] \end{aligned} \quad (5.26)$$

and only odd values of N for

$$\begin{aligned} & \int_0^1 dx x^{N-1} [F_3^{\nu P}(x) + F_3^{\bar{\nu} P}(x)] \\ & \int_0^1 dx x^{N-2} [F_2^{\nu P}(x) - F_2^{\bar{\nu} P}(x)] \end{aligned} \quad (5.27)$$

The parton model formula (5.25) provides the same results for the combinations quoted above, but it makes no restriction on the value of N : both even and odd N are allowed. In other words, the parton formula, besides reproducing the OPE result, provides also the analytic continuation of the latter to those values of N which are artificially forbidden in the OPE method. The rule is very simple :

$$\begin{aligned} -\frac{1}{2}\gamma(N,\alpha) &= P_{qq}(N,\alpha) + (-1)^N P_{q\bar{q}}(N,\alpha), \quad N \text{ even} \longrightarrow \\ &\longrightarrow P_{(+)}(N,\alpha) = P_{qq}(N,\alpha) + P_{q\bar{q}}(N,\alpha) \quad (\text{any } N) \end{aligned}$$

$$\begin{aligned} -\frac{1}{2}\gamma(N,\alpha) &= P_{qq}(N,\alpha) + (-1)^N P_{q\bar{q}}(N,\alpha), \quad N \text{ odd} \longrightarrow \\ &\longrightarrow P_{(-)}(N,\alpha) = P_{qq}(N,\alpha) - P_{q\bar{q}}(N,\alpha) \quad (\text{any } N) \end{aligned} \quad (5.28)$$

At the end of this Section we present the check of our results for the probabilities $P(x,\alpha)$ with the existing formulas for a two-loop anomalous dimension $\gamma(N,\alpha)$, derived by Floratos, Ross and Sachrajda⁹⁾. The original version of the result is very complicated : it contains single, double and triple sums and extends over a few pages. It has been shown recently²¹⁾ that all the multiple sums in the formula for $\gamma(N,\alpha)$ can be reduced to the following sums :

$$\begin{aligned} S_\alpha(N) &= \sum_{k=1}^N \frac{1}{k^\alpha} \\ S'_\alpha\left(\frac{N}{2}\right) &= \frac{1}{2} [1 + (-1)^N] S_\alpha\left(\frac{N}{2}\right) + \frac{1}{2} [1 - (-1)^N] S_\alpha\left(\frac{N-1}{2}\right) \\ \tilde{S}(N) &= \sum_{k=1}^N \frac{(-1)^k}{k^2} S_1(k) \end{aligned} \quad (5.29)$$

in terms of which the result can be written as follows :

$$\begin{aligned}
 \gamma^{(1)}(N) = & (C_F^2 - \frac{1}{2} C_F C_G) \left\{ 16 S_1(N) \frac{2N+1}{N^2(N+1)^2} + \right. \\
 & + 16 \left[2 S_1(N) - \frac{1}{N(N+1)} \right] \left[S_2(N) - S_2' \left(\frac{N}{2} \right) \right] + 64 \tilde{S}(N) + 24 S_2(N) \\
 & - 3 - 8 S_3' \left(\frac{N}{2} \right) - 8 \frac{3N^3 + N^2 - 1}{N^3(N+1)^3} - 16 (-1)^N \frac{2N^2 + 2N + 1}{N^3(N+1)^3} \Big\} \\
 & + C_F C_G \left\{ S_1(N) \left[\frac{536}{9} + 8 \frac{2N+1}{N^2(N+1)^2} \right] - 16 S_1(N) S_2(N) + \right. \\
 & + S_2(N) \left[-\frac{52}{3} + \frac{8}{N(N+1)} \right] - \frac{43}{6} - 4 \frac{151N^4 + 263N^3 + 97N^2 + 3N + 9}{9N^3(N+1)^3} \Big\} \\
 & + C_F N_F T_R \left\{ -\frac{160}{9} S_1(N) + \frac{32}{3} S_2(N) + \frac{4}{3} + \right. \\
 & \left. + 16 \frac{11N^2 + 5N - 3}{9N^2(N+1)^2} \right\}
 \end{aligned} \tag{5.30}$$

According to Eq. (5.8), we have to check the following relation (5.31)

$$-\frac{1}{8} \gamma^{(1)}(N) = P_{q\bar{q}}^{(1)}(N) + (-1)^N P_{q\bar{q}}^{(1)}(N) \tag{5.31}$$

which can be written in terms of the functions P_F, P_G, P_{NF}, P_A [Eqs. (4.52)-(4.55)] as follows

$$\begin{aligned}
 -\frac{1}{8} \gamma^{(1)}(N) = & (C_F^2 - \frac{1}{2} C_F C_G) (P_F(N) - P_F(1) + P_A(1) + (-1)^N P_A(N)) \\
 & + \frac{1}{2} C_F C_G (P_G(N) - P_G(1) + P_F(N) - P_F(1)) \\
 & + C_F N_F T_F (P_{NF}(N) - P_{NF}(1))
 \end{aligned} \tag{5.32}$$

We discuss here the calculation of the coefficient of $(C_F^2 - \frac{1}{2} C_F C_G)$ in order to illustrate how the complicated sums (5.29) are generated by taking the moments of probabilities $P_{q\bar{q}}^{(1)}(x), P_{q\bar{q}}^{(1)}(x)$.

The major part of the result in Eq. (5.30) is due to the function

$$\frac{1+x^2}{1+x} \int_{\frac{x}{1+x}}^{\frac{1}{1+x}} \frac{dx}{x} \ln \left(\frac{1-x}{x} \right)$$

appearing in $P_A(x)$. The moments can be derived by using the expansion

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$$

and integrating term by term by parts. Finally we get

$$\begin{aligned} P_A(N) = & 8(-1)^{N+1} [\tilde{S}(N) - \tilde{S}(\infty)] + 4 \left[2S_1(N) - \frac{1}{N(N+1)} \right] \eta(2, N+1) + \\ & + 4 \eta(3, N+1) + 2 \frac{2N^2 + 2N + 1}{N^3(N+1)^3} \end{aligned} \quad (5.33)$$

where

$$\eta(\alpha, m) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(i+m)^\alpha} \quad (5.34)$$

The functions $\eta(\alpha, m)$ can be related to the generalized Riemann ζ function

$$\zeta(\alpha, m) = \sum_{i=0}^{\infty} \frac{1}{(i+m)^\alpha} \quad (5.35)$$

through the relations :

$$\begin{aligned} \eta(\alpha, m) &= \frac{1}{2^\alpha} \left[\zeta\left(\alpha, \frac{m}{2}\right) - \zeta\left(\alpha, \frac{m+1}{2}\right) \right] \\ \zeta(\alpha, m) &= \frac{1}{2^\alpha} \left[\zeta\left(\alpha, \frac{m}{2}\right) + \zeta\left(\alpha, \frac{m+1}{2}\right) \right] \end{aligned} \quad (5.36)$$

Combining Eqs. (5.36), we get

$$\begin{aligned} \zeta(\alpha, m) + (-1)^m \eta(\alpha, m) &= \frac{1}{2^\alpha} \left[(1 + (-1)^m) \zeta\left(\alpha, \frac{m}{2}\right) + \right. \\ & \quad \left. + (1 - (-1)^m) \zeta\left(\alpha, \frac{m+1}{2}\right) \right] \end{aligned} \quad (5.37)$$

Now the arguments $m/2$, $(m+1)/2$ of the ζ functions of the right-hand side of Eq. (5.37) are integers so that finally we can express the function η by the functions (5.21) as follows

$$\eta(\alpha, N+1) = (-1)^{N+1} \left[S_\alpha(N) - \frac{1}{2^{\alpha-1}} S'_\alpha\left(\frac{N}{2}\right) - \left(1 - \frac{1}{2^{\alpha-1}}\right) \zeta(\alpha) \right] \quad (5.38)$$

where $\zeta(\alpha) \equiv \zeta(\alpha, 1)$ is the Riemann zeta function. Using (5.38) we can write Eq. (5.33) as follows

$$\begin{aligned} (-1)^N P_A(N) = & \left[2S_1(N) - \frac{1}{N(N+1)} \right] \left[-4S_2(N) + 2S'_2\left(\frac{N}{2}\right) + 2\zeta(2) \right] \\ & - 8\tilde{S}(N) - 4S_3(N) + S'_3\left(\frac{N}{2}\right) + (-1)^N \frac{2(2N^2+2N+1)}{N^3(N+1)^3} - 2\zeta(3) \end{aligned} \quad (5.39)$$

The calculation of $P_F(N)$ is straightforward and we only report the result :

$$\begin{aligned} P_F(N) - P_F(1) = & \left[2S_1(N) - \frac{1}{N(N+1)} \right] \left[2S_2(N) - 2\zeta(2) \right] - \frac{2(2N+1)}{N^2(N+1)^2} S_1(N) \\ & + 4S_3(N) - 3S_2(N) + 3\zeta(2) + \frac{3N^3+N^2+1}{N^3(N+1)^3} - \frac{23}{8} \end{aligned} \quad (5.40)$$

Combining (5.39) and (5.40) we get

$$\begin{aligned} P_F(N) - P_F(1) + P_A(1) + (-1)^N P_A(N) = & \\ = -\frac{1}{8} \left[\frac{16(2N+1)}{N^2(N+1)^2} S_1(N) + 16 \left(2S_1(N) - \frac{1}{N(N+1)} \right) \left(S_2(N) - S'_2\left(\frac{N}{2}\right) \right) + \right. \\ & + 64\tilde{S}(N) - 8S'_3\left(\frac{N}{2}\right) + 24S_2(N) - 3 - 8 \frac{3N^3+N^2+1}{N^3(N+1)^3} - \\ & \left. - 16(-1)^N \frac{2N^2+2N+1}{N^3(N+1)^3} \right] \end{aligned}$$

in agreement with the OPE result [Eq. (5.30)]. In a similar way, we can reproduce the coefficients of $\frac{1}{2}C_F C_G$ and $C_F N_F T_F$ in the formula (5.30).

