QUANTUM CHROMODYNAMICS AND ITS APPLICATIONS

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1. INTRODUCTION

1.1 The theory and its motivations

It is generally accepted that hadrons contain quarks as more fundamental constituents of matter. There are many pieces of evidence for this consensus, which we may group into four main classes:

- A) Spectroscopy -- both of the traditional light quarks and the modern heavier quarks, charmonium, etc.
- B) Chiral Dynamics -- PCAC, chiral symmetry and its breaking, the low-energy interactions of pions.
- C) Deep Inelastic Scattering -- the observation of approximate Bjorken scaling and the relations between νN and $(eN, \mu N)$ cross-sections.
- D) e^+e^- Annihilation -- the value of $\sigma_{\text{tot}}(e^+e^- \rightarrow \text{hadrons})$ and the observation of jets at high energies.

Our aim is to produce a calculable and predictive theory of the dynamics of quarks which includes and explains all these manifestations of quarks, as well as making testable predictions for future experiments. We are still far from this ultimate goal. However, many qualitative arguments lead us towards Quantum Chromodynamics¹⁾ (QCD) as the underlying theory of the strong interactions. So far we are severely limited in our ability to calculate with this theory, principally because its coupling strength is large in presently accessible kinematic régimes, which means that perturbation theory converges slowly, and that incalculable non-perturbative effects are of fundamental importance. Nevertheless, in the last few years we have attained substantial understanding of the asymptotic behaviour of QCD at large momenta²⁾, and it is now possible to start making semi-quantitative tests of this understanding. The main purpose of these lectures is to review the bases for QCD predictions at large momenta, and discuss how they may be (or have been) tested by experiment.

First we will briefly review the arguments leading to the primacy of QCD¹⁾ as a candidate theory of the strong interactions. We study in turn each of the above motivations A to D for the existence of quarks, and see how more detailed considerations lead us to suspect QCD may be the underlying dynamical theory.

A) Spectroscopy

The lowest-lying hadrons with any given set of quantum numbers (Q, I, B, S, C) are well described by a naive non-relativistic model of bound quarks. This model works well for J^{PC} assignments, static quantities such as magnetic moments, and transition matrix elements³⁾. The basis of these non-relativistic successes is still totally obscure, but if we accept it then we will see how details of the spectroscopy suggest that quark interactions are mediated by gluons which are

vectorial and coloured. It is a general feature of meson spectroscopy that for each $q_1\bar{q}_2$ system

$$m(^1S_0) < m(^3S_1) \quad (1.1)$$

e.g. $m_\pi < m_\rho$, $m_K < m_{K^*}$, $m_D < m_{D^*}$ The sign of the inequality (1.1) is the same as in atomic spectroscopy, where it is known to reflect the vector nature of the γ -exchange Coulomb potential and the ensuing inequality $E(f\uparrow\bar{f}\uparrow) > E(f\uparrow\bar{f}\downarrow)$ on the energies E of states with different spin orientations. If the $q\bar{q}$ interaction has anything to do with one-gluon exchange, which is totally unclear, then this gluon should be vectorial. Another argument for vectorial gluons is that spin 0 gluons would give identical forces between $q_1\bar{q}_2$ and q_1q_2 , so giving no understanding why $q\bar{q}$ states are bound, but not qq . Why should we believe that the vector gluons are coloured? Remember that a primary motivation for colour is the desire that the lowest-lying baryons (N, Δ) have quarks lying in a relative S-wave. If the spin-spin splitting between (N, Δ) is modelled by one-gluon exchange between quark pairs, then uncoloured (non-Abelian) gluons would make $m_N > m_\Delta$, since vector gluons imply for the Binding Energy BE

$$-BE(q\uparrow q\uparrow) = BE(q\uparrow q\downarrow) \quad (1.2)$$

and the sign of the latter is known [or deduced from (1.1)] to be negative. In the case of colour, any pair of qq must be in a $\underline{\bar{3}}$ of colour, in order that the full qqq system be a colour singlet. There is therefore the possibility that the Binding Energies may obey the inequality

$$BE(qq\uparrow q\uparrow) > BE(qq\downarrow q\uparrow) \quad (1.3)$$

and more detailed calculations show that this is indeed the case. There are many more detailed spectroscopic calculations using one-gluon exchange⁴⁾, which seem generally to work surprisingly -- e.g. predicting $m_\Sigma > m_\Lambda$ -- and provide confirmatory evidence for coloured vector gluons. A final remark is that the simplicity of charmonium spectroscopy⁵⁾, in particular the apparent successes of non-relativistic approximations and the smallness of the strong coupling as deduced from potential models of P-wave spin-spin splittings, suggests that the strong coupling actually gets smaller when heavier quarks are involved, a harbinger of asymptotic freedom.

B) Chiral Dynamics

The successes⁶⁾ of current algebra combined with PCAC in low-energy pion dynamics constitute evidence that hadrons look like quarks at short distances, as probed using current algebra commutators. Building on these successes, the anomalous current algebra calculation of the $\pi^0 \rightarrow 2\gamma$ decay rate⁷⁾ is evidence that quarks must

be coloured. Chiral symmetry is not exact, and there is evidence that it is broken "softly", i.e. that it becomes exact at very short distances because the symmetry-breaking terms in the Lagrangian have mass dimensions < 4 . All we know about chiral symmetry breaking suggests that it arises from terms in the strong interaction Lagrangian having the same algebraic properties as quark mass terms⁸⁾. In fact, the full set of chiral algebra commutators is only valid if gluons are vectorial rather than scalar, more theoretical evidence in favour of vector gluons.

C) Deep Inelastic Scattering

The observation of approximate Bjorken scaling⁹⁾ and the successes of the parton model¹⁰⁾ suggest that the strong interactions become weak at high Q^2 (small distances). The only field theory¹¹⁾ with this property of asymptotic freedom¹²⁾ is the non-Abelian vector gluon theory QCD. This is a very theoretical argument, but there is also a more direct phenomenological argument, based on the observed form of deep inelastic scaling violations to be discussed in Section 2.5, which suggests that gluons are indeed vectorial.

D) e^+e^- Annihilation

The cross-section ratio $R \equiv \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ is about 2 for $1.5 \text{ GeV} < E_{\text{cm}} = Q < 3.7 \text{ GeV}$, and about 4 at $Q = E_{\text{cm}}$ up to 32 GeV within experimental errors of $O(10 \text{ to } 20)\%$ ¹³⁾. Both these values are quite close to the formula

$$R = 3 \sum_{q: m_q < Q/2} \frac{e_q^2}{9} \quad (1.4)$$

where the factor of 3 comes from colour. Formula (1.4) gives

$$R = 3 \left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9} \right) = 2$$

below charm threshold, $R = 3^{1/3}$ between charm and bottom thresholds, and $R = 3^{2/3}$ above bottom threshold. In a theory where the effective strong coupling did not vanish as $Q^2 \rightarrow \infty$ one would expect on the basis of dimensional analysis¹⁴⁾ that R would become asymptotically constant, but not with the specific value given by (1.4). Furthermore, heuristically we would expect that if the strong interaction coupling did not vanish at large Q^2 , then a finite fraction of the total e^+e^- cross-section would contain multi-jet events coming from the strong bremsstrahlung of gluons and $q\bar{q}$ pair creation¹⁵⁾. On the other hand, one would expect an asymptotically free theory to yield mainly two-jet events at large Q^2 ¹⁶⁾. (This point will be justified in more detail in Lecture 7.) Experimentally, high-energy $e^+e^- \rightarrow \text{hadrons}$ annihilation is dominated by two-jet events¹³⁾, qualitative evidence in favour of asymptotic freedom.

At this point we may summarize our qualitative reasons for choosing a non-Abelian vector gluon theory of the strong interactions¹⁷⁾.

Property of the strong interaction field theory	Evidence from			
	Spectroscopy	Chiral dynamics	Deep inelastic	e^+e^- annihilation
Colour	✓	✓	—	✓
Vector gluons	✓	✓	✓	—
Coloured vectors	✓	—	—	—
Asymptotic freedom	✓	—	✓	✓

Since QCD is the non-Abelian vector gluon field theory of colour, and is the only strong interaction field theory to have asymptotic freedom, we proceed to study it. The Lagrangian is

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F_{\mu\nu}^a(x) F^{\mu\nu a}(x) + i \sum_i \bar{q}_i(x) D_{ij}^\mu q_j(x) - \sum_q m_q \bar{q}_i(x) q_i(x) \quad (1.5)$$

where the non-Abelian gauge field strength

$$F_{\mu\nu}^a(x) \equiv \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + g f_{abc} A_\mu^b(x) A_\nu^c(x) \quad (1.6)$$

with f^{abc} the structure constants of the SU(3) group, and the $D_{\mu ij}$ is the gauge-covariant derivative

$$\left. \begin{aligned} D_{\mu ij} &\equiv \partial_\mu \delta_{ij} - ig A_\mu^a T_{ij}^a : T_{ij}^a = \frac{1}{2} \lambda_{ij}^a \\ &[\lambda^a, \lambda^b] = 2i f^{abc} \lambda^c \end{aligned} \right\} \quad (1.7)$$

The Feynman rules for QCD are shown in Fig. 1, but will not be derived here. We refer the interested reader to other works¹⁸⁾ for the important subtleties surrounding the introduction, quantization and Feynman rules of the ghost fields. We proceed directly to the discussion of

1.2 Renormalization

In any field theory a bare vertex is inevitably renormalized by higher order corrections (cf. Fig. 2), and these in general depend on the momenta of the particles coming into the vertex. This means that we must introduce an effective momentum-dependent vertex

$$-i \gamma^\mu T_{ij}^a g(\mu) \Big|_{p_i^2 = -\mu^2} \quad (1.8)$$

and we now study the form of this renormalization and the way in which $g(\mu)$ depends on the renormalization scale μ ¹⁹⁾. In calculating higher order diagrams we generally encounter divergences, for example, in a scalar field theory the diagram in Fig. 3 is logarithmically divergent as the loop momentum $k \rightarrow \infty$:

$$\int \frac{d^4 k}{(2\pi)^4} \frac{\lambda^2}{(\frac{p}{2} + k)^2 (\frac{p}{2} - k)^2} \quad (1.9)$$

There are many ways of regularizing these infinities: let us consider the conceptually simplest method which introduces a cut-off momentum Λ in all integrations. Let us define the truncated Green functions of the theory

$$\Gamma_{n_B, n_F}(p_i) \equiv \prod_{i=1}^{n_B} [G_{2,0}(p_{B_i})]^{-1} \prod_{j=1}^{n_F} [G_{0,2}(p_{F_j})]^{-1} G_{n_B, n_F}(p_i) \quad (1.10)$$

where the $G_{n_B, n_F}(p_i)$ are the conventional Green functions. If we compute $\Gamma_{n_B, n_F}(p_i)$ in perturbation theory with a momentum cut-off Λ we obtain the finite quantity $\Gamma_{n_B, n_F}^U(p_i, g_0, \Lambda)$ where g_0 is the bare gauge coupling.

Let us for convenience work in the Landau gauge¹⁾ $\alpha = 0$. Then there is a heavy-duty field theory theorem which assures us that one can introduce universal Λ -dependent factors $\{Z_B(\Lambda), Z_F(\Lambda)\}$ with the property that all the Green functions

$$[Z_B(\Lambda)]^{n_B} [Z_F(\Lambda)]^{n_F} \Gamma_{n_B, n_F}^U(p_i, g_0, \Lambda) \quad (1.11)$$

simultaneously have finite limits as $\Lambda \rightarrow \infty$. Let us call these the renormalized Green functions Γ_{n_B, n_F}^R .

