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NON-ABELIAN ZILCH

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ABSTRACT

The non-Abelian analogues of the conserved tensors (zilch) of electromagnetism are either non-conserved or lead to conservation laws for time-non-local functions of the fields. This seems to rule out extension of duality or similar invariances to the non-Abelian regime.

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It is well known that free fields have an infinite number of conserved quantities, corresponding to the infinity of free oscillators they represent. Some time ago it was noted that simple local forms can be given for these, which were called "zilch" in the electromagnetic case¹⁾. One particular zilch of interest has the form

$$Z_{\text{abel.}}^{\mu\nu\sigma}(A) \equiv *F^{\mu\lambda} \overleftrightarrow{\partial}_\sigma F_\lambda{}^\nu + *F^{\nu\lambda} \overleftrightarrow{\partial}_\sigma F_\lambda{}^\mu \quad (1)$$

where $A \overleftrightarrow{\partial}_\rho B = 1/2(A \partial_\rho B - \partial_\rho AB)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $*F^{\mu\nu} \equiv 1/2 \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. The conserved charge corresponding to Z^{000} has an interesting physical interpretation as (essentially) the generator of duality transformation $F_{\mu\nu} \rightarrow *F_{\mu\nu}$. This is clear from its form

$$Q \equiv -\frac{1}{2} \int Z^{000} d^3x = \int (\vec{E} \cdot \nabla \times \vec{E} + \vec{B} \cdot \nabla \times \vec{B}) d^3x \quad (2)$$

Since $[\vec{E}, \vec{B}] = -\nabla \times \delta^3$, Q transforms the transverse fields $(\vec{E}, \vec{B})$ into $(\nabla^2 \vec{B}, -\nabla^2 \vec{E})$, and the ∇^2 factor may be removed by insertion of an appropriate factor ∇^{-2} in the integrand of (2).

Although it is known that duality transformations are not well defined in the non-Abelian case²⁾, it is nevertheless of interest to reconsider the analogues of (1) there, as a systematic way of seeing whether there are any conserved quantities which have an Abelian limit^{*)}. Consider therefore the non-Abelian equivalent of (1) which now includes a trace over internal indices. Here $F_{\mu\nu}(g;A) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ and ∂_ρ is replaced by the covariant derivative D_ρ . This non-Abelian Z possesses all of the algebraic properties of the Abelian Z and since it is a singlet, its divergence is an ordinary derivative which may be taken as the covariant one when operating on each F factor. This means that

$$Z_{\text{nonab.},g}^{\mu\nu\sigma} = \frac{1}{2} \text{Tr} \left(*F^{\mu\lambda} \square F_\lambda{}^\nu + *F^{\nu\lambda} \square F_\lambda{}^\mu - F^{\mu\lambda} \square *F_\lambda{}^\nu - F^{\nu\lambda} \square *F_\lambda{}^\mu \right) \quad (3)$$

just as in the Abelian case, except that $\square \equiv D_\mu D^\mu$ is the covariant d'Alembertian.

*) Only colour singlet zilches are of interest since covariant, unlike ordinary, conservation of a coloured vector does not yield a conserved charge.

It is at this point that the self-interaction of the non-Abelian case becomes relevant. For, unlike the Abelian situation in which $\square F_{\mu\nu} = 0$, propagation of the field strength is not along the (covariant) null-cone. The additional terms are easily obtained using the field equations and cyclic identities^{*)}

$$\begin{aligned} 0 &= D_\alpha D_\mu F^{\mu\nu} = D_\mu D_\alpha F^{\mu\nu} - ig [F_{\alpha\mu}, F^{\mu\nu}] = \\ &= -D_\mu (D^\mu F^\nu{}_\alpha + D^\nu F_\alpha{}^\mu) - ig [F_{\alpha\mu}, F^{\mu\nu}] = \\ &= -\square F^\nu{}_\alpha + 2ig [F^{\mu\nu}, F_{\alpha\mu}] \end{aligned} \quad (4a)$$

while similarly

$$\square *F_{\nu\alpha} = -ig \left([F_{\alpha\mu}, *F^\mu{}_\nu] - [F_{\nu\mu}, *F^\mu{}_\alpha] \right) \quad (4b)$$