6. - TIMELIKE PARTON DENSITIES : NEXT-TO-LEADING CORRECTIONS

6.1. - Space-time relations at the leading log level

The present Section is devoted to the presentation and discussion of the $O(\alpha^2)$ corrections to the probabilities $P(x, \alpha)$ in the timelike region. The general formalism for timelike processes is very similar to the spacelike case with the obvious replacement of parton densities by fragmentation functions. For example, the structure functions $F_i^{\ell H}$ in the single-particle inclusive e^+e^- annihilation are given by

$$F_i^{\ell H} = \sum_q [C_i^{\ell(+)} D_{(+)}^H + C_i^{\ell(-)} D_{(-)}^H] \quad (6.1)$$

where

$$D_{(\pm)}^H = D_q^H \pm D_{\bar{q}}^H = \Gamma_{(\pm)} D_{B,(\pm)}^H \quad (6.2)$$

are the fragmentation functions for the quark $\rightarrow$ hadron transition, defined as a product of the corresponding bare quantities and of the timelike parton densities $\Gamma_{(\pm)} = \Gamma_{qq} \pm \Gamma_{q\bar{q}}$, discussed in Section 2.8. (Note that for the electromagnetic current $C_i^{\ell(-)} = 0$ so that only the $\Gamma_{(+)}$ density contributes.)

The Q^2 evolution of the fragmentation functions is governed by timelike probabilities $P_{(\pm)}^T(x, \alpha)$, related to timelike densities $\Gamma_{(\pm)}^T$ by the same formulas as in the spacelike case [Eq. (4.7)] :

$$Q^2 \frac{\partial}{\partial Q^2} D_{(\pm)}^H \left(\frac{Q^2}{\mu^2}, x \right) = \int_x^1 \frac{dz}{z} P_{(\pm)}^T \left(\frac{x}{z}, \alpha(Q^2) \right) D_{(\pm)}^H \left(\frac{Q^2}{\mu^2}, z \right) \quad (6.3)$$

Therefore the only question we have to answer is how the quantities $P_{(\pm)}^T$ and $P_{(\pm)}^S$ are related to each other.

At the leading log level, time- and spacelike probabilities are identical (Gribov-Lipatov relation) ²²⁾ :

$$[P_{qq}^{(0)}(x)]_S = [P_{qq}^{(0)}(x)]_T \quad (6.4)$$

Another interesting relation valid at order α is the analytic continuation rule ²³⁾

$$\left[-x \hat{P}_{qq}^{(0)}\left(\frac{1}{x}\right) \right]_S = \left[\hat{P}_{qq}^{(0)}(x) \right]_T \quad (6.5)$$

according to which the timelike probability can be derived by making the analytic continuation of the spacelike probability $\left[P_{qq}^{(0)}(x) \right]_S$ to the non-physical region $x > 1$.

In the following we will discuss the validity of the relations (6.4) and (6.5) beyond leading order.

6.2. - Analytic continuation

We start with the analysis of the analytic continuation rule [Eq. (6.5)] which, assuming that it works beyond leading log level, would provide an efficient method for calculating timelike densities without repeating the involved two-loop calculation we have already done for the spacelike case.

The time- and spacelike densities Γ_S and Γ_T are given by the formulas :

$$\Gamma_S(x, \alpha, \frac{1}{\epsilon}) = Z_F \left[\delta(1-x) + x \int \frac{d^m k}{(2\pi)^m} \delta(x - \frac{kn}{pn}) \left[\frac{\pi}{4kn} \tilde{K} \frac{1}{1 - \tilde{P}\tilde{K}} \right] \right] \quad (6.6)$$

$$\Gamma_T(x, \alpha, \frac{1}{\epsilon}) = Z_F \left[\delta(1-z) + z^2 \int \frac{d^m k}{(2\pi)^m} \delta(z - \frac{pn}{kn}) \left[\frac{\pi}{4kn} \tilde{K} \frac{1}{1 - \tilde{P}\tilde{K}} \right] \right] \quad (6.7)$$

[see Eqs. (2.27) and (2.56)]. Substituting $z = 1/x$ in (6.6), we get

$$\Gamma_S\left(\frac{1}{z}, \alpha, \frac{1}{\epsilon}\right) = Z_F \left[\delta(1-z) + z \int \frac{d^m k}{(2\pi)^m} \delta(z - \frac{pn}{kn}) \left[\frac{\pi}{4kn} \tilde{K} \frac{1}{1 - \tilde{P}\tilde{K}} \right] \right] \quad (6.8)$$

Therefore, after substituting formally $\tilde{K} = K$, we would have

$$z \Gamma_S\left(\frac{1}{z}, \alpha, \frac{1}{\epsilon}\right) = \Gamma_T(z, \alpha, \frac{1}{\epsilon}) \quad (6.9)$$

In other words, the understanding of the analytic continuation rule (6.5) can be reduced to the understanding of the difference between the 2PI kernels K_0 and $\tilde{K}_0$ in the space- and timelike configuration.

The obvious difference, discussed in Section 2.8, is that $\tilde{K}_0(k_1, k_2)$ contains an extra factor $(k_{2n}/k_{1n})^\epsilon$ with respect to $K_0(k_1, k_2)$. There is, however, also a difference in the kinematical limits. Look, for example, at Fig. 9 : after rotating the timelike diagram in order to compare both kernels, we see that the energy of cut lines flows in opposite directions in K_0 and $\tilde{K}_0$. Consequently $x = (kn/pn) < 1$ in K_0 and $x = (kn/pn) > 1$ in $\tilde{K}_0$ so that the relation (6.9) cannot be satisfied as it stands for obvious kinematical reasons : if $0 < z < 1$, then $\Gamma_S(1/z) \equiv 0$ because of the phase space.

Let us therefore define a new density $\Gamma(x, \alpha, 1/\epsilon)$ as follows :

$$\Gamma(x, \alpha, \frac{1}{\epsilon}) = Z_F \left[\delta(1-x) + x \int \frac{d^m k}{(2\pi)^m} \delta(x - \frac{kn}{pn}) \left[\frac{\pi}{4km} \tilde{K} \frac{1}{1-\tilde{P}\tilde{K}} \right] \right] \quad (6.10)$$

Now the energy flows like in the Γ_T density, i.e., Γ is defined for $x > 1$ and is related to Γ_T by a trivial change of variables :

$$z \Gamma\left(\frac{1}{z}, \alpha, \frac{1}{\epsilon}\right) = \Gamma_T\left(z, \alpha, \frac{1}{\epsilon}\right) \quad (6.11)$$

Thus, in order to interpret the relation (6.9) we have to relate $\Gamma_S(x)$ (defined for $x < 1$) with $\Gamma(x)$ (defined for $x > 1$). The pictorial comparison of both densities is presented in Fig. 10, where the arrows indicate the direction of the positive energy flow. The factors $y_i^{-\epsilon} = (k_{in}/k_{i+1n})^{-\epsilon}$ will generate some extra contribution to the coefficient of the single pole coming from higher poles, like

$$\frac{1}{\epsilon^2} y_i^{-\epsilon} = \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \log y_i \quad (6.12)$$

We will calculate these effects explicitly later. For the moment, let us take $y_i^{-\epsilon} \equiv 1$ in Γ and discuss the effect of the difference in the direction of the energy flow. For this reason we parametrize the momenta of cut lines by

$$l_i = \left(z_i P + \frac{l_{i1}^2}{4z_i P}, \vec{l}_{i1}, z_i P - \frac{l_{i2}^2}{4z_i P} \right)$$

in the same way in both densities Γ_S and Γ so that $z_i > 0$ in Γ_S and $z_i < 0$ in Γ . Now, since uncut diagrams do not depend on the sign of the energy of any internal or external lines (analyticity property of Feynman diagrams), the only dependence of this type in our densities is due to the factors $\Theta(z_i)$ associated with cut lines ℓ_i .