The $\{Z_B, Z_F\}$ are dimensionless quantities, but have a non-trivial dependence on the dimensionful cut-off parameter Λ . Hence their definition must also involve some other scale μ :

$$Z_{B,F} = Z_{B,F}(g_0, \Lambda/\mu) \quad (1.12)$$

Since all Green functions Γ_{n_B, n_F}^R are finite, in particular the three-point vertices are finite, and we may use one of them to define a finite dimensionless coupling constant

$$g = g(g_0, \Lambda/\mu) \quad (1.13)$$

There are many different ways of specifying the renormalization factors Z_B, Z_F and the coupling g , and of introducing the renormalization scale μ . The simplest is perhaps to specify that at a momentum scale $p^2 = -\mu^2$, the propagators of the gluon and quark fields have the same form as in canonical free field theory:

$$Z_B : \Gamma_{2,0}^R \big|_{p^2 = -\mu^2} = Z_B^2 \Gamma_{2,0}^U = (-g_{\mu\nu} p^2 + p_\mu p_\nu) \quad (1.14)$$

$$Z_F : \Gamma_{0,2}^R \big|_{p^2 = -\mu^2} = Z_F^2 \Gamma_{0,2}^U = \not{p} \quad (1.15)$$

Once the renormalization prescription (1.14), (1.15) has been chosen, we can then choose a suitable definition of the renormalized coupling constant such as

$$g, \quad \Gamma_{1,2}^R \Big|_{p_i^2 = -\mu^2} \equiv Z_B^2 Z_F \Gamma_{1,2}^U \equiv g \quad (1.16)$$

This description of the renormalization prescription probably seems rather arid, so let us briefly consider a specific case. Suppose we return to our earlier scalar field theory example and evaluate the first loop of Fig. 4. This has a logarithmic divergence so that

$$\Gamma_{2,0}^U(p) = p^2 \left[1 + \frac{\lambda^2}{16\pi^2} \ln(\lambda^2/p^2) + \dots \right] \quad (1.17)$$

If we now introduce a renormalization factor

$$Z_B \equiv 1 - \frac{1}{2} \frac{\lambda^2}{16\pi^2} \ln(\lambda^2/\mu^2) + \dots \quad (1.18)$$

then we see that the product

$$\Gamma_{2,0}^R = p^2 \left[1 - \frac{\lambda^2}{16\pi^2} \ln(-p^2/\mu^2) + \dots \right] \quad (1.19)$$

and we have achieved a momentum space renormalization of the type indicated in formulae (1.14), (1.15). And so it goes.

An alternative to the cut-off procedure we have discussed is the technique of dimensional regularization²⁰). This consists of analytically continuing the integral (1.9) to non-integer n :

$$\int \frac{d^4 k}{(2\pi)^4} \lambda^2 \rightarrow \int \frac{d^n k}{(2\pi)^n} \lambda_\mu^2 \quad (1.20)$$

where a parameter $\tilde{\mu}$ is introduced to keep track of dimensions:

$$\lambda_\mu^2 \equiv \lambda^2 (\tilde{\mu})^{4-n}$$

The integral (1.9) becomes

$$I_n = \lambda_\mu^2 \frac{\Gamma(2-\frac{n}{2})}{(4\pi)^{n/2}} (-p^2)^{\frac{n}{2}-2} \quad (1.21)$$

and if we now take the limit as $\epsilon \equiv (4-n) \rightarrow 0$ we get

$$I_{4-\epsilon} \approx \frac{\lambda^2}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln 4\pi - \gamma_E - \ln(-p^2/\mu^2) \right\} + O(\epsilon) \quad (1.22)$$

where γ_E is Euler's constant. The minimal subtraction prescription for renormalization is now to remove from all the integrals of the theory the poles in $(1/\epsilon)$, an example of which is illustrated in (1.22). When this is done we see that the resulting renormalized inverse propagator is very similar to (1.19), except that there is an extra $(g^2/16\pi^2)(\ln 4\pi - \gamma_E)$ term, which means that the loop correction vanishes at

$$\beta^2 = -\tilde{\mu}^2 \exp(\gamma_E/4\pi) \quad (1.23)$$

This example illustrates how different renormalization prescriptions are essentially equivalent, but there are finite differences between them which must be considered carefully in higher order calculations (see Lecture 3).

1.3 The renormalization group equations

We have seen how the theory may be renormalized, introducing a momentum scale parameter μ . It is always the same theory, different choices of μ reparametrize it and many choices of μ are possible. The different parametrizations are related by the renormalization group equations^{21,22}). Consider the total change of a Green function $\Gamma_{n_B, n_F}^R(p_i, g, \mu)$ when μ is changed:

$$\begin{aligned} & \left(\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial g}{\partial \mu} \frac{\partial}{\partial g} \right) \Gamma_{n_B, n_F}^R(p, g, \mu) \\ &= \mu \lim_{\Lambda \rightarrow \infty} \left[n_B \frac{1}{Z_B} \frac{\partial Z_B}{\partial \mu} + n_F \frac{1}{Z_F} \frac{\partial Z_F}{\partial \mu} \right] Z_B^{n_B} Z_F^{n_F} \Gamma_{n_B, n_F}^U(p_i, g_0, \Lambda) \end{aligned} \quad (1.24)$$

Recall that Γ_{n_B, n_F}^U knows nothing about μ , so that only the Z_B and Z_F factors in $\Gamma_{n_B, n_F}^R \equiv Z_B^{n_B} Z_F^{n_F} \Gamma_{n_B, n_F}^U$ have non-zero μ derivatives. If we introduce

$$\gamma_B \equiv - \lim_{\Lambda \rightarrow \infty} \mu \frac{1}{Z_B} \frac{\partial Z_B}{\partial \mu} \quad (1.25)$$

$$\gamma_F \equiv - \lim_{\Lambda \rightarrow \infty} \mu \frac{1}{Z_F} \frac{\partial Z_F}{\partial \mu} \quad (1.26)$$

and

$$\beta \equiv \mu \frac{\partial g}{\partial \mu} \quad (1.27)$$

then Eq. (1.24) becomes

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + n_B \gamma_B + n_F \gamma_F \right] \Gamma_{n_B, n_F}^R(p_i, g, \mu) = 0 \quad (1.28)$$

the renormalization group equation.

This expresses the effect of a change in renormalization scale. We can parlay it into a statement about the effect of a change in the scale of the external momenta

p_i by using naive dimensional analysis. If the naive mass dimension of a Green function is D (e.g. for the inverse propagator $\Gamma_{2,0}^R \sim p^2$ we have $D = 2$) then

$$\Gamma_{n_B, n_F}^R(\lambda p_i, g, \mu) = \mu^D f(\lambda^2 p_i^2 / \mu^2) \quad (1.29)$$

Let us introduce the variable

$$t \equiv \ln \lambda \quad (1.30)$$

in terms of which we can write a differential equation expressing the scaling property (1.29):

$$\left(\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial \mu} - D \right) \Gamma_{n_B, n_F}^R(\lambda p_i, g, \mu) = 0 \quad (1.31)$$

If we combine the naive scaling equation (1.31) with the renormalization group equation (1.28) and eliminate the $\mu(\partial/\partial\mu)$ terms, we obtain

$$\left(-\frac{\partial}{\partial t} + \beta \frac{\partial}{\partial g} + D + n_B \gamma_B + n_F \gamma_F \right) \Gamma_{n_B, n_F}^R(\lambda p_i, g, \mu) = 0 \quad (1.32)$$

which expresses the way the renormalized Green functions vary with the external momentum scale.

We can solve²²⁾ Eq. (1.32) by first introducing an effective coupling constant $\bar{g}(g, t)$ which solves the equation

$$\frac{\partial \bar{g}(g, t)}{\partial t} = \beta(\bar{g}) : \quad \bar{g}(g, 0) = g \quad (1.33)$$

We see in a region where $\beta > 0$ that $\bar{g}(g, t)$ increases, whereas $\bar{g}(g, t)$ decreases in a region where $\beta < 0$ (cf. Fig. 4). It is easy to verify that Eq. (1.32) is solved by the following function of $\bar{g}(g, t)$:

$$\Gamma_{n_B, n_F}^R(\lambda p_i, g, \mu) = \lambda^D \Gamma_{n_B, n_F}^R(p_i, \bar{g}, \mu) \times \exp \left\{ \int_0^t dt' [n_B \gamma_B(\bar{g}(t')) + n_F \gamma_F(\bar{g}(t'))] \right\} \quad (1.34)$$

To discuss the content of this equation we must first analyse some possible behaviours of $\bar{g}(t)$ given by (1.33). Suppose that

$$\bar{g}(t) \xrightarrow[t \rightarrow \infty]{} g^* \text{ (fixed)} \quad (1.35)$$

In this case of an asymptotic fixed point, also

$$\gamma_{B,F}(\bar{g}(t')) \xrightarrow{t' \rightarrow \infty} \gamma_{B,F}(g^*) \quad (1.36)$$

Let us suppose that the approach (1.36) is sufficiently rapid that

$$\int_0^\infty dt' [\gamma_{B,F}(\bar{g}(t')) - \gamma_{B,F}(g^*)] \quad (1.37)$$

is finite. In this case

$$\exp \int_0^t dt' \gamma_{B,F}(\bar{g}(t')) \underset{t \rightarrow \infty}{\propto} \exp \int_0^t dt' \gamma_{B,F}(g^*) = \lambda^{\gamma_{B,F}(g^*)} \quad (1.38)$$

Therefore asymptotically

$$\Gamma_{n_B, n_F}^R(\lambda p_i, g, \mu) \underset{t \rightarrow \infty}{\propto} \Gamma_{n_B, n_F}^R(p_i, g^*, \mu) \lambda^{D + n_B \gamma_B(g^*) + n_F \gamma_F(g^*)} \quad (1.39)$$

We see from Eq. (1.39) that if $\gamma_{B,F}(g^*) \neq 0$, as occurs if $g^* \neq 0$, then the Green functions violate canonical scaling (λ^D) by powers of λ as $\lambda \rightarrow \infty$. If $g^* = 0$, so that $\gamma(g^*) = 0$, then we might expect canonical scaling to occur. However, this depends on the convergence of the integrals (1.37), and we will see in a moment that in QCD, which does indeed have $g^* = 0$ (asymptotic freedom), the integrals (1.37) in fact diverge, so that canonical scaling is violated logarithmically.

Before going on to discuss QCD it is useful to mention another renormalization property¹⁹⁾, and the related renormalization group equation, which is important in the discussion of deep inelastic scattering. There and elsewhere we will need to introduce composite operators which are products of quark fields, e.g.

$$O(x) \equiv \bar{q}_i(x) q_i(x) \quad \text{or} \quad J_\mu^a(x) \equiv \frac{1}{2} \bar{q}_i(x) \lambda_{ij}^a q_j(x) \quad (1.40)$$

We will be interested in Green functions involving these composite operators, such as $\Gamma_{n_B, n_F, 0}$ illustrated in Fig. 5. A heavy-duty theorem analogous to the previous one assures us that we may renormalize all the unrenormalized perturbative Green functions $\Gamma_{n_B, n_F, 0}^U$ by introducing an extra factor $Z_O(\Lambda/\mu)$:

$$\Gamma_{n_B, n_F, 0}^R(p_i, g, \mu) \equiv \lim_{\Lambda \rightarrow \infty} Z_B^{n_B} Z_F^{n_F} Z_O \Gamma_{n_B, n_F, 0}^U(p_i, g_0, \Lambda) \quad (1.41)$$

A convenient renormalization prescription, analogous to (1.14) to (1.16), is to specify that the operators take their free field theory values between external states with momenta $p^2 = -\mu^2$. The resulting renormalized Green function

$\Gamma_{n_B, n_F, 0}^R(p_i, g, \mu)$ obeys a renormalization group equation analogous to (1.28):

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + n_B \gamma_B + n_F \gamma_F + \gamma_0 \right] \Gamma_{n_B, n_F, 0}^R(p_i, g, \mu) = 0 \quad (1.42)$$

where γ_0 is defined analogously to γ_B, γ_F [(1.25), (1.26)]:

$$\gamma_0 \equiv \lim_{\Lambda \rightarrow \infty} \frac{1}{Z_0} \mu \frac{\partial Z_0}{\partial \mu} \quad (1.43)$$

which may be solved by using the analogue of (1.31) to get the analogues of (1.32) and (1.34).

Sometimes there are several operators O_i which have the same naive engineering mass dimension, e.g.

$$\left[\bar{q}_i(x) \gamma_\mu \overleftrightarrow{\partial}_\nu q_i(x) + (\mu \leftrightarrow \nu) \right] ; F_{\mu\nu}^a(x) F_{\sigma\tau}^a(x) \quad (1.44)$$

and these operators will in general mix when they are renormalized. This means that (1.41) is replaced by a matrix equation:

$$\Gamma_{n_B, n_F, O_i}^R = Z_B^{n_B} Z_F^{n_F} Z_{O_{ij}} \Gamma_{n_B, n_F, O_j}^U \quad (1.45)$$

in which case the previous analysis goes through with some notational complications due to indices

$$\gamma_{O_{ij}} \equiv \lim_{\Lambda \rightarrow \infty} \mu \left(\frac{\partial Z_0}{\partial \mu} \right)_{ik} (Z_0^{-1})_{kj} \quad (1.46)$$

We will later discuss some phenomenological aspects of operator mixing.

