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EQUIVALENCE OF THE Z_4 AND THE $Z_2 \times Z_2$ LATTICE GAUGE THEORIES

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ABSTRACT

We prove the equivalence of the Z_4 and $Z_2 \times Z_2$ lattice gauge theories in any dimension and discuss some of the consequences of this result.

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The lattice formulation of gauge theories¹⁾ has provided an elegant way to introduce a cut-off and to discuss confinement^{1),2)}. Abelian Z_N gauge theories are of interest for two reasons. On one hand, although they have a two or three-phase structure, they are simple enough to be used to develop techniques that may be useful for the further investigation of $U(1)$ or $SU(N)$ gauge theories³⁾. On the other hand, there is a close connection between the gauge theory of $SU(N)$ and of its centre Z_N ⁴⁾ and a better understanding of the Abelian theory may be important for further progress in the non-Abelian $SU(N)$ theory.

It is known that in two-dimensional spin systems the Z_4 model decouples into two independent Z_2 models⁵⁾. The corresponding gauge models may be explicitly solved in two dimensions where they exhibit the same relationship. There are indications that this equivalence carries over to four dimensions. Both theories are self-dual (Korthals-Altes, Yoneya⁴⁾, Drouffe⁶⁾) with the critical coupling of the Z_4 theory being twice that of the Z_2 theory. Monte Carlo results⁷⁾ suggest that both theories have first order phase transitions; duality arguments confirm this suggestion for the Z_2 theory but leave the Z_4 transition type undecided⁶⁾.

Recent strong coupling expansions for the string tension⁸⁾ show an equality up to 12th order for $d = 3$ and 4 . It is the aim of this paper to prove that the Z_4 theory with coupling constant β is fully equivalent to a $Z_2 \times Z_2$ theory with coupling constant $\beta/2$ in any dimension. All Z_4 quantities (free energy, string tension, etc.) may be expressed through the corresponding Z_2 quantities and the singularity structures of both theories are simply related to each other.

The partition function of a Z_N gauge theory with coupling β on a finite lattice with V points is²⁾

$$Z(Z_N, \beta, V) = \int \prod_l d u_l \prod_p \exp [\beta \operatorname{Re} \chi_1^{(N)}(u_p)] \quad (1)$$

where the product is over all links and plaquettes. The characters of Z_N are given by

$$\chi_p^{(N)} \left(e^{\frac{2\pi i n}{N}} \right) = e^{\frac{2\pi i p n}{N}} \quad (2)$$

and the Haar measure is defined by

$$\int_{Z_N} F(u) du \equiv \frac{1}{N} \sum_{\xi \in Z_N} F(\xi) \quad (3)$$

The plaquette variable U_p is the oriented product of its boundary link variables; reversing the orientation of a link is the same as taking its complex conjugate.

Specializing to Z_4 , we make use of the character expansion²⁾ to rewrite the exponential of the Z_4 action as follows

$$\begin{aligned} \exp[\beta \operatorname{Re} \chi_1(u_p)] &= \\ &= \cosh^2 \frac{\beta}{2} \left(1 + \tanh \frac{\beta}{2} (\chi_1(u_p) + \chi_3(u_p)) + \tanh^2 \frac{\beta}{2} \chi_2(u_p) \right) \end{aligned} \quad (4)$$

For later convenience, we also define

$$\chi_A(u) \equiv \chi_1(u) + \chi_3(u) = u + u^+ \quad (5a)$$

$$\chi_B(u) \equiv \chi_2(u) = u^2 = u^{+2} \quad (5b)$$

so the Z_4 partition function becomes^{*)}

$$\begin{aligned} \mathcal{Z}(Z_4, \beta, V) &= \left(\cosh \frac{\beta}{2} \right)^{Vd(d-1)} \cdot \\ &\cdot \int_{Z_4^V} \{du_\ell\} \prod_P \left(1 + t \chi_A(u_p) + t^2 \chi_B(u_p) \right) \end{aligned} \quad (6)$$

Expanding the product and exploiting the orthogonality relations between χ_A and χ_B , one obtains a sum of terms, each of which may be associated with a two dimensional polyhedral complex³⁾. Each such complex may be divided into regular components, i.e., two dimensional surfaces whose boundary is homeomorphic to a circle. Conversely, any complex may be constructed by joining regular components along common parts of their boundaries. In d dimensions, the number of surfaces meeting at any common piece of their boundaries cannot exceed $2(d-1)$.