The expression for Γ can then be obtained from the corresponding formula for Γ_S simply by replacing all factors $\Theta(z_i)$ by $\Theta(-z_i)$. Performing the integrals over $d^{m-2}\ell_{\mathbb{T}_i}$ and $d^{m-2}k_{\mathbb{T}_i}$, the results for Γ_S and Γ can be written in the form

$$\Gamma_S(x, \alpha, \frac{1}{\epsilon}) = \mathcal{Z}_F \left[\delta(1-x) + \int_{-\infty}^{+\infty} \frac{dk^2}{k^2} \Theta((1-x)k^2) \Phi_{(+)}(k^2/\mu^2, x, \epsilon) \right] \quad (6.13)$$

$$\Gamma(x, \alpha, \frac{1}{\epsilon}) = \mathcal{Z}_F \left[\delta(1-x) + \int_{-\infty}^{+\infty} \frac{dk^2}{k^2} \Theta((1-x)k^2) \Phi_{(-)}(k^2/\mu^2, x, \epsilon) \right] \quad (6.14)$$

where

$$\Phi_{(\pm)}(k^2/\mu^2, x, \epsilon) = \prod_i \int_{-\infty}^{+\infty} \frac{dz_i}{|z_i|} \Theta(\pm z_i) \delta(1 - \sum_k \frac{z_k}{1-x}) F(k^2/\mu^2, x, \{z_k\}, \epsilon) \quad (6.15)$$

and the indices i, j, k run over all cut lines. By making the rescaling $z_i = (1-x)\eta_i$, we get

$$\Phi_{(\pm)}(k^2/\mu^2, x, \epsilon) = \Theta[\pm(1-x)] \Phi(k^2/\mu^2, x, \epsilon)$$

where

$$\Phi(k^2/\mu^2, x, \epsilon) = \prod_i \int_0^1 \frac{d\eta_i}{\eta_i} \delta(1 - \sum_k \eta_k) F(k^2/\mu^2, x, \{(1-x)\eta_k\}, \epsilon) \quad (6.16)$$

so that Eqs. (6.13) and (6.14) can be rewritten as

$$\Gamma_S(x, \alpha, \frac{1}{\epsilon}) = \mathcal{Z}_F \left[\delta(1-x) + \Theta(1-x) \int_{-\infty}^0 \frac{dk^2}{k^2} \Phi(k^2/\mu^2, x, \epsilon) \right] \quad (6.17)$$

$$\Gamma(x, \alpha, \frac{1}{\epsilon}) = Z_F [\delta(1-x) + \theta(x-1) \int_0^\infty \frac{dk^2}{k^2} \Phi(\frac{k^2}{\mu^2}, x, \epsilon)] \quad (6.18)$$

Since in the derivation of Eqs. (6.17) and (6.18) all non-analytic effects due to the phase space have been explicitly taken into account, the function $\Phi(k^2/\mu^2, x, \epsilon)$ is expected to be an analytic function in the variables $k^2/\mu^2, x, \epsilon$. Timelike and spacelike densities are therefore related by the analytic continuation of the integrand Φ in both variables k^2/μ^2 and x . Therefore in general, one should not expect any simple connection in one variable as in the leading log case [Eq. (6.9)] : before integrating over k^2 in Eq. (6.18) we have to continue the function Φ from $k^2 < 0$ to $k^2 > 0$, crossing the threshold at $k^2 = 0$. Because of the singularities due to the emission of soft gluons, one usually gets extra contributions as in the well-known example of the electromagnetic vertex, discussed recently ²⁴⁾ in the context of the corrections to the Drell-Yan process :

$$\left[\frac{1}{\epsilon^2} \left(\frac{-k^2}{4\pi\mu^2} \right)^\epsilon \right]_T = \left[\frac{1}{\epsilon^2} \left(\frac{-k^2}{4\pi\mu^2} \right)^\epsilon \right]_S - \frac{\pi^2}{2} \quad (6.19)$$

In our case, the $k^2 = 0$ singularity contributes to the probability $P^T(x, \alpha)$ only if the over-all singularity of the diagram is like $1/\epsilon^3$. In fact, it takes place only for diagram D (uncut vertex) which gets an extra term $(\pi^2/2)(1+x^2)/(1-x)$ in the timelike region.

Another subtle point is related to the singularities at $x=1$, which have the same physical origin (soft gluon emission). After adding real and virtual contributions the singularity disappears but the (up to now) analytic functions of x , have to be considered as distributions. Consequently, as suggested by Gribov and Lipatov ²²⁾, the analytic combination through the point $x=1$ in the vector theories is impossible : e.g., the leading log probability $((1+x^2)/(1-x))_+$, when exponentiated, generates the density which behaves like $(1-x)^\lambda$ for $x \rightarrow 1$ with some non-integer value of λ so that it starts to develop an imaginary part when we cross the point $x=1$. In the axial gauge, the soft divergences of this type manifest as the $1/\ln$ singularities of the gluon propagator. As discussed in Section 3, we introduce the principal value regulator $(1/\ln) \rightarrow \ln/((\ln)^2 + \delta^2)$. As a consequence, as long as $\delta \neq 0$ the point

$x=1$ is regular ; e.g., the leading log density now behaves like $[(1-x)^2 + \delta^2]^{\lambda/2}$ when $x \rightarrow 1$.

Therefore we expect that the analytic continuation through the point $x=1$ is possible, assuming that we perform it before removing the regulator δ . On the other hand, in practical calculations of multiple integrals, it is much more convenient to extract $\ln \delta$ singularities and to take the limit $\delta \rightarrow 0$ in the finite part of the integrand. Then, however, the regions $x < 1$ and $x > 1$ are completely decoupled and should be considered independently. Let us illustrate it on the example of an integral which appears in the diagram D (uncut vertex) :

$$f(x, \delta) = \int_0^1 \frac{du}{u} \ln \frac{(1-x-u)^2 + \delta^2}{(1-x)^2 + \delta^2} - \frac{1}{2} \ln^2 [(1-x)^2 + \delta^2] \quad (6.20)$$

As long as $\delta \neq 0$, the point $x=1$ is regular. Therefore if we had the analytic expression for $f(x, \delta)$, derived for $x < 1$, we could go smoothly to the region $x > 1$. On the other hand, taking $\delta = 0$ we get

$$f(x, 0) = \int_0^{1-x} \frac{du}{u} \ln |1-u| - \frac{1}{2} \ln^2 |1-x| \quad (6.21)$$

In order to cancel the singular term $\ln^2 |1-x|$ we have to consider independently two cases : $x < 1$ and $x > 1$, this implies non-analyticity at $x=1$:

$$f(x, 0) = - \int_0^{1-x} \frac{du}{u} \ln |1-u| - \frac{\pi^2}{3} + \frac{\pi^2}{2} \Theta(x-1) \quad (6.22)$$

A detailed inspection of all the diagrams in Fig. 7 leads to the conclusion that the π^2 contribution of this type appears only in diagram D and cancels the π^2 contribution from the $k^2=0$ singularity in this diagram [e.g., Eq. (6.19)]. As a consequence, the full result for the spacelike probability given in Section 4 [Eqs. (4.50)-(4.55)] has the smooth behaviour at $x=1$ and can be directly used in the timelike case. In particular, all logarithms $\ln(1-x)$ should be continued as $\frac{1}{2} \ln [(1-x)^2 + \delta^2] = \ln |1-x|$.

Taking into account the over-all change of the sign coming from the dk^2 integral in Eqs. (6.17) and (6.18) and including the x^ϵ correction we finally get the following simple rule for the analytic continuation of non-singlet probabilities from the space- to the timelike region.

$$\left[\hat{P}_{\pm}(x, \alpha, \frac{1}{\epsilon}) \right]_{\mp} = \left[-x^{1+\epsilon} \hat{P}_{\pm}(\frac{1}{x}, \alpha, \frac{1}{\epsilon}) \right]_S \quad (6.23)$$

where P_S on the right-hand side in Eq. (6.23) should be derived by formal substitution $x \rightarrow 1/x$ in Eqs. (4.50)-(4.55). The only correction to the above rule comes from diagram I in Fig. 7, where, according to the discussion in Section 2, only the part $(kn/k_1n)^\epsilon$ of $x^\epsilon = (kn/k_1n)^\epsilon (k_1n/pn)^\epsilon$ should be included. All other diagrams (PK_0, PK_0^2) do not contain internal projectors P and the x^ϵ correction can be derived by multiplying the full result by x^ϵ , as indicated in Eq. (6.23).