1.4 Asymptotic freedom

Now we will see what all the fuss is about: we will compute γ_B, γ_F , and β in QCD and see that the theory is asymptotically free¹²⁾. Working in the Landau gauge $\alpha = 0$, we find by calculating the diagrams of Fig. 6 that

$$\Gamma_{2,0}^R(p, g, \mu) = (-g_{\mu\nu} p^2 + p_\mu p_\nu) \left[1 + \left(\frac{13}{6} C_A - \frac{4}{3} T_R \right) \frac{g^2}{16\pi^2} \ln\left(\frac{p^2}{\mu^2}\right) + \dots \right] \quad (1.47)$$

where the group-theoretical factors C_A and T_R are defined by

$$C_A \delta_{ad} = f_{abc} f_{bcd} : C_A = 3 \text{ for QCD} \quad (1.48)$$

and by

$$T_R \delta_{ab} = \text{Tr}(T^a T^b) : T_R = \frac{f}{2} \text{ for QCD} \quad (1.49)$$

where f is the number of quark flavours. If we put (1.47) into the appropriate renormalization group equation (1.28), we see immediately that

$$\gamma_B = \frac{g^2}{16\pi^2} \left(\frac{13}{6} C_A - \frac{4}{3} T_R \right) \quad (1.50)$$

Turning to the renormalization of the quark propagator in the Landau gauge we find that in second order

$$\Gamma_{0,2}^R(p, g, \mu) = \cancel{\gamma} \quad (1.51)$$

and so clearly

$$\gamma_F = 0 \quad (1.52)$$

Computing the $q\bar{q}$ gluon vertex, using the diagrams of Fig. 7, we find

$$\Gamma_{1,2}^R(p, g, \mu) = -ig T^a \gamma_\mu \left[1 - \frac{3}{4} C_A \frac{g^2}{16\pi^2} \ln(-p^2/\mu^2) + \dots \right] \quad (1.53)$$

Substituting this into the appropriate renormalization group equation (1.28) and using the previously obtained results (1.50) and (1.52) for γ_B and γ_F , we obtain

$$\beta = \frac{\beta_0 g^3}{16\pi^2} + \dots = -\frac{g^3}{16\pi^2} \left[\frac{11}{3} C_A - \frac{4}{3} T_R \right] + \dots \quad (1.54a)$$

$$= -\frac{g^3}{16\pi^2} \left[11 - \frac{2}{3} f \right] + \dots \quad (1.54b)$$

Equation (1.54) is our holy grail: we see that providing $f \leq 16$, β is negative for small couplings g . This means that $\bar{g}(g, t)$ decreases to zero as $t \rightarrow \infty$ as in Fig. 4. The solution of the effective coupling equation (1.33), using the value

$$\beta_0 = \left[\frac{11}{3} C_A - \frac{4}{3} T_R \right] \quad (1.55)$$

for the coefficient of the g^3 term in (1.54), is

$$\bar{g}^2(g, t) = \frac{g^2}{1 + \frac{\beta_0}{16\pi^2} g^2 t} + O(\bar{g}^4) \quad (1.56)$$

where we have kept the initial condition $\bar{g}(g, 0) = g$. The formula (1.56) often has its leading-order behaviour written in the form

$$\bar{\alpha}_s(Q^2 \equiv \lambda^2 \mu^2) \equiv \frac{\bar{g}^2(Q^2)}{4\pi} \approx \frac{12\pi}{(33-2f) \ln(Q^2/\Lambda^2)} + \dots \quad (1.57)$$

where this Λ (not to be confused with the previous cut-off) is the constant of integration of (1.33) found when only the g^3 term in β is retained. It reflects the fundamental momentum scale of the strong interactions.

So QCD is asymptotically free, with $\bar{g}^2(t) \rightarrow g^{*2} = 0$ as $t \rightarrow \infty$. We therefore expect something like canonical scaling behaviour (λ^D) as momenta $\lambda p \rightarrow \infty$. In this case the traditional parton model results^{9,10} for high momentum phenomena would be recovered. However, as we remarked earlier, $\bar{g}^2(t) \rightarrow 0$ very slowly $\sim 1/\ln t$, and this means that the integrals (1.37) in fact diverge. If we write (1.50) in the form

$$\gamma_B = \frac{C_B}{16\pi^2} g^2 + \dots \quad (1.58)$$

then we find that

$$\exp\left[\int_0^t dt' \gamma_B(\bar{g}(t'))\right] \approx \exp\left[\int_0^t \frac{dt'}{t'} \frac{C_B}{2\beta_0}\right] = \exp\left(\frac{C_B}{2\beta_0} \ln t\right) = t^{C_B/2\beta_0} \quad (1.59)$$

Hence canonical scaling is violated by logarithms at large momenta. In particular, the Landau gauge propagator

$$\Gamma_{2,0}^R(p, g, \Lambda) \approx (-g_{\mu\nu} p^2 + p_\mu p_\nu) \left[\ln(-p^2/\Lambda^2) \right]^{C_B/2\beta_0} \quad (1.60)$$

where $\tilde{\Lambda}$ is some scale parameter whose relation to the fundamental scale parameter Λ can only be determined by a calculation of non-leading logarithmic effects. Analogous logarithmic violations of scaling will hold in the deep inelastic structure functions^{2,3} and other physical quantities discussed in subsequent lectures.

1.5 Application to e^+e^- annihilation

Let us now discuss a first application^{2,4} of asymptotic freedom to e^+e^- annihilation, where we will reproduce the naive result (1.4) whose experimental verification has already been quoted as evidence for colour and asymptotic freedom. We consider the inverse photon propagator in Quantum Electrodynamics (QED) :

$$\Gamma^{\mu\nu}(p) = \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2} \right) \pi(p^2) \quad (1.61)$$

The vacuum polarization $\pi(p^2)$ has naive dimension 2, so that the relevant version of the scaling equation (1.32) is

$$\left[-\frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + \beta_e \frac{\partial}{\partial e} + 2 + 2\gamma_A \right] \pi(p^2) = 0 \quad (1.62)$$

where (just for fun) we have introduced a β function for the variation in electromagnetic coupling as well as strong coupling, and γ_A is the anomalous dimension of the photon field. The lowest order electromagnetic and (electromagnetic + strong) contributions to γ_A are indicated in Fig. 8. They yield

$$\gamma_A = A e^2 \left(3 \sum_q e_q^2 \right) \left[1 + B g^2 + \dots \right] + \dots \quad (1.63)$$

where the value of A will turn out to be irrelevant to us, and

$$B = \frac{3C_F}{16\pi^2} : C_F \delta_{ij} \equiv (T^a T^a)_{ij} \Rightarrow C_F = \frac{4}{3} \quad (1.64)$$

Solving the renormalization group equation (1.62) yields

$$\pi(p^2) = p^2 \exp \left[2 \int_0^t \gamma_A(\bar{g}(t'), \bar{e}(t')) dt' \right] \quad (1.65)$$

In the range of p^2 of present interest the variation in $\bar{e}(t)$ is negligible and α is small, so we may expand the exponential in (1.65) and retain only the first two powers of α :

$$\pi(p^2) \approx p^2 \left[1 + A e^2 \left(3 \sum_q e_q^2 \right) \left[t + \frac{8\pi^2 B}{\beta_0} \ln t + \dots \right] + \dots \right] \quad (1.66)$$

We are now ready to go to the $e^+e^- \rightarrow$ hadrons cross-section by taking the absorptive part of (1.66). To avoid the necessity of knowing A, we compute R, the ratio to $\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ in which the same A factor appears:

$$R \equiv \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = \left(3 \sum_q e_q^2 \right) \left[1 + \frac{2s(Q^2)}{\pi} + \dots \right] \quad (1.67)$$

We therefore recover the desired prediction (1.4) and also acquire an estimate of the subasymptotic correction to this high Q^2 prediction. At least in the region above charm threshold, this correction is probably less than the experimental uncertainty in the value of R ²⁵⁾, and the experimental consistency with (1.4) can indeed be taken as evidence in favour of QCD.

2. DEEP INELASTIC SCATTERING

2.1 Basic formalism

We study deep inelastic lepton-hadron cross-sections which have the generic form (see Fig. 9):

$$\sigma = L^{\mu\nu} W_{\mu\nu} \quad (2.1)$$

where $L^{\mu\nu}$ is a purely leptonic tensor, and $W_{\mu\nu}$ is the hadronic tensor whose dynamics interest us^{22,26}). It can be written in the form

$$W_{\mu\nu}(p, q) = \frac{1}{4\pi} \int d^4x e^{iq \cdot x} \langle p | [J_\mu^\dagger(x), J_\nu(0)] | p \rangle \quad (2.2)$$

It may be regarded as the absorptive part of a forward current-hadron scattering amplitude (see Fig. 10):

$$W_{\mu\nu} = \frac{1}{2\pi} \text{Im } T_{\mu\nu} : T_{\mu\nu} \equiv i \int d^4x e^{iq \cdot x} \langle p | T(J_\mu^\dagger(x) J_\nu(0)) | p \rangle \quad (2.3)$$

The weak and electromagnetic currents which we have available are either conserved or almost conserved: using current conservation we may write

$$\begin{aligned} T_{\mu\nu} = & - \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) T_1(q^2, \nu) + \frac{1}{m_H^2} \left(p_\mu - \nu \frac{q_\mu}{q^2} \right) \left(p_\nu - \nu \frac{q_\nu}{q^2} \right) T_2(q^2, \nu) \\ & - \frac{i}{2m_H^2} \epsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta T_3(q^2, \nu) \quad : \quad \nu \equiv p \cdot q \end{aligned} \quad (2.4)$$

The third invariant $T_3(q^2, \nu)$ would vanish if parity were conserved, and so may be non-zero only in charged and neutral weak interaction processes. Corresponding to the invariants $T_i(q^2, \nu)$ are their absorptive parts, the structure functions

$$W_i(q^2, \nu) = \frac{1}{2\pi} \text{Im } T_i(q^2, \nu) \quad (2.5)$$

which rejoice in the following simple crossing properties:

$$\left. \begin{aligned} W_{1,2}^{eH, \mu H}(q^2, \nu) &= - W_{1,2}^{eH, \mu H}(q^2, -\nu) \\ W_{1,2,3}^{\nu H}(q^2, \nu) &= - W_{1,2,3}^{\bar{\nu} H}(q^2, -\nu) \end{aligned} \right\} \quad (2.6)$$

The forward scattering amplitude (2.3) has all the usual analyticity properties (cuts in ν from $-q^2/2$ to $+\infty$, $-\infty$ to $-q^2/2$) which enable one to derive simple dispersion relations such as

$$\left. \begin{aligned} T_2^{eH, \mu H}(q^2, \nu) &= 4 \int_{q^2/2}^{\infty} d\nu' \frac{\nu' W_2^{eH, \mu H}(q^2, \nu')}{(\nu'^2 - \nu^2)} \\ T_3^{\nu H + \bar{\nu} H}(q^2, \nu) &= 4 \int_{q^2/2}^{\infty} d\nu' \frac{\nu' W_3^{\nu H + \bar{\nu} H}(q^2, \nu')}{(\nu'^2 - \nu^2)} \end{aligned} \right\} \quad (2.7)$$

which will be used later on.

We will study the Bjorken⁹⁾ limit

$$Q^2 \equiv -q^2 \rightarrow \infty, \quad \nu \rightarrow \infty : \quad x \equiv \frac{Q^2}{2\nu} \text{ fixed} \quad (2.8)$$

of the structure functions $W_i(q^2, \nu)$ (2.5). Looking back at the expression (2.2) we see that the phase-space factor $e^{iq \cdot x}$ promises to kill by rapid oscillation regions of the x-space integration where $q \cdot x$ is large. We can decompose the factor in the form

$$q \cdot x = \left(\frac{q_0 + q_3}{\sqrt{2}} \right) \left(\frac{x_0 - x_3}{\sqrt{2}} \right) + \left(\frac{q_0 - q_3}{\sqrt{2}} \right) \left(\frac{x_0 + x_3}{\sqrt{2}} \right) - \underline{q}_T \cdot \underline{x}_T \quad (2.9)$$

In the target rest frame

$$p = (m_H, 0, 0, 0), \quad q = \frac{1}{m_H} (\nu, 0, 0, \sqrt{\nu^2 + m_H^2} Q^2) \quad (2.10)$$

so that if we consider the Bjorken limit (2.8)

$$q_0 + q_3 \approx \frac{2\nu}{m_H} \rightarrow \infty, \quad q_0 - q_3 = \frac{\nu}{m_H} - \frac{1}{m_H} \sqrt{\nu^2 + m_H^2} Q^2 \approx m_H x \quad (2.11)$$

One would therefore expect from (2.9), on the basis of phase oscillations, that the relevant regions of the x-space integration would be

$$x_0 - x_3 \approx \frac{m_H}{2\nu}, \quad x_0 + x_3 \approx \frac{1}{m_H x} \quad (2.12)$$

This argument is sloppy and heuristic, but can be made more rigorous^{22, 27, 28)}.

Noting that

$$x^2 = (x_0 + x_3)(x_0 - x_3) - \underline{x}_T^2$$

and that $x^2 > 0$ is enforced by the causality of the commutator in (2.2), we see that

$$x^2 \lesssim \left[\frac{m_H}{2\nu} \right] \left[\frac{1}{m_H x} \right] = O\left(\frac{1}{Q^2}\right) \quad (2.13)$$

close to the light-cone $x^2 = 0$. We therefore conclude that the Bjorken limit of (2.2) is dominated by the light-cone behaviour of the underlying field theory, and to this we now turn our attention.