In the case of the Z_4 theory, each regular component carries either the index A or B. Surfaces of type A are orientable, whereas those of type B may not be due to the self-adjointness (cf. Eq. (5b); e.g., $\chi_A(ABC) \neq \chi_A(AB^+C)$ whereas $\chi_B(ABC) = \chi_B(AB^+C)$). It is easy to verify that only those complexes where an even number of type A surfaces are joined to an arbitrary number of type B surfaces at the intersection of their boundaries can possibly contribute to the Z_4 partition function⁶⁾(6). Each type A surface, consisting of P plaquettes, is accompanied by a factor t^P ; for type B, the factor is t^{2P} .

*) $t \equiv \tanh \beta/2$ throughout this paper.

Due to the local orientability of type A surfaces, two possibilities may arise. In the first case, the polyhedral complex may be built by joining together closed orientable type A surfaces which are homeomorphic to spheres with any number of handles cut along branch lines; an arbitrary number of type B surfaces may be added. Then, all regular components of type A may be oriented so that at any boundary piece half of them are directed in one direction and the other half in the opposite direction. One way to see this is that progressively more complicated complexes may be constructed by breaking up "old" branching lines and attaching "new" ones; in this process, the required orientation is always preserved (see Fig. 1).

In the second case, the complex is built up from some orientable and some non-orientable surfaces of type A (e.g., Klein-bottles) closed up to an integral along common branching lines. Then the non-orientable surfaces contribute along their boundary line factors like $\chi_A(UU)$ which may be replaced by $\chi_B(U)$. Stated otherwise, the non-orientable type A contribution may be treated like a type B surface with the same boundary. These two observations are of crucial importance for our equivalence proof.

The general expression for an integral over a boundary piece U bordering on α type A surfaces and β type B surfaces, which is encountered in the course of evaluating (6) reads

$$\int_{Z_4} \prod_{i=1}^{\alpha} \chi_A(UR_i) \prod_{j=1}^{\beta} \chi_B(US_j) dU \quad (7a)$$

where, by R_i and S_j , we denote the pieces of boundary connected to U. For Z_4 , the result of the integration is:

$$(7a) = 0 \quad \text{if } \alpha \text{ is odd}$$

$$= \sum_{\text{permutations}} \prod_{k=1}^{\alpha/2} Y_k \prod_{j=1}^{\beta} \chi_B(S_j) \delta \left[\left(\beta + \frac{\alpha}{2} + n \right) \bmod 2 \right] * \quad (7b)$$

otherwise

The sum includes all $2^{\alpha/2}$ permutations where Y_k is either $\chi_A(R_{2k-1}R_{2k})$ or $\chi_A(R_{2k-1}R_{2k}^+)$; n is the number of times a "twisted character" of type $\chi_A(RR^+)$ occurs in the product.

The Z_2 gauge theory has only one non-trivial representation, which is self-adjoint. An element $\tilde{U}$ of $Z_2 \times Z_2$ is represented as $(u, \bar{u})$. For the $Z_2 \times Z_2$ theory, the character expansion of the exponentiated action with coupling $\beta/2$ is

*) $\prod_{k=1}^0 Y_k \equiv 1$; $\delta(0) = 1$ and $\delta(1) = 0$.

$$\exp \left[\frac{\beta}{2} (u + \bar{u}) \right] = \cosh^2 \frac{\beta}{2} \left(1 + t(u + \bar{u}) + t^2 u \bar{u} \right) \quad (8)$$

In complete analogy with (3) and (4) we define

$$\hat{\chi}_A(\hat{u}) \equiv u + \bar{u} \quad (9a)$$

$$\hat{\chi}_B(\hat{u}) \equiv u \bar{u} \quad (9b)$$

and the partition function is

$$\begin{aligned} \mathcal{Z}(Z_2 \times Z_2, \frac{\beta}{2}, V) &= \left(\cosh \frac{\beta}{2} \right)^{Vd(d-1)} \\ &\cdot \int_{(Z_2 \times Z_2)^V} \{ d\tilde{u}_\ell \} \prod_P \left(1 + t \hat{\chi}_A(\tilde{u}_P) + t^2 \hat{\chi}_B(\tilde{u}_P) \right) \end{aligned} \quad (10)$$

with obvious notation. In particular,

$$\begin{aligned} \hat{\chi}_A(\tilde{u}_P) &= \hat{\chi}_A(\tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \tilde{u}_4) = u_P + \bar{u}_P = \\ &= u_1 u_2 u_3 u_4 + \bar{u}_1 \bar{u}_2 \bar{u}_3 \bar{u}_4 \end{aligned} \quad (11)$$