6.3. - Fermion number conservation

We demonstrate here that the relation between real and virtual diagrams [Eq. (4.21)] is satisfied also for timelike densities :

$$\frac{1}{Z_F} = \int_0^1 \left[\frac{1}{q\bar{q}}(x, \alpha, \frac{1}{\epsilon}) - \frac{1}{q\bar{q}}(x, \alpha, \frac{1}{\epsilon}) \right] dx \quad (6.24)$$

The proof is based on the dispersion relations for the full quark propagator $G(k)$. Let us define the function $Z_F(k^2)$ as follows :

$$\frac{1}{4km} \text{Tr} [\not{k} G(k)] = \frac{i}{k^2} Z_F(k^2) \quad (6.25)$$

so that $Z_F = Z_F(0)$ [see Eq. (3.6)]. Taking the discontinuity in k^2 on both sides of Eq. (6.25), we get

$$\frac{1}{2\pi i} \text{Disc} \frac{1}{4km} \text{Tr} [\not{k} G(k)] = -i\delta(k^2) Z_F + \frac{1}{2\pi} \frac{\text{Disc} [Z_F(k^2) - 1]}{k^2 - i\epsilon} \quad (6.26)$$

The first term on the right-hand side corresponds to the sum of all the cuts through the single fermion in G whereas the second one is given by cuts through the fermion self-energies. We can therefore write :

$$\begin{aligned} \frac{1}{2\pi i} \int_0^\infty dk^2 \frac{1}{4km} \text{Disc} [\chi G(k)]_\Sigma &= -\frac{1}{2\pi i} \int_0^\infty \frac{dk^2}{k^2 - i\epsilon} \text{Disc} [Z_F(k^2) - 1] \\ &= 1 - Z_F \end{aligned} \quad (6.27)$$

where the subscript Σ denotes that only cuts through the self-energies are taken into account. The last equality in Eq. (6.27) is the dispersion relation for $1 - Z_F(0) = 1 - Z_F$. As can be seen from Fig. 11, there is a one-to-one correspondence between various diagrams contributing to the first moment of timelike density Γ_T and various cuts contributing to $\text{Disc} [\chi / 4km]_\Sigma$ (except for the three-fermion cut, which will be considered later).

There is, however, a difference in kinematics : in Γ_T the direction of p is fixed and we integrate over k_T , whereas in $\text{Disc} [\chi / 4km]_\Sigma$ the direction of k is fixed and we integrate over p_T . By changing the variable $k_T^2 = p_T^2 = z^2 k^2$, the $d^n k$ integral for Γ_T can be written as follows :

$$\begin{aligned} \Gamma_T(z, \alpha, \frac{1}{\epsilon}) &= z^\epsilon \int_0^\infty \frac{dk^2}{k^2} dk_\perp^2 \left(\frac{k_\perp^2}{4\pi\mu^2} \right)^{\epsilon/2} T(k^2; s, u) \frac{1}{\Gamma(1+\frac{\epsilon}{2})} \\ &= \int_0^\infty \frac{dk^2}{k^2} dp_\perp^2 \left(\frac{p_\perp^2}{4\pi\mu^2} \right)^{\epsilon/2} T(k^2; s, u) \frac{1}{\Gamma(1+\frac{\epsilon}{2})} \end{aligned} \quad (6.28)$$

where

$$\begin{aligned} s &= (p-k)^2 = (1 - \frac{1}{z})(-zk^2) - z k_\perp^2 = (1-z)k^2 - \frac{p_\perp^2}{z} \\ u &= (p+k)^2 = (1 + \frac{1}{z})(zk^2) + z k_\perp^2 = (1+z)k^2 + \frac{p_\perp^2}{z} \end{aligned} \quad (6.29)$$

The last integral in Eq. (6.28) is equivalent to the integral

$$\Gamma_T(z, \alpha, \frac{1}{\epsilon}) = \frac{1}{2\pi i} \int_0^\infty \frac{dk^2}{k^2} T(k^2; s, u) \frac{d^m p}{(2\pi)^m} 2\pi \delta^{(+)}(p^2) \quad (6.30)$$

performed in the frame

$$(6.31)$$

$$k = \left(P + \frac{k^2}{4P}, 0, P - \frac{k^2}{4P} \right)$$

$$\phi = \left(\mp P + \frac{p^2}{4\mp P}, \vec{p}_\perp, \mp P - \frac{p^2}{4\mp P} \right) \quad (6.31)$$

i.e., it has precisely the structure of the dispersive cut on the left-hand side in Eq. (6.27).

Here the z^ϵ correction is crucial : without it, the change of variables would produce an extra factor $z^{-\epsilon}$ which, in our normalization is exactly cancelled by z^ϵ . In other words, the z^ϵ correction is equivalent to the change of frame discussed above : the parton density Γ_T can be expressed by the same 2PI kernel K_0 as in Γ_S if we fix the direction of the incoming (off-shell) parton and integrate over the transverse momentum of the final (on-shell) parton.

Let us finally consider the three-fermion cut, which is related to three different diagrams in Γ_T . The diagrams 1 and 2, 3 in the timelike region are given by the expressions [in units of $1/e(\alpha/2\pi)^2(C_F^2 - \frac{1}{2}C_F C_G)$]:

$$\Gamma_{2,3}(x) = \frac{1+x^2}{1-x} \left[-\ln x \ln(1-x) + \frac{3}{4} \ln x + \frac{\pi^2}{6} + L(x) \right] - \frac{7}{4}(1+x) \ln x - \frac{7}{2} + 4x \quad (6.32)$$

where $L(x) \equiv \int_0^x \frac{du}{u} \ln(1-u)$

$$\Gamma_1(x) = \left[\hat{\Gamma}_{q\bar{q}}^1(x) \right]_S \quad (6.33)$$

[Eqs. (6.32) and (6.33) have been obtained from the spacelike results by using the analytic continuation rule (6.23)], and one can easily verify that

$$\int_0^1 dx \Gamma_1(x) = \int_0^1 dx \Gamma_{2,3}(x) \quad (6.34)$$

Therefore the contribution of the cut can be written as

$$\int_0^1 dx \Gamma_2(x) = \int_0^1 dx \Gamma_1(x) - 2 \int_0^1 dx \Gamma_2(x) - \int_0^1 dx \Gamma_1(x) \quad (6.35)$$

Finally, we get from Eq. (6.27) :

$$1 - \mathcal{Z}_F = \mathcal{Z}_F \left[\int_0^1 dx \hat{\Gamma}_{q\bar{q}}(x) - \int_0^1 dx \hat{\Gamma}_{\bar{q}q}(x) - 1 \right] \quad (6.36)$$

which is equivalent to Eq. (6.24).

6.4. - The Gribov-Lipatov relation

According to the discussion presented in Section 6.2, in order to calculate the timelike density we can use the analytic continuation rule (6.23) (with the correction for diagram I in Fig. 7). It is instructive to collect all x^e effects together, i.e., we write Eq. (6.23) as follows

$$\left[\hat{\Gamma}_{1/e}(x, \alpha) \right]_{\pi} = \left[-x \hat{\Gamma}_{1/e}\left(\frac{1}{x}, \alpha\right) \right]_S + \Gamma_e(x) \quad (6.37)$$

where $\hat{\Gamma}_{1/e}$ is the coefficient of the single pole of $\hat{\Gamma}$ and Γ_e contains all the x^e corrections. The contributions to $\Gamma_e(x)$ from all relevant diagrams are given in Table II. After summing all the terms, almost everything cancels and we get a simple result

$$\Gamma_e(x) = \left(\frac{\alpha}{2\pi}\right)^2 C_F \left(\frac{11}{3} C_G - \frac{2}{3} N_F\right) \frac{1+x^2}{1-x} \ln x \quad (6.38)$$

Note that Γ_e is proportional to $\beta_0 = \frac{11}{3} C_G - \frac{2}{3} N_F$ [Eq. (2.43)] which indicates that :

- a) the cancellations of various terms from Table II are due to the peculiarities of the light-like gauge discussed in Section 3 ;
- b) $\Gamma_e(x)$, as a typical scheme dependent effect, will probably cancel with the analogous term from the "short-distance" cross-section (see Section 8).

The final result for the timelike probability can be written in the form

$$\left[P_{(\pm)}(x, \alpha) \right]_{\pi} = \left[P_{(\pm)}(x, \alpha) \right]_S + \Delta(x) \quad (6.39)$$

where

$$\Delta(x) = \left(\frac{g}{2\pi}\right)^2 C_F^2 \left[4 \frac{1+x^2}{1-x} \ln x \ln(1-x) + \left(\frac{6}{1-x} - 5 - x\right) \ln x + \left(1+x - 2 \frac{1+x^2}{1-x}\right) \ln^2 x \right] \quad (6.40)$$

Since the fermion number conservation rule is satisfied both for spacelike and timelike densities [e.g. (4.21) and (6.24)], we have

$$\int_0^1 dx \left[P_{(-)}(x, \alpha) \right]_{\Gamma} = \int_0^1 dx \left[P_{(-)}(x, \alpha) \right]_S = 0 \quad (6.41)$$

which implies

$$\int_0^1 dx \Delta(x) = 0 \quad (6.42)$$

One can verify by explicit integration of Eq. (6.40) that the relation (6.42) is satisfied, which provides a nice check of the calculation in the timelike region.

The moments of $\Delta(x)$ can be written in the factorized form

$$\Delta(N) = 2 \left[-2S_1(N) + \frac{3}{2} + \frac{1}{N(N+1)} \right] \left[2S_2(N) - \frac{\pi^2}{3} - \frac{2N+1}{N^2(N+1)^2} \right] \quad (6.43)$$

so that, inverting back to the x space, $\Delta(x)$ can be written in the form of a convolution

$$\Delta(x) = 2 \int_0^1 dy \left(\frac{1+y^2}{1-y} \right)_+ \int_0^1 dz \frac{1+z^2}{1-z} \ln z \delta(x-yz) \quad (6.44)$$

The origin of Eq. (6.42) is now very transparent.