2.2 Light-cone behaviour^{14,22,27,28)}

Green functions in field theory are all generally singular on the light-cone. The simplest example is the free scalar field propagator, which has the form

$$\Delta_F(x, m^2) \underset{x^2 \rightarrow 0}{\approx} \frac{i}{4\pi^2} \frac{1}{-x^2 + i\epsilon} + \dots \quad (2.14)$$

close to the light-cone, independent of the mass m . We need to find an organizing principle for these light-cone singularities, and this is provided by operator product expansions. Consider the product $T[J(x/2) J(-x/2)]$ of two local normal ordered composite operators $J(x) \equiv :\phi^2(x):$ made up out of scalar fields. The operator $J(x)$ is represented diagrammatically in Fig. 11a: how would a pair of them be connected up in a general Green function? If we neglect interactions, three possibilities occur as shown in Fig. 11b. These may be written as

$$i T\left(J\left(\frac{x}{2}\right) J\left(-\frac{x}{2}\right)\right) = -2 \left[\Delta_F(x, m^2) \right]^2 + 4i \Delta_F(x, m^2) : \phi\left(\frac{x}{2}\right) \phi\left(-\frac{x}{2}\right) : + : \phi^2\left(\frac{x}{2}\right) \phi^2\left(-\frac{x}{2}\right) : \quad (2.15)$$

The dots around the q-number bilocal operators $:\phi(x/2)\phi(-x/2):$, etc., indicate that all light-cone singularities have been removed: they are explicit as the c-number light-cone singular propagator factors $\Delta_F(x, m^2)$. The generic light-cone expansion structure²⁹⁾

$$i T(A(x) B(y)) \underset{(x-y)^2 \rightarrow 0}{\approx} \sum_N C_N(x-y) O_N(x, y) \quad (2.16)$$

with singular c-number functions and regular bilocal operators $O_N(x, y)$ is a general feature of free field theory. Since $:\phi(x/2)\phi(-x/2):$ in Eq. (2.15) is regular as $x \rightarrow 0$, we may expand it in a Taylor series:

$$: \phi\left(\frac{x}{2}\right) \phi\left(-\frac{x}{2}\right) : = \sum_N \frac{1}{2^N} x_{\mu_1} \dots x_{\mu_N} \phi(0) \overset{\leftrightarrow}{\partial}_{\mu_1} \dots \overset{\leftrightarrow}{\partial}_{\mu_N} \phi(0) \quad (2.17)$$

Substituting this expansion into (2.15) gives us the generic structure of a short-distance expansion in terms of local operators $O_{\mu_1 \dots \mu_N}^{(0)}$:

$$i T(A(x) B(0)) \underset{x \rightarrow 0}{\approx} \sum_{i, N} C_{i, N}(x) x^{\mu_1} x^{\mu_2} \dots x^{\mu_N} O_{\mu_1 \dots \mu_N}^{(0)}(0) \quad (2.18)$$

The form of (2.18) -- sum of terms with c-number singular functions multiplying local operators¹⁴⁾ -- characterizes the short-distance behaviour of any free or interacting field theory¹⁹⁾.

What differentiates between field theories is the set of local operators $O_{\mu_1 \dots \mu_N}^i(0)$ and the degrees of singularity of the $C_{i,N}(x)$. In free field theory the degrees of the singularities come directly from naive dimensional power counting: there are no explicit mass parameters in the leading singularities like that in the propagator (2.14). We therefore have

$$C_{i,N}(x) \approx \left(\frac{1}{x}\right)^{d_{C_{i,N}}} \quad (2.19)$$

where

$$d_A + d_B = d_{C_{i,N}} - N + d_{O_N^i} \quad (2.20)$$

where N is the spin of the operator $O_N^i(0)$. Therefore we have the degree of singularity

$$d_{C_{i,N}} = (d_A + d_B) - (d_{O_N^i} - N) \quad (2.21)$$

Meditation on (2.21) immediately reveals that the most singular terms in the short-distance expansion (2.18) will be those with the minimum value of

$$d_{O_N^i} - N \equiv \text{Twist} \quad (2.22)$$

The minimum value of twist appearing in field theory is generally two:

$$\left. \begin{aligned} O_{\mu_1 \dots \mu_N} &\equiv \frac{1}{N} \left[\bar{q}(0) \overleftrightarrow{\partial}_{\mu_1} \overleftrightarrow{\partial}_{\mu_2} \dots \overleftrightarrow{\partial}_{\mu_N} q(0) + \text{perms} \right] \\ \text{and } O_{\mu_1 \dots \mu_N} &\equiv \phi(0) \overleftrightarrow{\partial}_{\mu_1} \dots \overleftrightarrow{\partial}_{\mu_N} \phi(0) \end{aligned} \right\} \quad (2.23)$$

both have twist two.

However, in field theory the degrees of singularity of the $C_{i,N}(x)$ generally get modified from their free field values (2.21). An example is afforded by the QCD gluon propagator (1.60) that we have already met:

$$\Gamma_{2,0}^R(p) \underset{p^2 \rightarrow \infty}{\approx} (-g_{\mu\nu} p^2 + p_\mu p_\nu) \left[\ln(-p^2/\Lambda^2) \right]^{C_B/2\beta_0} \quad (2.24)$$

If we Fourier transform this we clearly get a logarithmic violation of naive power-law behaviour (2.19) in x , and this may be expected to happen to many other short-distance singularities. In a theory other than QCD, in which there is a non-zero fixed-point coupling g^* and scaling in momentum space is violated by a power (1.39), we might expect and do indeed find power-law violations of the canonical short-distance power behaviour (2.21).

Just how are the short-distance singularities and operator product expansion (2.18) related to the deep inelastic structure functions (2.5)? This we study in the next section.

2.3 Moment sum rules^{30,31)}

In order to simplify the discussion by avoiding the (inessential?) complications due to spin, we will start by returning to our spinless currents $J(x) = :\phi^2(x):$ made out of spinless fields. We write the operator product expansion in the form

$$i T(J(x)J(0)) = \sum_{i=0}^{\infty} \sum_{N=0}^{\infty} [x^{\mu_1} \dots x^{\mu_N} C_{i,N}(x)] O_{\mu_1 \dots \mu_N}^i(0) \quad (2.25)$$

with i = twist-2. Taking the matrix elements of (2.25) in a target of momentum p and making the Fourier transform (2.3) we get

$$T(x, q^2) \equiv i \int d^4x e^{iq \cdot x} \langle p | T(J(x)J(0)) | p \rangle = \sum_i \sum_N \langle p | O_{\mu_1 \dots \mu_N}^i(0) | p \rangle \times q^{\mu_1} \dots q^{\mu_N} \tilde{C}_{i,N}(q^2) \quad (2.26)$$

where we have introduced $\tilde{C}_{i,N}(q^2)$, the Fourier transform of $C_{i,N}(x)$ defined by

$$\int d^4x e^{iq \cdot x} x^{\mu_1} \dots x^{\mu_N} C_{i,N}(x) \equiv q^{\mu_1} \dots q^{\mu_N} \tilde{C}_{i,N}(q^2) \quad (2.27)$$

If naive dimensional counting were correct so that

$$C_{i,N} \approx \left(\frac{1}{x^2}\right)^{1-i} \quad (2.28)$$

then the asymptotic behaviour of $\tilde{C}_{i,N}(q^2)$ would be $O(1/q^2)^{1+N+i}$. In practice, $\tilde{C}_{i,N}(q^2)$ will generally deviate from this canonical behaviour, and we introduce a reduced singular function $\bar{C}_{i,N}(q^2)$ to express this deviation:

$$\tilde{C}_{i,N}(q^2) \equiv \left(\frac{1}{q^2}\right)^{1+N+i} \bar{C}_{i,N}(q^2) \quad (2.29)$$

Let us turn now to the matrix elements of the local operators $O_{\mu_1 \dots \mu_N}^i(0)$. We suppose that these are expressed in a basis of symmetric, traceless operators of definite spin N . Then the matrix elements take the form

$$\langle p | O_{\mu_1 \dots \mu_N}^i(0) | p \rangle = \bar{O}_N^i \left[p_{\mu_1} \dots p_{\mu_N} - \sum_{i,j} g_{\mu_i \mu_j} p_{\mu_1} \dots \hat{p}_{\mu_i} \dots \hat{p}_{\mu_j} \dots p_{\mu_N} + \dots \right] \quad (2.30)$$

For the moment we will forget all except the first term in the right-hand side of Eq. (2.30): we will return to them later. Substituting (2.29) and (2.30) into the current correlation function of interest (2.26) we obtain

$$T(x, q^2) = \frac{1}{q^2} \sum_{i,N} \bar{C}_{i,N}(q^2) \bar{O}_N^i \left[\frac{(p \cdot q)^N + \dots}{(q^2)^{N+i}} \right] \quad (2.31)$$

[The dots come from the forgotten terms in (2.30).] In the Bjorken scaling limit $x = -q^2/2p \cdot q$ fixed; $q^2, p \cdot q \rightarrow \infty$ we can neglect all the higher-twist terms in (2.31) and find

$$T(x, q^2) \underset{-q^2 \rightarrow \infty}{\approx} \frac{1}{q^2} \sum_N \bar{C}_{0,N}(q^2) \bar{O}_N^0 \left(-\frac{1}{2}\right)^N x^{-N} \quad (2.32)$$

Now we seek to pick out the individual x^{-N} terms in the expansion (2.32). We remember from Section 2.1 that $T(v, q^2)$ is analytic in the complex v plane apart from cuts in $-q^2/2 \leq |v| < \infty$. Transforming to the complex x plane (keeping q^2 fixed) we find analyticity apart from a cut in the region $|x| \leq 1$, as indicated in Fig. 12. We can use this analyticity to write down a dispersion relation in x , integrating along the contour C indicated in Fig. 12. We obtain

$$\frac{1}{2\pi i} \oint_C dx x^{N-1} T(x, q^2) \underset{-q^2 \rightarrow \infty}{\approx} \frac{1}{q^2} \left(-\frac{1}{2}\right)^N \bar{C}_{0,N}(q^2) \bar{O}_N^0 \quad (2.33)$$

Now we remember that the absorptive part of $T(x, q^2)$ is just the structure function $W(x, q^2)$ analogous to the spinning case in (2.2) and (2.3). We therefore have

$$\int_0^1 dx x^{N-1} W(x, q^2) \underset{-q^2 \rightarrow \infty}{\approx} \left(-\frac{1}{q^2}\right) \left(-\frac{1}{2}\right)^{N-2} \bar{O}_N^0 \bar{C}_{0,N}(q^2) \quad (2.34)$$

This fundamental equation tells us that the asymptotic behaviour of the x moments of the deep inelastic structure functions are controlled by the asymptotic q^2 dependence of the functions $\bar{C}_{0,N}(q^2)$, which are just the Fourier transforms (2.27) of the c -number singular functions $C_{0,N}(x)$ appearing in the short-distance operator product expansion (2.25). Before proceeding to a computation of the singular behaviour in QCD, two remarks should be made.