In two dimensions, $\mathcal{Z}$ is already given by the factor in front of the integral and Z_4 and $Z_2 \times Z_2$ are easily seen to coincide. In $d > 2$, we again expand the product over the plaquettes in (10). As in the case of Z_4 , the contributions correspond to closed two dimensional polyhedral complexes with orientable and non-orientable type A and B surfaces; to each complex in Z_4 there corresponds exactly one in $Z_2 \times Z_2$.

The operation which corresponds to complex conjugation in Z_4 is the interchange of u and $\bar{u}$ and we accordingly define $\tilde{U}^+ = (\bar{u}, u)$ if $\tilde{U} = (u, \bar{u})$. The "twisted $Z_2 \times Z_2$ " characters" are

$$\begin{aligned} \hat{\chi}_A(\tilde{u} \tilde{v}^+) &= u \bar{v} + \bar{u} v = \hat{\chi}_A(\tilde{u}^+ \tilde{v}) \\ \hat{\chi}_B(\tilde{u} \tilde{v}^+) &= u \bar{u} \bar{v} v = \hat{\chi}_B(\tilde{u} \tilde{v}) \end{aligned} \quad (12)$$

Observe that $\tilde{\chi}_A(\tilde{U}\tilde{U}^+) = 2\tilde{\chi}_B(\tilde{U})$ and $\tilde{\chi}_A(\tilde{U}\tilde{U}) = 2$ whereas exactly the opposite relations hold in the Z_4 theory! This implies that a non-orientable closed surface of type A, cut along branching lines with other surfaces, now contributes a factor like $\tilde{\chi}_A(\tilde{U}\tilde{U}^+)$ which in turn equals $\tilde{\chi}_B(\tilde{U})$ in $Z_2 \times Z_2$.

The expression which is analogous to (7a) acquires the form

$$\begin{aligned} \int_{Z_2 \times Z_2} \prod_{i=1}^{\alpha} \tilde{\chi}_A(\tilde{u}\tilde{R}_i) \prod_{j=1}^{\beta} \tilde{\chi}_B(\tilde{u}\tilde{S}_j) d\tilde{u} &= \\ &= 0 \quad \text{if } \alpha \text{ is odd} \\ &= \sum_{\text{permutations}} \prod_{k=1}^{\alpha/2} \tilde{Y}_k \prod_{j=1}^{\beta} \tilde{\chi}_B(\tilde{S}_j) \delta\left[(\beta + \frac{\alpha}{2} + n^*) \bmod 2\right] \\ &\quad \text{otherwise} \end{aligned} \quad (13)$$

Again, the sum runs over all $2^{\alpha/2}$ permutations where $\tilde{Y}_k$ is either $\chi_A(\tilde{R}_{2k-1}\tilde{R}_{2k})$ or $\tilde{\chi}_A(\tilde{R}_{2k-1}\tilde{R}_{2k}^+)$ but, in contradistinction to Z_4 , n^* is the number of times an "untwisted character" $\tilde{\chi}(\tilde{R}\tilde{R})$ occurs in the product.

For the proof of our main result, we need the following identities ($\alpha \geq 4$)

$$\begin{aligned} Z_4 : \int \prod_{k=1}^{\alpha/2} [\chi_A(u R_{2k-1}) \chi_A(u^+ R_{2k})] \prod_{j=1}^{\beta} \chi_B(u S_j) du &= \\ = \int \sum_{\text{permutations}} \chi_B^{\beta}(u_1) \prod_{k=1}^{\alpha/2} [\chi_A(u_k R_{2k-1}) Y_k du_k] &\cdot \\ \cdot \prod_{j=1}^{\beta} \chi_B(S_j) \delta\left[(\frac{\alpha}{2} + n) \bmod 2\right] & \quad (14a) \end{aligned}$$