By looking at Table I, one sees that the G-L relation is violated by all the diagrams except for the one contributing to $\Gamma_{q\bar{q}}$. However, after adding the results, miraculous cancellations occur: the functions $P_G(x)$ and $P_{N_F}(x)$ [Eqs. (4.53)-(4.54)] break the G-L relation only by the term $-\Gamma_e(x)$ which is cancelled after including the x^e correction so that finally both $P_G(x)$ and $P_{N_F}(x)$ satisfy the G-L relation.

The correction term $\Delta(x)$ is built up from the contributions coming from the ladder (diagrams H and I in Fig. 7), the self-energy (diagram E) and the vertex (diagram C). Since $\Delta(x)$ has the form of the convolution of two kernels $(1+x^2)/(1-x)$, it is reasonable to expect that diagrams C and E are responsible only for the internal "plus", whereas the physical origin of the violation of the Gribov-Lipatov relation is entirely due to ladder diagrams. In fact, the function $\Delta(x)$ [without "plus" for $(1+y^2)/(1-y)$] can be easily derived by considering the difference in the kinematics of the ladder diagram in the spacelike and timelike configurations.

For this reason it is sufficient to take the double pole part in diagrams H and I in Fig. 7 and include the appropriate kinematical factor generating the subleading correction. In the spacelike region one has the following expression

$$\begin{aligned} [\Gamma_S(x)]_{H,I} &= \left(\frac{\alpha}{2\pi}\right)^2 C_F^2 \int_0^1 dy dz \delta(x-yz) \hat{P}^{(0)}(y) \hat{P}^{(0)}(z) \cdot \\ &\quad \cdot \left[\frac{2}{\epsilon^2} |1-y|^{\epsilon/2} \left|1-\frac{z}{y}\right|^{\epsilon/2} - \frac{4}{\epsilon^2} |1-z|^{\epsilon/2} \right] \end{aligned} \quad (6.45)$$

where $\hat{P}^{(0)}(y) = \frac{1+y^2}{1-y}$

Since there is no x^ϵ effect in the C_F^2 part after including also the contributions from diagrams E and C, one can calculate directly the time-like density :

$$\Gamma_T(x) = -x \Gamma_S\left(\frac{1}{x}\right) = \Gamma_S(x) + \left(\frac{\alpha}{2\pi}\right)^2 C_F^2 \frac{2}{\epsilon} \int dy dz \delta(x-yz) \hat{P}^{(0)}(y) \hat{P}^{(0)}(z) \ln z \quad (6.46)$$

The second term in Eq. (6.46) is generated by the replacement

$$(1-x)^{\epsilon/2} \longrightarrow \left|1-\frac{1}{z}\right|^{\epsilon/2} = |1-z|^{\epsilon/2} |z|^{-\epsilon/2}$$

in the subtraction diagram I.

Therefore, apart from the "plus" for the probabilities $\hat{P}^{(0)}(z)$ which is provided by self-energy diagrams, the breaking of the Gribov-Lipatov relation can be traced back to simple kinematical effects.

7. - "SHORT-DISTANCE" CROSS-SECTIONS

In order to complete the calculation of the next-to-leading corrections for DIS and SIA we still need the $O(\alpha)$ contribution to short-distance cross-sections. As before, we perform an explicit calculation for DIS and the formula for SIA will be derived by analytic continuation.

Expanding the series in Eq. (2.27c) to order α we have

$$\frac{1}{1 - (1 - R)K_0} = 1 + (1 - R)K_0 \quad (7.1)$$

i.e., we have to calculate the full structure functions and subtract off the pole. The calculation is now a children's play with respect to the two-loop problem for parton densities so that we limit ourselves to the presentation of the results.

We use the standard decomposition of $W_{\mu\nu}(p, q)$:

$$W_{\mu\nu}(p, q) = \frac{1}{4\pi} \int d^4x e^{iqx} \langle p | [j_\mu^+(x), j_\nu(0)] | p \rangle = \sum_{i=1}^3 e_{\mu\nu}^{(i)}(p, q) P_i(x, q^2) \quad (7.2)$$

where

$$\begin{aligned} e_{\mu\nu}^{(1)}(p, q) &= -(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) \\ e_{\mu\nu}^{(2)}(p, q) &= (-q^2) \left(\frac{p_\mu}{pq} - \frac{q_\mu}{q^2} \right) \left(\frac{p_\nu}{pq} - \frac{q_\nu}{q^2} \right) \\ e_{\mu\nu}^{(3)}(p, q) &= -i \epsilon_{\alpha\beta\mu\nu} \frac{p^\alpha q^\beta}{pq} \end{aligned} \quad (7.3)$$

so that $(p_1, p_2, p_3) = (F_1, \frac{1}{x} F_2, F_3)$.

The same decomposition will be used for the short-distance tensor

$$C_{\mu\nu}(p, q) = \sum_{i=1}^3 e_{\mu\nu}^{(i)}(p, q) C_i(x, q^2)$$

The basic tensors are invariant under the rescaling $p \rightarrow xp$ and therefore the full structure function and the short-distance cross-section coefficients C_i are related by :

$$g_i = C_i \Gamma \quad (i=1, 2, 3)$$

Since everybody is familiar with the letter F for the structure functions, we keep the generic notation $F_i = C_i \Gamma$ everywhere when the particular value of i is irrelevant. One should, however, always replace F_i by ρ_i (i.e., F_2 by $\frac{1}{x}F_2$) in all practical applications of our results.

The final result for DIS in the $\overline{MS}$ scheme and for $\mu^2 = Q^2/4\pi$ reads

$$C_2^S(x, \alpha(Q^2)) = \delta(1-x) + \frac{\alpha(Q^2)}{2\pi} C_F \left[\frac{1+x^2}{1-x} \left(\ln \frac{1-x}{x} - \frac{3}{4} \right) + \frac{1}{4} (9+5x) \right]_+ \quad (7.4)$$

$$C_L^S(x, \alpha(Q^2)) = C_1 - 2C_2 = \frac{\alpha(Q^2)}{2\pi} C_F \quad (7.5)$$

The contributions of particular diagrams are given in Table III.

By taking moments we can verify that Eqs. (7.4), (7.5) agree with the OPE results ²⁵⁾ :

$$C_2^S(N, \alpha) = 1 + \frac{\alpha}{4\pi} C_F \left[2S_1^2(N) + \left(3 - \frac{2}{N(N+1)} \right) S_1(N) - 2S_2(N) + \frac{2}{N^2} + \frac{3}{N} + \frac{4}{N+1} - 9 \right] \quad (7.6)$$

$$C_L^S(N, \alpha) = \frac{\alpha}{4\pi} C_F \frac{2}{N} \quad (7.7)$$

The corresponding result for SIA can be obtained by analytic continuation. If we denote $-q=k$ in DIS and $q=k$ in SIA, the problem of continuation is now formally identical as for the function Φ in Eqs. (6.17) and (6.18).

Using the same decomposition of $W_{\mu\nu}$ [Eq. (7.2)], we get the following rules for the "naïve" analytic continuation

$$\begin{aligned} \hat{C}_2^A(x) &= -x^{1+\epsilon} \hat{C}_2^S\left(\frac{1}{x}\right) \\ \hat{C}_L^A(x) &= \hat{C}_L^S\left(\frac{1}{x}\right) \end{aligned} \quad (7.8)$$

where

$$\begin{aligned} C_2^S(x) &= \hat{C}_2^S(x) - \delta(1-x) \int_0^1 dy \hat{C}_2^S(y) \\ C_2^A(x) &= \hat{C}_2^A(x) - \delta(1-x) \int_0^1 dy \hat{C}_2^S(y) \end{aligned} \quad (7.9)$$

Contrary to the case of the analytical continuation for parton densities, we get here a correction to the naïve rule coming from the singularity at $Q^2 = 0$: the analytic continuation of the electromagnetic vertex from $Q^2 < 0$ to $Q^2 > 0$ gives an extra contribution $(\alpha/2\pi) C_F \pi^2 \delta(1-x)$ which is exactly of the same origin as the analogous correction to the Drell-Yan process recently discussed. The x^e effect contributes $(\alpha/2\pi) C_F^2 ((1+x^2)/(1-x)) \ln x$ and the final result reads :

$$C_2^A(x) = C_2^S(x) + \frac{\alpha}{2\pi} C_F \left[3 \frac{1+x^2}{1-x} \ln x - \frac{7}{2}(1+x) + \pi^2 \delta(1-x) \right] \quad (7.10)$$

$$C_L^A(x) = C_L^S(x) = \frac{\alpha}{2\pi} C_F \quad (7.11)$$

The single-particle inclusive "short-distance" cross-section is given by

$$\frac{d\sigma}{dx d\cos\theta} = \sigma_0 \left[\frac{3}{2} C_L^A(x) + \frac{3}{8} (1+\cos^2\theta) C_2^A(x) \right] \quad (7.12)$$

where $\sigma_0 = \frac{4\pi}{3} \alpha^2$ is the Born cross-section and θ is the centre-of-mass angle between the quark and the beam.