A first concerns the (inessential?) complications due to spin arising if we have vector currents and fundamental quark fields. Let us consider a flavour non-singlet term in the product of two vector currents $i T[J_\mu(x) J_\nu(0)]_a$, which may be decomposed into two towers of Lorentz covariant tensors multiplying a unique set of twist-two local operators of spin N :

$$i T(\mathcal{J}_\mu(x) \mathcal{J}_\nu(0))_a \approx \sum_N \left[-g_{\mu\nu} i^N \partial_{\mu_1} \dots \partial_{\mu_N} \mathcal{F}_{1a}^N(x^2) + g_{\mu_1}^\mu g_{\mu_2}^\nu i^{N-2} \partial_{\mu_3} \dots \partial_{\mu_N} \mathcal{F}_{2a}^N(x^2) \right] O_a^{\mu_1 \dots \mu_N}(0) \quad (2.35)$$

where the $\mathcal{F}_{\frac{1}{2}a}^N(x^2)$ are c-number singular functions and

$$O_a^{\mu_1 \dots \mu_N}(0) \equiv \frac{i^{N-1}}{N} \left[\bar{q}(0) \frac{\lambda_a}{2} \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_N} q(0) + \text{perms} \right] \quad (2.36)$$

In the expression (2.36) λ_a is just an SU(f) flavour matrix, and the D^{μ_i} are the colour gauge-covariant derivatives introduced in Eq. (1.7). If we looked at flavour singlet combinations in $i T[\mathcal{J}_\mu(x) \mathcal{J}_\nu(0)]$ we would also need the gluon operators

$$F_{\mu_1 \alpha}(x) \overset{\leftrightarrow}{D}_{\mu_2} \dots \overset{\leftrightarrow}{D}_{\mu_{N-1}} F_{\alpha \mu_N}(x) \quad (2.37)$$

but these do not enter into the flavour non-singlet combination (2.35). If we introduce the Fourier transforms $\tilde{\mathcal{F}}_{\frac{1}{2}a}^N(q^2)$ analogously to the $\tilde{C}_{0,N}(q^2)$ (2.27), and then the dimensionally reduced functions

$$\left. \begin{aligned} \bar{\mathcal{F}}_{1a}^N(q^2) &\equiv (q^2)^N \tilde{\mathcal{F}}_{1a}^N(q^2) \\ \bar{\mathcal{F}}_{2a}^N(q^2) &\equiv (q^2)^{N-1} \tilde{\mathcal{F}}_{2a}^N(q^2) \end{aligned} \right\} \quad (2.38)$$

then we can go through a repeat of the analytic gymnastics (2.30) to (2.34) to finally obtain moment sum rules

$$\left. \begin{aligned} M_N^{(1)}(-q^2) &\equiv \int_0^1 dx x^{N-1} W_1^a(x, q^2) = \frac{1}{2^{N+2}} \bar{\mathcal{F}}_{1a}^N(-\frac{q^2}{\mu^2}, g^2(\mu)) \bar{O}_a^N \\ M_N^{(2)}(-q^2) &\equiv \int_0^1 dx x^{N-2} W_2^a(x, q^2) = \frac{m_H}{2^{N+1}} \bar{\mathcal{F}}_{2a}^N(-\frac{q^2}{\mu^2}, g^2(\mu)) \bar{O}_a^N \end{aligned} \right\} \quad (2.39)$$

A similar sum rule for $vW_3^a(x, q^2)$ can be obtained by considering the expansion of a vector and an axial current:

$$M_N^{(3)}(-q^2) \equiv \int_0^1 dx x^{N-1} W_3^a(x, q^2) = \frac{m_H}{2^{N+2}} \bar{\mathcal{F}}_{3a}^N(-\frac{q^2}{\mu^2}, g^2(\mu)) \bar{O}_a^N \quad (2.40)$$

We have explicitly indicated the renormalization scale and coupling constant which appear in the $\tilde{\mathcal{F}}_{ia}^N(q^2)$ in interacting field theory. Formulae (2.39) and (2.40) will be the moment sum rules actually used in QCD analyses of deep inelastic structure functions.

A second remark about the moments (2.34) or (2.40) concerns the subasymptotic corrections we have thrown away. Some of these were the contributions of higher-twist operators, which we saw explicitly in Eq. (2.31) were suppressed asymptotically by powers of q^2 . We have very few handles on them and little idea how to estimate, let alone calculate, these contributions. Another set of subasymptotic corrections to our formulae comes from the dots in Eq. (2.31) which are connected with the secondary terms on the right-hand side of (2.30). Contracting these with the $q^{\mu_1} \dots q^{\mu_N}$ in Eq. (2.26) we see explicitly that they give corrections to the moment formulae which are $O(m_H^2/q^2)^P$. Their significance is that whereas the simple moments (2.34), (2.39) and (2.40) project the structure functions on to definite spin at infinite q^2 , at finite q^2 they mix up all spins $\leq N$. The effect of these discarded terms from (2.30) is to continue the exact spin projection down to finite q^2 . If one takes careful account of these terms one can construct the appropriate finite q^2 moments. A simpler procedure is to use $O(4)$ analysis to do the correct partial-wave projection³²⁾. An example of the sort of definite spin moment one gets at finite q^2 is³³⁾

$$M_N^{(3)}(q^2) = \int_0^1 dx x^{N-1} W_3^a(x, q^2) \rightarrow \int_0^1 \frac{dx}{x^2} \tilde{z}^{N+1} W_3^a(x, q^2) \left[\frac{1 + (N+1) \sqrt{1 + 4m_H^2 x^2 / q^2}}{(N+2)} \right] \quad (2.41a)$$

where

$$\tilde{z} \equiv \frac{2x}{1 + \sqrt{1 + 4m_H^2 x^2 / q^2}} \quad (2.41b)$$

Because of their spin simplicity, these Nachtmann moments are theoretically attractive. There is also a prejudice that the $O(m_H^2/q^2)$ corrections to asymptotic predictions which they enshrine may be more important than the higher-twist corrections, at least for moments with low values of N .

2.4 Q^2 dependences of moments

We have seen that the computation of the $Q^2 \equiv -q^2$ dependences of the moments reduces to that of the functions $\bar{C}_{0,N}(q^2)$ [or $\bar{F}_{1a}^N(q^2)$] which are the Fourier transforms of c-number singular functions appearing in the operator product expansion (2.25) or (2.35). Because the operator product expansion is universal, we may compute the behaviour of the singular functions by sandwiching the operators between any convenient external states^{1,22)}. The most convenient choice is a quark target: $\langle 0 | T[\bar{\psi}(-p) J(x) J(0) \psi(p)] | 0 \rangle_a$. For the moment (ha! ha!) we neglect the spins of the currents. Like any other Green function, this one obeys a renormalization group equation:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + 2\gamma_F \right) \langle 0 | T(\bar{\psi}(-p) J(x) J(0) \psi(p)) | 0 \rangle_a = 0 \quad (2.42)$$

We have not put into (2.42) an anomalous dimension term for the currents $J(x)$: this is because the physical vector currents $J_\mu^a(x)$ have zero anomalous dimensions. This is because, being generators of exact (or softly broken) symmetries such as SU(3) flavour, the fundamental current algebra commutators

$$\delta(x_0) [J_0^a(0), J_\mu^b(x_0, x)] = i f^{abc} \delta^4(x) J_\mu^c(x_0, x) \quad (2.43)$$

fix their normalizations so that they cannot be renormalized. If we now substitute into (2.42) the operator product expansion (2.25) we get

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + 2\gamma_F \right) \sum_{i,N} x^{\mu_1} \dots x^{\mu_N} C_{i,N}(x) \langle 0 | T(\bar{\psi}(-p) O_{\mu_1 \dots \mu_N}^i(0) \psi(p)) | 0 \rangle = 0 \quad (2.44)$$

This expression can be decomposed according to the spin of the operators and their twists, obtaining

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + 2\gamma_F \right) C_{i,N}(x) \langle 0 | T(\bar{\psi}(-p) O_{\mu_1 \dots \mu_N}^i(0) \psi(p)) | 0 \rangle = 0 \quad (2.45)$$

This renormalization group equation can be compared with that which we can write down directly (1.42) for a Green function involving the operator $O_{\mu_1 \dots \mu_N}^i(0)$:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + 2\gamma_F + \gamma_{O_N^i} \right) \langle 0 | T(\bar{\psi}(-p) O_{\mu_1 \dots \mu_N}^i(0) \psi(p)) | 0 \rangle = 0 \quad (2.46)$$

If we compare the two equations (2.45) and (2.46), and take their difference, we get a non-trivial renormalization group equation for $C_{i,N}(x)$:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - \gamma_{O_N^i} \right) C_{i,N}(x) = 0 \quad (2.47)$$

The anomalous dimension $\gamma_{O_N^i}$ appearing in (2.47) tells us the degree of the non-canonical singularity in $C_{i,N}(x)$. If the operators appearing whose product we were studying had anomalous dimensions γ_A, γ_B , then we would have found the replacement

$$\gamma_{O_N^i} \rightarrow \gamma_{O_N^i} - \gamma_A - \gamma_B \quad (2.48)$$

in Eq. (2.47), but this does not concern us here because currents have zero anomalous dimensions.

All we need to do now is to compute the anomalous dimensions $\gamma_{O_N^i}$ by evaluating $\langle 0 | T[\bar{\psi}(-p) O_{\mu_1 \dots \mu_N}^i(0) \psi(p)] | 0 \rangle$ and using Eq. (2.46). By Lorentz invariance and symmetry the matrix elements of the interesting operators $O_a^{\mu_1 \dots \mu_N}(0)$ (2.36) take the form

$$\begin{aligned} & \langle 0 | T(\bar{\psi}(-p) O_a^{\mu_1 \dots \mu_N}(0) \psi(p)) | 0 \rangle \\ &= \sum_i \frac{1}{N} [p^{\mu_1} \dots p^{\mu_{i-1}} \delta^{\mu_i} p^{\mu_{i+1}} \dots p^{\mu_N} + \text{perms}] \tilde{O}_a^N(-p^2/\mu^2, g^2) \end{aligned} \quad (2.49)$$

We evaluate the matrix element by using a source

$$\Delta_{\mu_1} \dots \Delta_{\mu_N} O_a^{\mu_1 \dots \mu_N} : \Delta^2 = 0 \quad (2.50)$$

which yields the effective Feynman rules indicated in Fig. 13. The lowest order g^2 diagrams which must be calculated are shown in Fig. 14a. They yield a reduced operator matrix element

$$\tilde{O}_a^N(-p^2/\mu^2, g^2(\mu)) = 1 + \frac{g^2(\mu)}{32\pi^2} \gamma_0^N \ln(p^2/\mu^2) + \dots \quad (2.51)$$

where we have used the renormalization prescription that the operator have its free field matrix element between quark states of momenta $p^2 = -\mu^2$. The γ_0^N are found to be

$$\gamma_0^N = 2C_F \left[1 - \frac{2}{N(N+1)} + 4 \sum_{j=2}^N \frac{1}{j} \right] : C_F = \frac{4}{3} \quad (2.52)$$

and it is easy to deduce from the renormalization group equation (2.46) that

$$\gamma_{0_N} = \frac{\gamma_0^N}{16\pi^2} g^2 + \dots \quad (2.53)$$

The interesting functions $\bar{F}_{ia}^N[-p^2/\mu^2, g^2(\mu^2)]$ are essentially (2.38) the Fourier transforms of (2.47). Their renormalization group equations and their solutions are equally obvious:

$$\bar{F}_{ia}^N(-q^2/\mu^2, g^2(\mu)) = \bar{F}_{ia}^N(1, g^2(q^2)) \exp \left[- \int_{g^2(\mu^2)}^{g^2(q^2)} d\bar{g}' \frac{\gamma_{0_N}(\bar{g}')}{\beta(\bar{g}')} \right] \quad (2.54)$$

To see the leading behaviour of (2.54), let us specify μ^2 sufficiently large that we are always in a perturbative régime for $-q^2 > \mu^2$, so that

$$\bar{F}_{ia}^N(1, \bar{g}^2(\mu)) \rightarrow \bar{F}_{ia}^N(1, 0) \quad (2.55)$$

and we only need retain the $O(g^2)$ term γ_0^N [(2.52), (2.53)] in the anomalous dimension γ_{0_N} and the $O(g^3)$ term in the β function (1.54). We then find the following asymptotic behaviours for the moments of the deep inelastic structure functions [(2.39), (2.40)]:

$$M_N(q^2) = \bar{f}_{ia}^N(1,0) \left(\frac{\bar{g}^2}{g^2} \right)^{\gamma_0^N/2\beta_0} \bar{O}^N \quad (2.56)$$

So that using the asymptotic form (1.57) of the effective strong coupling we get

$$M_N(q^2) \approx M_N(\mu^2) \left(\frac{\ln Q^2/\Lambda^2}{\ln \mu^2/\Lambda^2} \right)^{-\frac{4}{33-2f} \left[1 - \frac{2}{N(N+1)} + 4 \sum_{j=2}^N \frac{1}{j} \right]} \quad (2.57)$$

The solution for moments of singlet combinations of structure functions can be written as

$$M_N(q^2) = A_N [g^2(q^2)]^{\gamma_+^N/2\beta_0} + B_N [g^2(q^2)]^{\gamma_-^N/2\beta_0} \quad (2.58)$$

with A^N and B^N unpredicted constants. The $\gamma_{\pm}^N$ are the eigenvalues of the lowest order singlet anomalous dimension matrix, which is²³⁾ calculated from the diagrams of Fig. 14b to be

$$\begin{pmatrix} \frac{8}{3} \left[1 - \frac{2}{N(N+1)} + 4 \sum_{j=2}^N \frac{1}{j} \right] & -4f \frac{N^2+N+2}{N(N+1)(N+2)} \\ -\frac{16}{3} \frac{N^2+N+2}{N(N^2-1)} & 6 \left[\frac{1}{3} - \frac{4}{N(N-1)} - \frac{4}{(N+1)(N+2)} + 4 \sum_{j=2}^N \frac{1}{j} \right] + \frac{4f}{3} \end{pmatrix} \quad (2.59)$$

This completes our discussion of the leading-order QCD results for the moments of deep inelastic scattering structure functions. These are the fundamental predictions for the logarithmic scaling violations expected for non-singlet combinations of deep inelastic structure functions²³⁾. Singlet combinations of structure functions have a more complicated asymptotic behaviour involving two "leading" powers of $\ln Q^2$, corresponding to the two different eigenvalues of the matrix of anomalous dimensions of the mixing operators (1.44) and their high-spin partners [cf. (2.36) and (2.37)].