As a result, Z fails to be conserved

$$\begin{aligned} Z^{\mu\nu\rho}_{\text{nonab.},g} &= -\frac{i}{2}g \text{Tr} \left(2 *F_\lambda{}^\rho [F^{\mu\lambda}, F^\nu{}_\rho] - \right. \\ &\quad \left. - *F^\mu{}_\lambda [F^{\lambda\rho}, F_\rho{}^\nu] - *F^{\nu\lambda} [F_{\lambda\rho}, F^{\rho\mu}] \right) \end{aligned} \quad (5)$$

The right-hand side of (5) does not vanish (even to linearized order) as may be seen from, e.g., the $\int Z^{000} d^3x$ component which is proportional to

$$\epsilon^{ijk} \text{Tr} \int \left(E_i [E_j, E_k] + B_i [B_j, E_k] \right) d^3x$$

Clearly, higher zilches, which have higher derivatives within the bilinear form (1) are a fortiori non-conserved.

The considerations given here also hold for gravitational zilches bilinear in curvatures. The only conserved quantities for the non-linear cases, then, are

*) An amusing consequence of (4) is that for static fields or in Euclidean signature, the sign of $i\text{Tr} F_{\mu\nu} [F_{\nu\alpha}, F_{\alpha\mu}]$ is determined (on-shell) since $\int F_{\nu\alpha} \square F_{\nu\alpha}$ is negative.

the stress-tensor and the Bel-Robinson tensor respectively, because these are bilinear in field strength (curvature) with no explicit derivatives (parity conserving zilches are also excluded).

Even though the "direct" approach by covariantization has failed, there is another possible way to construct infinitely many conserved quantities. For this purpose, consider the actions, including gauge-fixing terms (required to define the transformation below)

$$S_0(\xi, A') \equiv \text{Tr} \int \left\{ \frac{1}{4} F_{\mu\nu}(g=0; A')^2 + \xi (\partial_\mu A'_\mu)^2 \right\} d^4x \quad (6a)$$

$$S(\xi, g, A) \equiv \text{Tr} \int \left\{ \frac{1}{4} F_{\mu\nu}(g; A)^2 + \xi (\partial_\mu A_\mu)^2 \right\} d^4x \quad (6b)$$

Here S_0 represents the sum of Maxwell actions while S is the full non-Abelian action. It was shown in Ref. 3) (in a different context) that there exist a non-local and non-linear transformation $A \rightarrow A'(\xi, g, A)$ which is formally invertible and explicitly known to $O(g^2)$ ³⁾, such that

$$S_0(\xi, A'(\xi, g, A)) = S(\xi, g, A) \quad (7)$$

From (7) it follows immediately that (with summations implicit)

$$\frac{\delta S}{\delta A(x)} = \frac{\delta S_0}{\delta A'(y)} \cdot \frac{\delta A'(y)}{\delta A(x)} \quad (8a)$$

$$\frac{\partial S}{\partial \xi} = \frac{\partial S_0}{\partial \xi} + \frac{\delta S_0}{\delta A'(y)} \cdot \frac{\partial A'(y)}{\partial \xi} \quad (8b)$$

Hence, the full sets of field equations

$$\frac{\delta S_0}{\delta A'(x)} = 0 \quad \frac{\partial S_0}{\partial \xi} = 0 \quad (9a)$$

$$\frac{\delta S}{\delta A(x)} = 0 \quad \frac{\partial S}{\partial \xi} = 0 \quad (9b)$$

of the Abelian and non-Abelian systems are completely equivalent (since the functional matrix $\delta A'/\delta A$ is regular). Therefore, for any Abelian zilch quantity $Z_{\mu} \dots (A')$ which is conserved by (9a), we are guaranteed that

$$\hat{Z}_{\mu} \dots (A) \equiv Z_{\mu} \dots (A'(A)) \quad (10)$$

is also conserved on account of (9b). Consequently, the above construction has provided us with a set of conserved quantities $\hat{Z}(A)$ entirely within the non-Abelian context. Unfortunately, these $\hat{Z}$ are intrinsically time-non-local. We have not been able to remove this non-locality and therefore do not know whether (10) really contains any non-trivial information about conservation.

The present exercise does not, of course, disprove the existence of conserved quantities for non-Abelian fields, nor even of all local candidates which have an Abelian limit. It only confirms that, if they exist, they cannot be obtained through any obvious extension of familiar Abelian conservation laws, including that corresponding to duality invariance. Had it existed, such an extension might have accounted for the otherwise mysterious absence of certain counterterms in coupled Einstein-Yang-Mills theory⁴⁾ of the form $gk^2 f F_{\mu\nu} [\tilde{F}_{\nu\alpha}, F_{\alpha\mu}]$ which was one motivation for this search.

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