$$\begin{aligned} Z_2 \times Z_2 : \int \prod_{k=1}^{\alpha/2} [\tilde{\chi}_A(\tilde{u}\tilde{R}_{2k-1}) \tilde{\chi}_A(\tilde{u}\tilde{R}_{2k})] \prod_{j=1}^{\beta} \tilde{\chi}_B(\tilde{u}\tilde{S}_j) d\tilde{u} &= \\ = \int \sum_{\text{permutations}} \tilde{\chi}_B^{\beta}(\tilde{u}_1) \prod_{k=1}^{\alpha/2} [\tilde{\chi}_A(\tilde{u}_k \tilde{R}_{2k-1}) \tilde{Y}_k d\tilde{u}_k] &\cdot \\ \cdot \prod_{j=1}^{\beta} \tilde{\chi}_B(\tilde{S}_j) \delta\left[(\frac{\alpha}{2} + n^*) \bmod 2\right] & \quad (14b) \end{aligned}$$

where Y_k equals either $\chi_A(U_k R_{2k})$ or $\chi_A(U_k^+ R_{2k})$ (similarly for $\tilde{Y}_k$); otherwise, the notation is the same as before. Note that the orientation of the Z_4 complex has been chosen in accordance with our previous observation. The relations (14) increase the number of integration variables by $(\alpha/2) - 1$ at each junction.

By repeated application of (14a,b) we may now reformulate all integrals of the form (7a) and (13) so that each variable occurs twice, at most, in a type A character. The "old" variables appear only in the "twisted" combination UU^+ in Z_4 and in the "untwisted" combination $\tilde{U}\tilde{U}$ in $Z_2 \times Z_2$ while all "new" variables occur in both ways. From this formulation, one immediately recognizes the one-to-one correspondence between Z_4 and $Z_2 \times Z_2$: for each expression in Z_4 with k_1 "twisted" and k_2 "untwisted" combinations there exists one and only one expression in $Z_2 \times Z_2$ with k_1 "untwisted" and k_2 "twisted" combinations of the corresponding boundary variables. The generic form of the sub-integrals is then

$$\int_{Z_4} \chi_A(uX) \chi_A(u^+Y) \chi_B^\beta(u) du = \begin{cases} \chi_A(XY) & \beta \text{ even} \\ \chi_A(XY^+) & \beta \text{ odd} \end{cases} \quad (15a)$$

$$\int_{Z_2 \times Z_2} \tilde{\chi}_A(\tilde{u}\tilde{X}) \tilde{\chi}_A(\tilde{u}\tilde{Y}) \tilde{\chi}_B^\beta(\tilde{u}) d\tilde{u} = \begin{cases} \tilde{\chi}_A(\tilde{X}\tilde{Y}) & \beta \text{ even} \\ \tilde{\chi}_A(\tilde{X}\tilde{Y}^+) & \beta \text{ odd} \end{cases} \quad (15b)$$

The equivalence remains unaffected by the integrations since the relative twists within XY ($\tilde{X}\tilde{Y}$) undergo the same changes in both theories.

Therefore, both theories agree term by term when expanded on a finite lattice and we arrive at our main result

$$\begin{aligned} \mathcal{Z}(Z_4, \beta, V) &= \mathcal{Z}(Z_2 \times Z_2, \frac{\beta}{2}, V) = \\ &= \mathcal{Z}^2(Z_2, \frac{\beta}{2}, V) \end{aligned} \quad (16)$$

Being true for any finite V , this result remains valid in the thermodynamic limit $V \rightarrow \infty$. For the free energy

$$F(Z_N, \beta) = \lim_{V \rightarrow \infty} \frac{1}{V} \log Z(Z_N, \beta, V) \quad (17)$$

we obtain from (16)

$$F(Z_4, \beta) = 2 F(Z_2, \frac{\beta}{2}) \quad (18)$$

From (18), we rederive the well-known relation between the critical couplings (at the self-dual point)

$$\beta_{cr}(Z_4) = 2 \beta_{cr}(Z_2) = \log(1 + \sqrt{2}) \quad (19)$$

Furthermore, the order of the phase transition is the same in both theories, which settles one of the questions cited in the introduction. Finally, one may, with little effort, derive from (16) the following relation for the Wilson-loop expectation values¹⁾

$$W_C(Z_4, \beta) = W_C(Z_2, \frac{\beta}{2}) \quad (20)$$

where

$$W_C(Z_N, \beta) \equiv \left\langle \operatorname{Re} \chi_1 \left(\prod_{\ell \in C} u_\ell \right) \right\rangle_{Z_N, \beta} \quad (21)$$

This explains the result of Ref. 8). There is strong evidence for a roughening transition in the string tension^{8),9)}; again, the critical couplings in Z_4 and Z_2 must obey a relation like (19).

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fig. 1