The $O(\alpha)$ calculation of the cross-section (7.12) has been performed recently by Altarelli, K. Ellis, Martinelli and Pi. After writing Eq. (7.12) in terms of longitudinal and transverse parts as in Ref. 26), we have

$$\frac{d\sigma}{dx d\cos\theta} = \sigma_0 \left[\frac{3}{4} C_L^A(x) (1-\cos^2\theta) + \frac{3}{8} (1+\cos^2\theta) C_T^A(x) \right] \quad (7.13)$$

where $C_T^A(x) = C_2^A(x) + 2C_L^A(x)$ and we get the same results for C_T^A, C_L^A as in the paper quoted above.

8. - ALTERNATIVE PROBABILITIES AND NUMERICAL ESTIMATES

8.1. - An alternative definition of evolution probabilities

Leading order probabilities have the nice property of being universal and directly related to the evolution of the structure functions and/or fragmentation functions. As we have seen, this property does not persist to higher orders. In this case, one can define probabilities [see Eq. (4.9)] which do not govern the evolution of the whole structure function but only of hadronic densities and which possess an approximate universality. They are different between structure functions which probe either a symmetric or an antisymmetric combination of quarks and antiquarks and between space and timelike regions. One can analyze the latter difference as a breaking either of the Gribov-Lipatov relation or of analytic continuation rule, as we have already seen in Section 5.

These breakings, however, do not reflect in principle any physical effect since they occur for quantities, our probabilities, which are not experimentally observable, their effect on the evolution of structure functions being mixed with that of $O(\alpha_s)$ corrections to specific short distance cross-sections.

However, a new definition of an experimentally observable probability, the full probability $\mathcal{P}(x, \alpha)$, can be adopted, represented by the function which rules the Q^2 evolution of a specific, reference combination of structure functions. Adler's combination is the very natural candidate for this purpose : the associated full probability is defined by :

$$Q^2 \frac{\partial}{\partial Q^2} \frac{F_2^{v-\bar{v}, P}}{x}(x, Q^2) \equiv Q^2 \frac{\partial}{\partial Q^2} \mathcal{P}_2^{v-\bar{v}, P}(x, Q^2) = \int_x^1 \frac{dy}{y} \mathcal{P}(\alpha, y) \mathcal{P}_2^{v-\bar{v}, P}\left(\frac{x}{y}, Q^2\right) \quad (8.1)$$

$$\mathcal{P}(\alpha, x) = \left[\mathbb{P}_e(\alpha, x) - \frac{\beta_0}{2} \left(\frac{\alpha}{2\pi}\right)^2 C_2^{(1)}(x) \right]_S \quad (8.1)$$

where $\left[\mathbb{P}_e(-)(\alpha, x) \right]_S$ is defined by Eq. (4.8) and $\left[C_2^{(1)}(x) \right]_S$ is related to $C_2^S(x)$ of Eq. (7.4) by

$$C_2^S(x) = \delta(1-x) + \frac{\alpha}{2\pi} \left[C_2^{(1)}(x) \right]_S \quad (8.2)$$

The unique property of the full probability which justifies its name is that its first moment is normalized to all orders in power of $\ln Q^2$, expressing Adler's conservation rule

$$\int_0^1 \mathcal{P}(\alpha, x) dx = 0 \quad (8.3)$$

The formal solution of Eq. (8.1) in terms of moments reads :

$$\begin{aligned} \mathcal{P}_2^{\nu-\bar{\nu}, P}(N, Q^2) &= \mathcal{P}_2^{\nu-\bar{\nu}, P}(N, Q_0^2) \exp \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \mathcal{P}[\alpha(k^2), N] \\ &= \mathcal{P}_2^{\nu-\bar{\nu}, P}(N, Q_0^2) \left[\frac{1 + \beta_1 \frac{\alpha(Q^2)}{4\pi}}{1 + \beta_1 \frac{\alpha(Q_0^2)}{4\pi}} \right]^{\frac{\beta_0}{\beta_1} A_2^{\nu-\bar{\nu}, P}(N)} \cdot \left[\frac{\alpha(Q^2)}{\alpha(Q_0^2)} \right]^{-\frac{2P^{(0)}(N)}{\beta_0}} \end{aligned} \quad (8.4)$$

where

$$\begin{aligned} A_2^{\nu-\bar{\nu}, P}(N) &= 2[C_2^{(1)}(N)]_S + \frac{2\beta_1}{\beta_0^2} P^{(0)}(N) - 4 \frac{[P_{(-)}^{(1)}(N)]_S}{\beta_0} \\ \alpha(Q^2) &= \frac{4\pi}{\beta_0 \ln Q^2/\Lambda^2} \left[1 - \frac{\beta_1 \ln(\ln Q^2/\Lambda^2)}{\beta_0^2 \ln Q^2/\Lambda^2} \right] \\ \beta_0 &= \frac{1}{3} [11C_G - 2N_F] \\ \beta_1 &= \frac{34}{3} C_G^2 - \frac{10}{3} N_F C_G - 2N_F C_F \end{aligned}$$

If we expand the correction to a leading log result up to $O(\alpha)$ in the last equation, we obtain the traditional formula :

$$\mathcal{P}_2^{\nu-\bar{\nu}, P}(N, Q^2) = \mathcal{P}_2^{\nu-\bar{\nu}, P}(N, Q_0^2) \left[\frac{\alpha(Q^2)}{\alpha(Q_0^2)} \right]^{-\frac{2P^{(0)}(N)}{\beta_0}} \left[1 + \frac{\alpha(Q^2) - \alpha(Q_0^2)}{4\pi} A_2^{\nu-\bar{\nu}, P}(N) \right] \quad (8.5)$$

The exponentiated form of the solution [Eq. (8.4)] may indeed represent a good approximation for higher order corrections. In fact, we know that among short-distance cross-section contributions, at least two particular classes do probably exponentiate :

- i) the terms which restore the scheme independence of the result since they have to match the corresponding quantities contained in parton densities which do explicitly exponentiate ;
- ii) some large corrections like π^2 or $\ln^2 N$ which are related to soft gluon emissions and perhaps can be resummed to all orders by a simple exponentiation ²⁷⁾.

For the full probability one can repeat the test of Gribov-Lipatov and of the analytic continuation rule between space- and timelike regions. However, the property of the Adler combination of structure functions, the conservation law, is peculiar to this quantity and does not find a correspondence in the timelike region. Therefore, for the comparison between the space- and timelike region in terms of observable quantities we have to choose the evolution functions of a new combination of structure functions. A probabilistic interpretation for these functions appears now more difficult ; nevertheless, for simplicity of notation, we will still denote by $\mathcal{S}^{S,T}$ these functions.

For the spacelike case, we define

$$\begin{aligned} Q^2 \frac{\partial}{\partial Q^2} \left[\frac{F_2^{ep}(x, Q^2)}{x} - \frac{F_2^{en}(x, Q^2)}{x} \right] &\equiv Q^2 \frac{\partial}{\partial Q^2} \mathcal{F}_2^{e, p-n}(x, Q^2) = \\ &= \int_x^1 \frac{dy}{y} \mathcal{S}^S(y, \alpha(Q^2)) \mathcal{F}_2^{e, p-n}\left(\frac{x}{y}, Q^2\right) \\ \mathcal{S}^S(y, \alpha) &= \left[P_G(\alpha, y) - \frac{\beta_0}{2} \left(\frac{\alpha}{2\pi}\right)^2 C_2^{(1)}(y) \right]_S \end{aligned} \quad (8.6)$$

and for the timelike one :

$$\begin{aligned} Q^2 \frac{\partial}{\partial Q^2} \left[x^2 F_2^{e\nu \rightarrow K^+ \chi}(x, Q^2) - x^2 F_2^{e\nu \rightarrow K^0 \chi}(x, Q^2) \right] &\equiv Q^2 \frac{\partial}{\partial Q^2} \mathcal{F}_2^{A, K^+ - K^0}(x, Q^2) = \\ &= \int_x^1 \frac{dy}{y} \mathcal{S}^T(y, \alpha(Q^2)) \mathcal{F}_2^{A, K^+ - K^0}\left(\frac{x}{y}, Q^2\right) \\ \mathcal{S}^T(y, \alpha) &= \left[P_G(\alpha, y) - \frac{\beta_0}{2} \left(\frac{\alpha}{2\pi}\right)^2 C_2^{(1)}(y) \right]_T \end{aligned} \quad (8.7)$$

the function $\overline{P}_{(+)}^{(1)}(\alpha, y)_{\mathbb{T}}$ is related to $\overline{P}_{(+)}^{(1)}(\alpha, y)_{\mathbb{S}}$ according to Eq. (6.39) ; $\overline{C}_2^{(1)}(x)_{\mathbb{T}}$ is defined by

$$C_2^A = \delta(1-x) + \frac{\alpha}{2\pi} [C_2^{(1)}(x)]_{\mathbb{T}} \quad (8.8)$$

with C_2^A as in Eq. (7.10) and it is related to $\overline{C}_2^{(1)}(x)_{\mathbb{S}}$ by :

$$[C_2^{(1)}(x)]_{\mathbb{T}} = [C_2^{(1)}(x)]_{\mathbb{S}} + C_F \left[3 \frac{1+x^2}{1-x} \ln x - \frac{x}{2}(1+x) + \pi^2 \delta(1-x) \right] \quad (8.9)$$

In order to make the space-time comparison it is convenient to separate inside $\mathcal{S}$ the part proportional to $\delta(1-x)$ as follows

$$\mathcal{S}(x, \alpha) = \hat{\mathcal{S}}(x, \alpha) + \mathcal{S}_V \cdot \delta(1-x) \quad (8.10)$$

Combining Eqs. (6.39) and (8.9) we find the following relations :

$$\hat{\mathcal{S}}^{\mathbb{T}}(x, \alpha) = -x \hat{\mathcal{S}}^{\mathbb{S}}\left(\frac{1}{x}, \alpha\right) \quad (8.11)$$

$$\mathcal{S}_V^{\mathbb{T}} = \mathcal{S}^{\mathbb{S}} + \beta_0 C_F \pi^2 \quad (8.12)$$

While the Gribov-Lipatov relation is still violated for physical quantities like our $\mathcal{S}$'s, the analytic continuation rule holds provided it is performed in both Q^2 and x variables : the change from negative to positive Q^2 gives in fact the correction term in Eq. (8.12).