2.5 Phenomenology of non-singlet moments

Before continuing with some more formalism, it seems appropriate to give a brief discussion of the phenomenology of non-singlet structure function moments. For the reasons discussed at the end of Section 2.3, we will use Nachtmann moments^{32,33)} of the structure functions, principally using $vW_3(x, q^2)$ and the specification (2.41) of $M_N^{(3)}(-q^2)$. Two things have been done³⁴⁾ to check the QCD-predicted form (2.57). If we define

$$d_N \equiv \frac{4}{33-2f} \left[1 - \frac{2}{N(N+1)} + 4 \sum_{j=2}^N \frac{1}{j} \right] \quad (2.60)$$

then the form (2.57) implies for the logarithms of the moments

$$\ln M_N^{(i)}(-q^2) \approx -d_N \ln \ln Q^2/\mu^2 + (\text{constant}) \quad (2.61)$$

If we plot two such logarithms against each other, we should get a straight line with slope d_N/d_N . Experimentalists³⁴⁾ have done this (see Fig. 15) and found slopes in agreement with the slopes d_N/d_N , expected from QCD (2.60) even for Q^2 down to 1 or 2 GeV^2 . This consistency with QCD includes other vector gluon theories which predict the same d_N/d_N in lowest order of the coupling. Figure 15 shows that scalar gluon theories seem to be disfavoured³⁵⁾. Another test of (2.57) is to raise the moments $M_N^{(i)}(-q^2)$ to the powers $-1/d_N$, in which case they should rise like $\ln Q^2$:

$$\left[M_N^{(i)}(-q^2) \right]^{-1/d_N} \approx (\text{constant}) \left[\ln Q^2/\mu^2 \right] \quad (2.62)$$

The experimental data³⁴⁾ shown in Fig. 16 again agree with the QCD expectation for $Q^2 \geq 1$ or 2 GeV^2 . As mentioned in Sections 1.3 and 2.2, a theory with a non-zero fixed point $g^* \neq 0$ would violate scaling by powers of Q^2 . Such a theory would also predict a power-law behaviour for the $\left[M_N^{(i)}(Q^2) \right]^{-1/d_N}$. Shown in Fig. 16 is an attempt³⁵⁾ to fit these quantities with a power-law violation of scaling between $Q^2 = 1$ and 10 GeV^2 . It does not seem to fit the data as well as the QCD logarithm (2.62).

These two tests are just examples of the way in which the QCD predictions (2.57) may be tested by deep inelastic scattering experiments. They indicate consistency of QCD with experiment, but do not conclusively demonstrate its validity. There has been an intense debate on the significance of these tests: Are the values of Q^2 too low? ³⁶⁾ Are higher-twist effects important? ³⁷⁾ Are the log-log plots "bound" to give QCD slopes? ³⁸⁾ Should one use rather singlet combinations of structure functions? ³⁹⁾ Should one use structure functions directly rather than their moments? ⁴⁰⁾ There is no consensus on these questions, and you are referred to the original literature ³⁴⁻³⁹⁾ and to recent personal reviews⁴⁰⁾ and idiosyncratic debates⁴¹⁾ on the problem.

2.6 Probabilistic interpretation of the leading-order predictions

An extremely interesting and physically appealing interpretation of the above leading-order predictions for deep inelastic structure functions was given by Altarelli and Parisi⁴²⁾. We start by considering non-singlet combinations of structure functions, and we will return later to the singlet case. We write Eq. (2.57) in terms of the variable $t \equiv \ln q^2/q_0^2$ as

$$M_N(t) = M_N(0) \left[\frac{\alpha_s(t)}{\alpha_s(0)} \right]^{\gamma_0^N / 2\beta_0} \quad (2.63)$$

which satisfies the following equation

$$\frac{d}{dt} M_N(t) = - \frac{\alpha(t)}{8\pi} \gamma_0^N M_N(t) \quad (2.64)$$

Since the moment of a convolution of two functions is the product of the moments of the functions, if we can find a function whose moments are γ_0^N , we can invert Eq. (2.64). Thus we define a function $P_{q \rightarrow q}(z)$, such that

$$\int_0^1 dz z^{N-1} P_{q \rightarrow q}(z) = - \frac{\gamma_0^N}{4} \quad (2.65)$$

and then inverting (2.64) we get

$$\frac{d\tilde{q}(x,t)}{dt} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \tilde{q}(y,t) P_{q \rightarrow q}\left(\frac{x}{y}\right) \quad (2.66)$$

where $\tilde{q} \equiv q - \bar{q}$, which is a non-singlet combination of structure functions. Equation (2.66) is the "evolution equation" for $\tilde{q}$. $P_{q \rightarrow q}(x)$ is the variation/unit of the probability density of finding a quark in a quark with fraction x/y of its momentum. In its infinitesimal form Eq. (2.66) is

$$\begin{aligned} \tilde{q}(x,t) + d\tilde{q}(x,t) = & \int_0^1 dy \int_0^1 dz \delta(zy-x) \tilde{q}(y,t) \times \\ & \times \left[\delta(z-1) + \frac{\alpha(t)}{2\pi} P_{q \rightarrow q}(z) dt \right] \end{aligned} \quad (2.67)$$

so that to calculate $P_{q \rightarrow q}(x)$ directly we have to calculate the coefficient of $\log q^2$ in the one-loop diagrams for deep inelastic scattering on a quark. This function must now satisfy (2.65), where the γ_0^N are given in (2.52). The second term in the square brackets is there because, in QCD, quarks can radiate gluons with non-zero energy, see Fig. 17a.

We now study the singlet combinations of structure functions. We define Q^S to be the flavour singlet quark distribution

$$Q^S(x, q^2) = \sum_{i=1}^f \left[q^i(x, q^2) + \bar{q}^i(x, q^2) \right] \quad (2.68)$$

and the gluon distribution is denoted by $G(x, q^2)$. We now get a coupled set of evolution equations:

$$\frac{dQ^s(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[Q^s(y,t) P_{q \rightarrow q}\left(\frac{x}{y}\right) + G(y,t) P_{g \rightarrow q}\left(\frac{x}{y}\right) \right] \quad (2.69a)$$

$$\frac{dG(x,t)}{dt} = \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[Q^s(y,t) P_{q \rightarrow g}\left(\frac{x}{y}\right) + G(y,t) P_{g \rightarrow g}\left(\frac{x}{y}\right) \right] \quad (2.69b)$$

The Q^s distribution varies with t now, not only because of gluon bremsstrahlung as in the non-singlet case, but also because gluons can convert into $q\bar{q}$ pairs (Fig. 17b) in a flavour-independent way, so that this variation of Q^s now depends also on the gluon distribution G . Similarly the variation of G with t depends on Q^s (because of bremsstrahlung Fig. 17c) and G (because gluons can also radiate gluons, Fig. 17d). The moments of $P_{q \rightarrow q}$, $P_{g \rightarrow q}$, $P_{q \rightarrow g}$ and $P_{g \rightarrow g}$ are, up to a factor of -4 as in (2.65), equal to the entries in the anomalous dimension matrix (2.59).

Thus we have a nice probabilistic interpretation⁴²⁾ of the violations of Bjorken scaling.

3. HIGHER ORDER CALCULATIONS

3.1 Introduction

Given a value of Λ from the analysis of Section 2.5, we can deduce the value of the running coupling constant by using Eq. (1.57), and it turns out that

$$\alpha_s(Q^2 = 10 \text{ GeV}^2) \approx 0.2 \text{ to } 0.4$$

Thus α_s turns out to be fairly large at accessible values of Q^2 , and, therefore, it becomes important to check that all the predictions of Section 2 are not invalidated by large higher order corrections. A second motivation for the calculation of these corrections is that they may be expected to provide additional and more precise tests of QCD. The next-to-leading contributions have now been calculated^{43,44}) and in this section we will outline the calculation and discuss the consequences of the results.

In QED higher order corrections to various measurable quantities have been calculated for several decades now, so why do we consider it necessary to outline the calculations of the corrections in QCD? The answer is that whereas in QED, when we calculate the contributions from higher order terms in the perturbative expansion, we evaluate just the Feynman diagrams of the corresponding order in the coupling constant, in QCD at each stage we must include new contributions from every order of perturbation theory (this will be explained in detail in Lecture 4), and sum these contributions. This summation is done most conveniently by using the operator product expansion and the renormalization group. Thus, although the techniques used in the evaluation of individual Feynman diagrams are the same as in QED, the quantities which have to be evaluated are somewhat different.

3.2 Calculation of the ingredients necessary to make predictions which include the higher order corrections

For simplicity we restrict ourselves here to the case of "non-singlet" combinations of structure functions (such as $F_2^{\text{np}} - F_2^{\text{nn}}$ or F_3^{VA} , where A is an isoscalar target). The generalization to the singlet case is conceptually straightforward and will be discussed below. For the Nth moment of these non-singlet contributions, $M_N^{(i)}(q^2)$, we have seen in Lecture 2 that we can write

$$M_N^{(i)}(q^2) = \bar{F}_i^N(1, g(q^2)) \exp \int_{g(\mu^2)}^{g(q^2)} \frac{\gamma_{0N}(g')}{\beta(g')} dg' \bar{O}^N \quad (3.1)$$

where γ_{0N} , β and $\bar{F}_i^N$ are calculable power series in their arguments and $\bar{O}^N$ is independent of q^2 . We write

$$\gamma_0^N(g) = \gamma_0^N \frac{g^2}{16\pi^2} + \gamma_1^N \frac{g^4}{(16\pi^2)^2} + \dots \quad (3.2)$$

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} + \dots \quad (3.3)$$

and

$$\bar{f}_i^N(1, g) = 1 + \epsilon_i^N \frac{g^2}{16\pi^2} + \dots \quad (3.4)$$

From (3.1) to (3.4) we can see that as $q^2 \rightarrow \infty$, the behaviour with q^2 is predicted to be

$$M_N^{(i)}(q^2) = (\text{const.}) [\alpha(q^2)]^{\gamma_0^N/2\beta_0} \approx (\text{const.}) \left(\frac{1}{\ln Q^2/\Lambda^2} \right)^{\gamma_0^N/2\beta_0} \quad (3.5)$$

which is equivalent to Eq. (2.57). There are logarithmic corrections to (3.5); the prediction including the next-to-leading corrections can be written

$$M_N^{(i)}(q^2) = (\text{const.}) [\alpha(q^2)]^{\gamma_0^N/2\beta_0} \left[1 + \frac{\alpha(q^2)}{4\pi} a_i^N \right] \quad (3.6)$$

where

$$a^N = \frac{\gamma_1^N}{2\beta_0} - \frac{\beta_1 \gamma_0^N}{2\beta_0^2} + \epsilon^N \quad (3.7)$$

Thus, in order to make predictions which include the effects of the "next-to-leading" terms, we must know the three new quantities β_1 , γ_1^N , and ϵ^N . We now outline the calculation of these quantities.

a) β_1

β_1 was first calculated by Caswell⁴⁵⁾ and by Jones⁴⁶⁾, who found

$$\beta_1 = 102 - \frac{38}{3} f \quad (3.8)$$

where f is the number of flavours. To calculate β_1 it is convenient to regulate the ultraviolet divergences by dimensional regularization and to perform the renormalization of the coupling constant by using the "minimal subtraction" scheme of 't Hooft and Veltman²⁰⁾ (hereafter this scheme will be called the $\overline{\text{MS}}$ scheme). We recall that in this scheme²⁰⁾

$$\beta(g) = \frac{\epsilon}{2} g + g^3 \frac{\partial a_1}{\partial g^2} \quad (3.9)$$

where

$$\epsilon = 4 - n \quad (3.10)$$

and n is the number of dimensions in which the integrals are performed. a_1 is defined by

$$Z_g = 1 + \sum_{i=1}^{\infty} \frac{a_i}{\epsilon^i} \quad (3.11)$$

where Z_g is the renormalization constant of the coupling constant. Thus the calculation of β_1 involves the calculation of the coefficient of $1/\epsilon$ (i.e. the subleading ultraviolet divergence) in all the two-loop diagrams which renormalize the coupling constant (for example, those of Fig. 18). Since the MS scheme satisfies all the Taylor-Slavnov identities, the Z_g for the three-gluon coupling and for the quark-quark-gluon coupling are equal.