8.2. - Numerical estimates

We have seen that beyond leading order scaling violations in space-like and timelike region are not the same. We present here a numerical test of the difference between the scaling violations induced by full probabilities $\mathcal{S}^{\mathbb{S}}$ and $\mathcal{S}^{\mathbb{T}}$ previously defined.

We consider the following quantities :

$$R^S(Q^2, Q_0^2, N) \equiv \frac{\mathcal{F}_2^{e, p-n}(N, Q^2)}{\mathcal{F}_2^{e, p-n}(N, Q_0^2)} = \left[1 + \frac{\alpha(Q^2) - \alpha(Q_0^2)}{4\pi} A_2^{e, p-n}(N) \right] \left(\frac{\alpha(Q^2)}{\alpha(Q_0^2)} \right)^{-\frac{2P^{(e)}(N)}{\beta_0}}$$

$$R^A(Q^2, Q_0^2, N) \equiv \frac{\mathcal{F}_2^{A, k^+-k^0}(Q^2, N)}{\mathcal{F}_2^{A, k^+-k^0}(Q_0^2, N)} = \left[1 + \frac{\alpha(Q^2) - \alpha(Q_0^2)}{4\pi} A_2^{A, k^+-k^0}(N) \right] \left(\frac{\alpha(Q^2)}{\alpha(Q_0^2)} \right)^{-\frac{2P^{(A)}(N)}{\beta_0}}$$
(8.13)

and we define their ratio :

$$R(Q^2, Q_0^2, N) \equiv \frac{R^A(Q^2, Q_0^2, N)}{R^S(Q^2, Q_0^2, N)} \quad (8.14)$$

The latter expresses a measure of the relative scaling violations between timelike and spacelike regions. Its explicit form depends on whether one adopts the form given by Eq. (8.4) or by Eq. (8.5) for the solution of the renormalization group equation.

In the first case, one gets :

$$R^1(Q^2, Q_0^2, N) = \exp \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} [\mathcal{P}^T(N, \alpha(k^2)) - \mathcal{P}^S(N, \alpha(k^2))] \quad (8.15)$$

in the second one :

$$R^2(Q^2, Q_0^2, N) = 1 + \frac{\alpha(Q^2) - \alpha(Q_0^2)}{4\pi} [A_2^{e, p-n}(N) - A_2^{A, k^+-k^0}(N)] \quad (8.16)$$

Notice that both R^1 and R^2 are independent of the renormalization prescription : a redefinition of α by higher order terms leads to corrections beyond our approximation.

In Fig 12, we present the numerical values of the two expressions R_1 and R_2 for $Q_0^2 = 5 \text{ GeV}^2$ and $Q^2 = 200 \text{ GeV}^2$ as a function of N . A change of Q_0^2 or Q^2 affects only logarithmically the results : for example, replacing $Q^2 = 200 \text{ GeV}^2$ with $Q_1^2 = 50 \text{ GeV}^2$ would raise the value of R_1 for $N=10$ from 0.75 to 0.80. The difference between the expansion to $O(\alpha_s)$ [Eq. (8.16)] and the exponentiated form [Eq. (8.15)] is not

dramatic, even in extreme cases such as $N=10$: anyway the full correction remains small enough to be considered a perturbative modification of the leading result. This situation is to be contrasted with that for corrections to leading scaling violations of space- and timelike regions separately, Eqs. (8.13). The quantities $A_2^{e,p-n}(N)$ and $A_2^{A,K^+-K^0}(N)$ defined as in Eq. (8.4) become very large for large N ; this rapidly spoils the perturbative character of the correction. In Fig. 13 we plot $A_2^{e,p-n}$ and A_2^{A,K^+-K^0} as a function of N : the properly normalized correction entering in Eqs. (8.13) is obtained by multiplying the plotted values by $(\alpha(Q^2) - \alpha(Q_0^2))/4$. One can see that the difference

$$D_2(N) \equiv A_2^{A,K^+-K^0}(N) - A_2^{e,p-n}(N)$$

plotted in the same figure flattens when N increases, the large N behaviour of full probabilities between the space- and timelike regions being identical.

Asymptotically, $D_2(N)$ tends to a constant value essentially determined by the π^2 term in Eq. (8.9). Space-time comparison may be therefore a good way for a systematic investigation of higher twist effects. In standard cases, higher twist corrections which are down by powers of Q^2 are important, where also other corrections down only by a power of $\log Q^2$ become large. In this case, as we have seen, the difference between space-like and timelike scaling violations remains a perturbative correction even for very large moments. We finally give the analytic expression for $D_2(N)$

$$\begin{aligned} D_2(N) &= 2 \left\{ [C_2^{(A)}(N)]_T - [C_2^{(A)}(N)]_S \right\} - \frac{4}{\beta_0} \Delta(N) = \\ &= 2 \left[2 S_2(N) - \frac{\pi^2}{3} - \frac{2N+1}{N^2(N+1)^2} \right] \left[3C_F - \frac{4C_F^2}{\beta_0} (-2S_1(N) + \frac{3}{2} + \right. \\ &\quad \left. + \frac{1}{N(N+1)}) \right] - C_F \left(7 \frac{2N+1}{N(N+1)} - 2\pi^2 \right) \end{aligned} \quad (8.17)$$

with $\Delta(N)$ defined as in Eq. (6.43).

9. - CONCLUSIONS

We have described in the paper a calculation scheme which extends beyond leading order a physical interpretation for anomalous dimensions of Wilson operators in terms of moments of the bremsstrahlung spectrum of quarks and gluons accelerated by external currents. The method is not only a physically more transparent way for calculating next-to-leading corrections which are already known but it turns out to be an essential tool for solving new problems. We have in fact derived the pattern of scaling violations in "timelike" processes, like single particle inclusive distributions in e^+e^- annihilation, where the operator product expansion technique cannot be applied.

The analysis we carried out stimulates, in turn, new problems, perhaps accessible with this method. Two elements appear in fact to play a special rôle in building up the final result : kinematics and soft gluon emissions. The former is responsible for the violation of the Gribov-Lipatov relation between spacelike and timelike anomalous dimensions while large contributions in the final result like π^2 or $2\pi^2 N$ can be traced back to the latter.

The dominance of terms of infra-red origin might suggest the necessity of an improved perturbative expansion - very well known in the problem of radiative corrections in QED - where leading infra-red corrections are exactly resummed and hard emissions are treated perturbatively. Furthermore, the relevance of kinematical considerations for relating spacelike and timelike anomalous dimensions at fixed next-to-leading order might be very general and suggest a way to solve the problem of finding an exact relation, valid to all orders in powers of $\log Q^2$, between spacelike and timelike scaling violations.