Once we know β_1 , we can find a more accurate solution of (1.33) than that given in Eq. (1.56); we now find

$$\frac{\alpha_s(q^2)}{4\pi} = \frac{1}{\beta_0 \ln Q^2/\Lambda^2} - \frac{\beta_1}{\beta_0^3} \frac{\ln \ln Q^2/\Lambda^2}{\ln^2 Q^2/\Lambda^2} \quad (3.12)$$

where Λ has been chosen so that there are no terms of order $1/(\ln^2 q^2/\Lambda^2)$.

b) γ_1^N

The calculation of γ_1^N (43), the two-loop contribution to the anomalous dimensions of the twist-two non-singlet operators [the operators are given by Eq. (2.36)], has many similarities with that of β_1 . We recall from (1.43) that

$$\gamma^N = \mu \frac{\partial}{\partial \mu} \ln Z_{O_N} \quad (3.13)$$

where Z_{O_N} is the renormalization constant of the operator, and on the right-hand side all bare quantities are kept fixed. In the MS scheme (43)

$$\gamma^N = -g^2 \frac{\partial Z_1^N}{\partial g^2} \quad (3.14)$$

where Z_1^N is the coefficient of $1/\epsilon$ in Z_{O_N} . Thus, to obtain γ_1^N we must calculate the coefficient of $1/\epsilon$ (subleading ultraviolet divergence) in all the two-loop

diagrams which renormalize O_N (e.g. those of Fig. 19). The Feynman rules for the vertices are given in Refs. 43 and 44. The results of the calculation of γ_1^N are presented diagram by diagram (in the Feynman gauge)⁴³⁾, and these results have recently been rewritten in the relatively compact form⁴⁷⁾

$$\begin{aligned}
 \gamma_1^N = & \left(C_F^2 - \frac{C_F C_A}{2} \right) \left\{ 16 S_1(N) \frac{2N+1}{N^2(N+1)^2} + 16 \left[2 S_1(N) - \frac{1}{N(N+1)} \right] \right. \\
 & \times \left[S_2(N) - S'_2\left(\frac{N}{2}\right) \right] + 64 \tilde{S}(N) + 24 S_2(N) - 3 - 8 S'_3\left(\frac{N}{2}\right) \\
 & \left. - 8 \frac{3N^3 + N^2 - 1}{N^3(N+1)^3} - 16 (-1)^N \frac{2N^2 + 2N + 1}{N^3(N+1)^3} \right\} + \\
 & + C_F C_A \left\{ S_1(N) \left[\frac{536}{9} + 8 \frac{2N+1}{N^2(N+1)^2} \right] - 16 S_1(N) S_2(N) - \frac{43}{6} \right. \\
 & \left. + S_2(N) \left[-\frac{52}{3} + \frac{8}{N(N+1)} \right] - 4 \frac{151N^4 + 263N^3 + 97N^2 + 3N + 9}{9N^3(N+1)^3} \right\} \\
 & + C_F T_R \left\{ -\frac{160}{9} S_1(N) + \frac{32}{3} S_2(N) + \frac{4}{3} + 16 \frac{11N^2 + 5N - 3}{9N^2(N+1)^2} \right\}
 \end{aligned} \quad (3.15)$$

where

$$S_i(N) \equiv \sum_{j=1}^N \frac{1}{j^i} \quad (3.16)$$

$$\tilde{S}(N) \equiv \sum_{j=1}^N \frac{(-1)^j S_1(j)}{j^2} \quad (3.17)$$

and

$$S'_i\left(\frac{N}{2}\right) = \frac{1+(-1)^N}{2} S_i\left(\frac{N}{2}\right) + \frac{1-(-1)^N}{2} S_i\left(\frac{N-1}{2}\right) \quad (3.18)$$

C_F, C_A are the eigenvalues of the quadratic Casimir operators for the fundamental and adjoint representations (they are equal to $4/3$ and 3 for QCD) and T_R is equal to half the number of flavours.

The above calculation was performed in the MS scheme, and it should be remembered that γ_1^N depends on the prescription used to renormalize the operators⁴³⁾. To see this let us define the two schemes:

Scheme a):

$$Z_{O_N, a} = 1 - \frac{1}{2} \frac{g^2}{16\pi^2} \gamma_0^N \ln p^2/\mu^2 + O(g^4) \quad (3.19a)$$

and scheme b):

$$Z_{O_N, b} = 1 - \frac{g^2}{16\pi^2} \left(\alpha + \frac{1}{2} \gamma_0^N \ln p^2/\mu^2 \right) + O(g^4) \quad (3.19b)$$

where we have chosen μ^2 to be the point at which the coupling constant is defined.

Recalling (3.13) we find that

$$\gamma_{1, a}^N - \gamma_{1, b}^N = -2\alpha\beta_0 \quad (3.20)$$

Of course, physics must be independent of how we choose to renormalize the operators, i.e. the a^N of Eqs. (3.6) and (3.7) must be renormalization-prescription-independent. We shall see below that this is indeed the case.

c) ϵ_i^N 43, 48)

To calculate the ϵ_i^N we use the simple fact that the $\bar{F}_i^N$ are functions which depend only on the currents which we are expanding in terms of local operators, and are therefore independent of the states between which we sandwich the currents. Thus, in order to calculate the $\bar{F}_i^N$ we choose the most convenient states, in this case quark states of momentum p . We consider deep inelastic scattering on a quark of momentum p , [for which Eq. (3.1) is valid] and expand both sides of (3.1) in terms of $g(\mu^2)$ (which we write below as g). For the left-hand side we calculate all the appropriate tree and one-loop diagrams (Fig. 20) and for the right-hand side we use Eqs. (3.2) to (3.4) and also Eq. (1.56) in the form

$$g^2(q^2) = g^2 - \frac{g^4}{16\pi^2} \beta_0 \ln q^2/\mu^2 + O(g^6) \quad (3.21)$$

We then find that Eq. (3.1) can be expanded as

$$\begin{aligned} & 1 - \frac{g^2}{16\pi^2} \left(\frac{1}{2} \gamma_0^N \ln q^2/p^2 - \rho_i^N \right) \\ &= \left(1 + \epsilon_i^N \frac{g^2}{16\pi^2} \right) \times \left(1 - \frac{1}{2} \frac{g^2}{16\pi^2} \gamma_0^N \ln q^2/\mu^2 \right) \times \\ & \quad \times \left(1 + \frac{1}{2} \frac{g^2}{16\pi^2} \gamma_0^N \ln p^2/\mu^2 + \frac{g^2}{16\pi^2} \delta^N \right) \end{aligned} \quad (3.22)$$

where the three factors on the right-hand side correspond to $\tilde{F}_i^N$, $\exp \int_g^{g(q^2)} (\gamma_{0N}/\beta) dg'$ and $\tilde{O}^N$, respectively. The only unknown quantity in (3.22) is ϵ_i^N and therefore we can obtain it from this equation.

As we have already mentioned above, δ^N depends on the way we choose to renormalize the operators, and hence physics (in this case the q^2 behaviour of the moments, i.e. a_i^N) should be independent of δ^N . We have seen in Eq. (3.20) that δ^N corresponds to a contribution

$$2\delta^N \beta_0 \quad (3.23)$$

to γ_1^N , and to a contribution of

$$-\delta^N \quad (3.24)$$

to ϵ_i^N . Hence, from Eq. (3.7) we see that, indeed, the a_i^N are independent of δ^N . Nevertheless, it is crucial to remember that if we calculate the γ_1^N in a particular scheme (like the MS scheme we have been using), then we must use the appropriate δ^N to extract the ϵ_i^N .

3.3 Results and an ambiguity

The results of the calculation are presented in detail elsewhere⁴⁹). Here, in Table 3.1, we only present the results for a_i^N for $i = \nu W_2$ (since we are still considering non-singlet combinations of structure functions, it is really $\nu W_2^{\text{NP}} - \nu W_2^{\text{un}}$) and taking there to be four flavours.

Table 3.1

Results of the calculation of a_i^N [defined in Eq. (3.6)] for the νW_2 structure function. The calculation was performed in the minimal subtraction renormalization scheme.

N	a_i^N	N	a_i^N
1	0	6	30.73
2	9.045	7	34.75
3	16.02	8	38.22
4	21.75	9	41.38
5	26.64	10	44.29

The relative size of the corrections to the leading terms are the entries of Table 3.1 multiplied by $\alpha_s(q^2)/4\pi$, which is usually taken to be $\sim 2\%-3\%$ at $q^2 \sim 10 \text{ GeV}^2$. Thus it seems that for $N = 4$ and above, the corrections are large: 50% or more.

Does this mean that these higher order corrections should be detectable in the experimental data?

Not necessarily!! This is because when we expand a physically measurable quantity in terms of a power series in $\alpha_s(q^2)$, we must, of course, specify what we mean by $\alpha_s(q^2)$. In QED, $\alpha \equiv e^2/4\pi$ is universally accepted as being the coupling constant defined through the "Thompson Limit of Compton Scattering". Such a limit does not exist in QCD, so we have to specify precisely what we mean by $\alpha_s(q^2)$, and everyone is free to choose his or her own definition. In particular, one can relate two definitions by a power series of the form

$$\alpha_{s,1}(q^2) = \alpha_{s,2}(q^2) \left[1 + \rho \alpha_{s,2}(q^2) + \dots \right] \quad (3.25)$$

so that, using Eq. (3.6), we see that a_i^N can be written in the form

$$a_i^N = b_i^N + \rho d_N \quad (3.26)$$

where the b_i^N are renormalization-scheme-independent numbers, whereas ρ is an arbitrary (but N independent) number, and d_N (2.60) is equal to $\gamma_0^N/2\beta_0$. Thus we see that the values of the a_i^N in one particular scheme may not be physically significant. Clearly, one would like to choose a scheme in which the corrections which have not been calculated are reasonably small, but it is impossible to be sure of this without calculating still further corrections. Thus, one is forced either to guess what is a reasonable scheme, or to make predictions which are independent of ρ . We come back to this in Section 3.5, but now we digress to demonstrate that

3.4 β_1 is independent of the definition of α_s

Because it is always possible to define α_s in many ways, any two of which can be related by (3.25), one may suspect that in any quantity which can be expanded as a power series in α_s , it is only the term independent of α_s and the first term which depends on α_s which are independent of the definition of α_s . This is indeed true for quantities whose definition is independent of the definition of α_s (such as the ratio R in e^+e^- annihilation, γ_0^N , etc.) but not for the β function, which we recall is defined by

$$\beta(g) = \frac{\partial g(\mu)}{\partial \ln \mu} \quad (3.27)$$

and hence depends on the prescription used to define α_s . In addition to β_0 it is found that β_1 is also a physically significant, unambiguous number [given by (3.8)]. To illustrate this, let us define two schemes, labelled by a and b and related by*)

$$g_a = g_b (1 + \rho g_b^2 + \dots) \quad (3.28)$$

*) ρ here is slightly different from that used in (3.25).

We write the β functions in the two schemes in the form

$$\beta_a(g_a) = -\frac{\beta_0^a g_a^3}{16\pi^2} - \frac{\beta_1^a g_a^5}{(16\pi^2)^2} + \dots \quad (3.29a)$$

and

$$\beta_b(g_b) = -\frac{\beta_0^b (g_b^3)}{16\pi^2} - \frac{\beta_1^b g_b^5}{(16\pi^2)^2} + \dots \quad (3.29b)$$

Using (3.28) we write

$$\begin{aligned} \beta_a &\equiv \frac{\partial g_a}{\partial \ln \mu} = \frac{\partial g_b}{\partial \ln \mu} (1 + 3\rho g_b^2) \\ &= \beta_b(g_b) (1 + 3\rho g_b^2) \\ &= (1 + 3\rho g_b^2) \left(-\frac{\beta_0^b g_b^3}{16\pi^2} - \frac{\beta_1^b g_b^5}{(16\pi^2)^2} \right) \\ &= (1 + 3\rho g_b^2) \left(-\frac{\beta_0^b g_a^3 (1 - 3\rho g_a^2)}{16\pi^2} - \frac{\beta_1^b g_b^5}{(16\pi^2)^2} \right) \quad (3.30) \\ &= -\frac{\beta_0^b g_a^3}{16\pi^2} - \beta_1^b \frac{g_a^5}{(16\pi^2)^2} \end{aligned}$$

Comparing (3.30) and (3.29a) we deduce that

$$\beta_0^a = \beta_0^b \quad \text{and} \quad \beta_1^a = \beta_1^b \quad (3.31)$$

We leave it as an exercise for the reader to convince herself (himself) that the coefficients of higher order terms in β are not universal, but are ρ dependent.