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$\Gamma_{qq}(x, \alpha, 1/\varepsilon)$		C_F^2						$\frac{1}{2} C_F C_G$					$\frac{1}{2} N_F C_F$
							SUM					SUM	
A	$\frac{1+x^2}{1-x}$	$7 - \frac{2}{3} \pi^2$	$-7 + \frac{2}{3} \pi^2$	0	0	0	0	$7 - \frac{2}{3} \pi^2$	0	$-11 + \pi^2$	$\frac{103}{9} - \frac{2}{3} \pi^2$	$\frac{67}{9} - \frac{1}{3} \pi^2$	$-\frac{10}{9}$
	$\frac{1+x^2}{1-x} \ln^2 x$	-2	1	-1	2	0	0	-1	1	1	0	1	0
	$(1+x) \ln x$	0	$-\frac{7}{2}$	2	-1	0	$-\frac{5}{2}$	$\frac{7}{2}$	-2	$\frac{1}{2}$	0	2	0
	$1-x$	3	-11	0	3	0	-5	11	0	-1	$\frac{10}{3}$	$\frac{40}{3}$	$-\frac{4}{3}$
1/ ε	$(1+x) \ln^2 x$	0	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	0	0	0	0
	$\frac{1+x^2}{1-x} \ln^2 (1-x)$	0	0	0	0	0	0	0	0	2	-2	0	0
	$\frac{1+x^2}{1-x} \ln x \ln (1-x)$	-4	2	0	0	0	-2	-2	0	6	-4	0	0
	$\frac{1+x^2}{1-x} \ln x$	0	$-\frac{3}{2}$	0	0	0	$-\frac{3}{2}$	$\frac{3}{2}$	0	$-\frac{3}{2}$	$\frac{11}{3}$	$\frac{11}{3}$	$-\frac{2}{3}$
	$\frac{1+x^2}{1-x} \ln (1-x)$	3	-3	4	-4	0	0	3	-4	5	-4	0	0
	$(1-x) \ln x$	-4	2	0	3	0	1	-2	0	2	0	0	0
	$(1-x) \ln (1-x)$	0	0	0	0	0	0	0	0	4	-4	0	0
	$I_1 \frac{1+x^2}{1-x}$	4	-4	0	0	0	0	4	0	-8	4	0	0
	$I_0 \frac{1+x^2}{1-x}$	0	0	4	-4	0	0	0	-4	8	-4	0	0
	$I_0 \frac{1+x^2}{1-x} (\ln x + \ln(1-x))$	-4	4	0	0	0	0	-4	0	8	-4	0	0
1/ ε^2	$I_0(1-x)$	-4	4	0	0	0	0	-4	0	8	-4	0	0
	$\frac{1+x^2}{1-x}$	6	-6	0	0	6	6	6	0	-6	$\frac{22}{3}$	$\frac{22}{3}$	$-\frac{4}{3}$
	$\frac{1+x^2}{1-x} \ln x$	-8	4	0	0	0	-4	-4	0	4	0	0	0
	$\frac{1+x^2}{1-x} \ln (1-x)$	0	0	0	8	0	8	0	0	8	-8	0	0
	$(1+x) \ln x$	0	0	0	2	0	2	0	0	0	0	0	0
	$1-x$	0	0	0	4	-8	-4	0	0	0	0	0	0
	$I_0 \frac{1+x^2}{1-x}$	-8	8	0	8	-8	0	-8	0	16	-8	0	0

- TABLE I -

$C_F^2 2 \ln X \left[4 \frac{1+X^2}{1-X} (I_0 + \ln \frac{1-X}{X}) + (1+X) \ln X - 2 (1-X) \right]$
$C_F^2 2 \ln X \left[-4 \frac{1+X^2}{1-X} (I_0 + \ln \frac{1-X}{X} + \frac{1}{2} \ln X) + 2 (1-X) - (1+X) \ln X \right]$
$(C_F^2 - \frac{1}{2} C_F C_G) 2 \ln X \left[\frac{1+X^2}{1-X} \right] (3 - 2 \ln X - 4 I_0)$
$\frac{1}{2} C_F C_G \frac{1+X^2}{1-X} 2 \ln X (8 I_0 - 6 \ln X - 3 + 4 \ln (1-X))$
$\frac{1}{2} C_F C_G \frac{1+X^2}{1-X} 2 \ln X (-4 I_0 + 4 \ln X + \frac{11}{3} - 4 \ln (1-X))$
$C_F^2 2 \ln X \frac{1+X^2}{1-X} (-3 + 4 \ln X + 4 I_0)$
$\frac{1}{2} N_F C_F \frac{1+X^2}{1-X} (-\frac{4}{3}) \ln X$

- TABLE II -

$\omega \left[-\frac{8}{\epsilon^2} \delta(1-x) + \frac{1}{\epsilon} (6\delta(1-x)) - \left(\frac{\pi^2}{3} + 8 \right) \delta(1-x) \right] + \frac{1}{\epsilon} (4I_0 - 2) \delta(1-x)$
$\omega \left[\frac{8}{\epsilon^2} \delta(1-x) + \frac{1}{\epsilon} \left((-4I_0 - 4) \delta(1-x) \right) + (-2I_1 + \frac{\pi^2}{3} + 6) \delta(1-x) - 2 \left(\frac{x}{1-x} \right)_+ \right]$
$\frac{1}{\epsilon} (-\delta(1-x))$
$\omega \left[\frac{1}{\epsilon} \delta(1-x) - \frac{1}{2} \delta(1-x) + \frac{1}{2} \left(\frac{1}{1-x} \right)_+ \right]$
$\omega \left[\frac{1}{\epsilon} \left((4I_0 - 3) \delta(1-x) + 2 \left(\frac{1+x^2}{1-x} \right)_+ \right) + (2I_1 + 1) \delta(1-x) + (1-x)_+ + \left(\frac{1+x^2}{1-x} \frac{1-x}{x} \right)_+ \right]$
$\omega \left[\frac{3}{2} \delta(1-x) + 3 (x)_+ \right]$
$\frac{1}{\epsilon} \left[(-4I_0 + 3) \delta(1-x) - 2 \left(\frac{1+x^2}{1-x} \right)_+ \right]$

- TABLE III -

TABLE CAPTIONS

Table I Detailed contribution of various diagrams to $\Gamma_{qq}(x, \alpha, 1/\epsilon)$.
Appropriate colour factors are shown in the first line. Among
single pole terms, those of type A satisfy the Gribov-Lipatov
relation while those of type B break it.

Table II The contribution of different diagrams to $\Gamma_\epsilon(x)$ defined in
Section 6.

Table III The $O(\alpha)$ corrections to short-distance cross-section σ_2^S
in units of $(\alpha/2\pi)C_F$. The quantity ω is defined by

$$\omega \equiv \left(\frac{Q^2}{4\pi\mu^2} \right)^{\epsilon/2} \frac{\Gamma(1+\epsilon/2)}{\Gamma(1+\epsilon)}$$

The incoming photon is represented by a dashed line.

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FIGURE CAPTIONS

- Figure 1 The matrix element squared for the photon (dashed line) space-like-quark interaction, its development in terms of 2PI kernels C_0 and K_0 and its final factorized form.
- Figure 2 The product of kernels : $AB = C$.
- Figure 3 The hadronic structure function F^H is obtained by convoluting partonic quantities $(C \cdot \Gamma)$ with "bare" parton densities $(q^{B,H})$. The quantity $\Gamma \cdot q^{B,H}$ is then interpreted as the physical parton density q^H .
- Figure 4 The same as in Fig. 1 for the photon timelike-quark interaction.
- Figure 5 Lowest order 2PI kernel K_0 and lowest order contributions to Z_F .
- Figure 6 One-loop virtual diagrams entering into the non-singlet next-to-leading corrections. The variables x_i are defined by $x_i = (k_i \cdot n) / (P \cdot n)$ where k_i is the momentum of the corresponding line and P an external reference momentum.
- Figure 7 Diagrams contributing to next-to-leading corrections in the light-like gauge to scaling violations of non-singlet combinations of hadronic structure functions. Corrections are divided into fully virtual (Z_F) quark into quark (Γ_{qq}) and quark into antiquark transitions ($\Gamma_{q\bar{q}}$). Diagrams A to G involve the 2PI kernel K_0 at order α^2 , diagram H is quadratic in the first order kernel K_0 and must be properly subtracted (diagram I) through projectors P .
- Figure 8 Equivalence of the diagram with crossed fermion lines to the diagram with crossed boson lines : the s channel discontinuity of the dotted box four-point amplitude A_s is equivalent to the u channel discontinuity of the corresponding A_u .

Figure 9 The difference in the positive energy flow (double arrow) between "rotated" spacelike Γ_S and timelike Γ_T .

Figure 10 A pictorial comparison of densities Γ_S and Γ defined in Section 6 : arrows indicate the direction of the positive energy flow. For Γ a factor y_i^e is present at each iteration of the 2PI kernels K_0 .

Figure 11 The correspondence between diagrams contributing to the discontinuity of the projected self-energy part of the fermion propagator $\text{Disc}[\chi G/4\kappa]_\Sigma$ (right-hand side) and to the first moment of Γ_T (left-hand side).

Figure 12 The quantity

$$R = \frac{\mathcal{F}_2^{A, k \pm k^0}(Q^2, N)}{\mathcal{F}_2^{A, k \pm k^0}(Q_0^2, N)} \cdot \frac{\mathcal{F}_2^{s, p-n}(Q^2, N)}{\mathcal{F}_2^{s, p-n}(Q_0^2, N)}$$

is plotted for typical values of Q_0^2 and Q^2 ($Q_0^2 = 5 \text{ GeV}^2$, $Q^2 = 200 \text{ GeV}^2$) as a function of N . Curves R^1 and R^2 refer respectively to the exponentiated form of the result and to its expansion to $O(\alpha)$.

Figure 13 The N dependence of $A_2^{A, K^+ - K^0}(N)$, $A_2^{e, p-n}(N)$ defined in Eqs. (8.13) and of their difference $D_2(N)$.

fig.4

fig.5

fig.6

$\Gamma_{q\bar{q}}$		A	PK_0
Γ_{qq}		B
	C
	D
	E
	F
	G	
Γ_{qq}		H	$P(K_0^2)$
	I	$-P(K_0PK_0)$
z_F

fig.7

fig.8

fig. 9

fig.10

fig.11

fig. 12

fig.13