3.5 Are the effects of the higher order corrections measurable?

We start by looking for the effects of higher order corrections which are independent of the definition of the coupling constant, i.e. of ρ ⁵⁰). One such effect is the behaviour with N of the quantity $R^N(q^2, q_0^2)$, which is defined by

$$R^N(q^2, q_0^2) = \left[\frac{M_N(q^2)}{M_N(q_0^2)} \right]^{1/d_N} \quad (3.32)$$

where we have suppressed the index i . The QCD prediction is

$$R^N(q^2, q_0^2) - R^T(q^2, q_0^2) = \frac{\alpha_s(q^2)}{\alpha_s(q_0^2)} (\alpha_s(q^2) - \alpha_s(q_0^2)) \left[\frac{a^N}{d_N} - \frac{a^T}{d_T} \right] \quad (3.33)$$

where from Eq. (3.26) we see that the expression in square brackets is independent of ρ and is positive for $N > T$. Thus for fixed q^2 and q_0^2 , R^N should decrease with N , and this is a consequence of the higher order corrections. In Fig. 21 we show the behaviour of $R^N(45 \text{ GeV}^2, 6.5 \text{ GeV}^2)$ and $R^N(45 \text{ GeV}^2, 10 \text{ GeV}^2)$ with N (defined by moments of the F_3 structure function) and the theoretical predictions corresponding to the three values $\Lambda = 0.3, 0.5$ and 0.7 . The experimental data are from Ref. 51. We see that the experimental errors are too large, even to check whether R^N falls or rises with N , and the data are certainly not inconsistent with the theoretical predictions. Hence, even though the entries in Table 3.1 are fairly large, we see that they lead to an unmeasurably small effect, at least as far as the N dependence of R^N goes for the q^2 ranges considered in Fig. 21. If we reduce q_0^2 , the magnitude of the effect is increased, but then we have to worry about "higher-twist" effects, i.e. terms which violate Bjorken scaling by a power of q^2 .

There are several other (related) ways of trying to find (ρ independent) effects of higher order corrections in moments of structure functions. One of these is to check whether the value of the parameter Λ , determined from the data using the lowest order formula

$$R^N(q^2, q_0^2) = \left[\frac{\ln q_0^2/\Lambda^2}{\ln q^2/\Lambda^2} \right] \quad (3.34a)$$

has the correct N dependence. Strictly speaking, one should extract Λ using a slightly more complicated formula than (3.34a), that is

$$R^N(q^2, q_0^2) = \left[\frac{\ln q_0^2/\Lambda^2}{\ln q^2/\Lambda^2} \right] \frac{\left[1 - \frac{\beta_1}{\beta_0^2} \frac{\ln \ln q^2/\Lambda^2}{\ln q^2/\Lambda^2} \right]}{\left[1 - \frac{\beta_1}{\beta_0^2} \frac{\ln \ln q_0^2/\Lambda^2}{\ln q_0^2/\Lambda^2} \right]} \quad (3.34b)$$

where β_0, β_1 are respectively the coefficients of $-g^3/16\pi^2, -g^5/(16\pi^2)^2$ in the β function. Because of the slow variation of $\log \log q^2/\Lambda^2$ with q^2 , at least for q^2

substantially larger than Λ^2 , the N dependence of the Λ 's obtained in this way should be very nearly the same as those using Eq. (3.34a). We have checked that this, indeed, is the case for the existing data. The fact that we neglect the a_N in Eqs. (3.34a), (3.34b) means that the higher order corrections will now manifest themselves as an N dependence in Λ . We find

$$\Lambda_N = \Lambda \exp \left(\frac{2\pi a_N}{\beta_0 d_N} \right) \quad (3.35)$$

A very nice feature of Eq. (3.35) is that Λ_N/Λ_T is independent of ρ and therefore is a clean prediction of QCD.

This definition of Λ_N is different from that of Bardeen et al.⁴⁸⁾, which was subsequently used by Duke and Roberts⁵²⁾; in their definition only a part*) of the higher order corrections was absorbed into an N dependent Λ . Using the definition of Bardeen et al.⁴⁸⁾, Λ_N must be extracted, not by using the lowest order formula (3.34a) or (3.34b), but a more complicated formula which includes the "unabsorbed" higher order terms. We find our definition preferable, not only because it is simpler, but primarily because only in our case are both the theoretical prediction and the procedure for extracting Λ_N from the data explicitly independent of ρ .

In Table 3.2 we tabulate Λ_N/Λ_2 for the first few values of N assuming there to be four (or six) flavours (the dependence on the number of flavours is very weak). The entries are somewhat different depending on whether one uses F_2 or F_3 to extract the Λ 's. For $N = 2$, the value of Λ extracted using F_2 is predicted to be 37% larger than that extracted using F_3 ; this prediction is independent of the number of flavours.

From Table 3.2 we see that the parameter Λ_N should change substantially^{34,51)} with N . In Fig. 22 we plot the values of Λ_N extracted from existing neutrino^{34,51)} and muon⁵⁴⁾ data. On the same figure we plot the theoretical prediction; this curve may be displaced vertically at will, since the theory only predicts the shape and not its normalization. At first sight, the two sets of neutrino data look inconsistent, but on closer examination⁵⁵⁾ it has been found that systematic uncertainties in the analysis (such as the number of flavours used in the formulae, whether or not to include the pseudoelastics and the correction for Fermi motion) account for about half the discrepancy, so that the final discrepancy in the data is not very significant (see also Ref. 40). Moreover, the discrepancy does not stem from the different ranges of Q^2 in the two experiments⁵⁵⁾. Bearing in mind that

*) From the paper of Duke⁵²⁾ one might wrongly conclude that Duke and Roberts have the same definition as ours for Λ_N . In fact this is not so, and their Λ_N is defined using the definition of Bardeen et al.⁴⁸⁾. The procedure we are using has previously been proposed by Bačič⁵³⁾.

Table 3.2

Predicted values of Λ_N/Λ_2 , for four flavours;
the values are different if one uses F_2 or F_3 to obtain the Λ 's.
The values in parentheses are for six flavours.

N	F_2	F_3
2	1.00 (1)	1.00 (1)
3	1.19 (1.17)	1.41 (1.39)
4	1.33 (1.30)	1.67 (1.63)
5	1.45 (1.41)	1.87 (1.82)
6	1.56 (1.51)	2.04 (1.98)
7	1.66 (1.60)	2.18 (2.11)
8	1.74 (1.68)	2.31 (2.23)
9	1.82 (1.75)	2.43 (2.34)
10	1.89 (1.82)	2.53 (2.43)

these systematic uncertainties are not included in the errors, it is clear that the neutrino data are not yet good enough to confirm or disprove the theoretical predictions. The values of Λ_N have also been extracted from the moments of $F_2^{\mu p} - F_2^{\mu n}$ (54), and show an increase of Λ_N with N in remarkable agreement with the predictions of the table (see Fig. 22b). This striking agreement is somewhat surprising in view of the fact that these values of Λ_N are determined mostly by the low- q^2 region. In this region one might expect higher-twist non-scaling contributions to be still relevant, especially for the higher moments, and there is no reason to expect these to reproduce Fig. 22b. We leave the reader to interpret the significance of Fig. 22, and look forward to the publication of neutrino and muon data which are more accurate and, in the muon case, at higher values of q^2 .

Both the above examples were presented in terms of moments; the same conclusions are obtained when we look at the effects of higher order corrections to the structure functions themselves^{37,39,56}). Even though in a particular renormalization scheme the corrections may seem large, the major part of these corrections can be absorbed into a redefinition of the coupling constant, i.e. of Λ . For example⁴⁰), if one uses the MS scheme, when one compares the ep and μp data to the QCD predictions which include the higher order corrections, one obtains a value of Λ which is about half of that obtained by comparing the lowest order formula to the data. On the other hand, the best fits in the two cases are comparable, though there is a slight improvement if one includes the corrections.

3.6 Higher order corrections to singlet combinations of structure functions

So far, we have studied non-singlet combinations of structure functions such as $F_2^{\mu p} - F_2^{\mu n}$ and $F_3^{\nu A}$ (where A is an isoscalar target), where the effects of diagrams

such as that in Fig. 23 are zero. For flavour singlet combinations of structure functions, for each moment there are two operators of twist two which contribute, namely

$$\bar{\psi} \gamma^{\mu_1} \overleftrightarrow{D}^{\mu_2} \dots \overleftrightarrow{D}^{\mu_N} \psi + \text{perms} \quad (3.36)$$

and

$$F_{\mu_1 \alpha} D_{\mu_2} \dots D_{\mu_{N-1}} F_{\alpha \mu_N} \quad (3.37)$$

which mix under renormalization. Thus, the anomalous dimensions now form a 2×2 matrix, and its evaluation⁴⁴⁾ is a generalization of the non-singlet case and involves the calculation of the coefficient of the $1/\epsilon$ term in four sets of two-loop diagrams; a sample diagram from each set is shown in Fig. 24. The anomalous dimensions have been calculated⁴⁴⁾; the only comment we would like to make is that there is a nice consistency check on the calculation. Because the MS scheme preserves the Ward identities, the energy momentum tensor

$$\frac{i}{4} [\bar{\psi} \gamma^{\mu_1} \overleftrightarrow{D}^{\mu_2} \psi + \text{perm}] + [F^{\mu_1 \mu_2} F^{\mu_3 \mu_4} g_{\mu_3 \mu_4}] \quad (3.38)$$

is not renormalized. This means that for $N = 2$

$$\gamma_{ff}^{N=2} + \gamma_{fg}^{N=2} = 0 \quad (3.39a)$$

and

$$\gamma_{gf}^{N=2} + \gamma_{gg}^{N=2} = 0 \quad (3.39b)$$

for all representations of quarks and gluons; this proves to be a highly non-trivial test of the calculation.

The ϵ_i^N corresponding to operator (3.36) is the same as in the non-singlet case, but one has to calculate the ϵ_i^N corresponding to operator (3.37). This has been done in Refs. 44 and 48.

In general, a structure function is a mixture of singlet and non-singlet terms. The phenomenology of the higher order corrections to, for example, F_2^{up} has been studied by Ross⁵⁷⁾. The conclusions are similar to those in the non-singlet case, namely that the measurable effects of the higher order corrections are small.

3.7 Comments

We have seen in this section that the higher order corrections to deep inelastic scattering have now been calculated. They turn out to be small, and so do not invalidate the lowest order picture (discussed in Lecture 2) to which we have

become accustomed. This is certainly a non-trivial result. Because they are small, it is very hard to find any positive sign of these effects unless one is prepared to go to low values of q^2 ($\gtrsim 1 \text{ GeV}^2$) and hope that even here the "higher-twist" effects are negligible. It is just possible⁵⁰⁾ that future high q^2 [$q^2 > 6.5 \text{ GeV}^2$ *) say] data will be able to verify or disprove the existence of the predicted higher order terms.

Many of our parton model ideas, which were still valid in QCD when only the leading terms were included, e.g. the Callan-Gross relation

$$F_2 = 2x F_1 \quad (3.40)$$

and various parton model sum rules, no longer hold when higher order corrections are included. One notable exception is the Adler sum rule

$$\int_0^1 \frac{dx}{x} [F_2^{\bar{\nu}P} - F_2^{\nu P}] = 2 \quad (3.41)$$

which has no logarithmic corrections, but other parton model sum rules do have $O[\alpha_s(q^2)]$ corrections^{48,58)}. The violation of the Callan-Gross relation leads to a non-zero longitudinal structure function. Since QCD predicts the q^2 behaviour of structure functions -- in particular the longitudinal structure function F_L -- and since the only parameters to be determined from the data are the operator matrix elements, once these are determined from fits to νW_2 (or W_1 or νW_3) we can predict F_L absolutely up to corrections of order $\alpha_s(q^2)$ and m^2/q^2 . F_L is very difficult to measure⁵⁹⁾ and not very well determined, but the theoretical predictions tend to lie below the data⁵⁶⁾. It would be very interesting to have better data for this process.

*) This value is arbitrarily chosen. It is the smallest value for which moments of structure functions are given⁵¹⁾ by the CERN-Dortmund-Heidelberg-Saclay (CDHS) group.

4. HARD SCATTERING PROCESSES

4.1 Introduction

In this section we will review why we now believe that we can make predictions about the asymptotic behaviour of hard scattering processes. The operator product expansion and renormalization group techniques discussed in the preceding lectures are not in general directly applicable, so we have to search for a new approach. The class of processes which we will consider in this lecture are those in which there are three mass scales:

- a) a large mass scale Q^2
- b) a small mass scale (of the order of hadronic masses) p^2 , and
- c) the renormalization scale μ^2 .

We will be interested in the limit $Q^2 \rightarrow \infty$, p^2 fixed; physical cross-sections are, of course, independent of μ . In the relevant process there may be more than one variable which is $O(Q^2)$ (e.g. in deep inelastic scattering there is q^2 and $p \cdot q$), but then the ratio of these variables should stay fixed as we take the limit $Q^2 \rightarrow \infty$. Similarly, we assume that all hadronic masses are of the same order p^2 .

We start by reproducing the results of Lecture 2 for the deep inelastic structure functions by using diagrammatic techniques. Later, we will see that the same techniques can be generalized to other hard scattering processes. There is another approach⁶⁰⁾ to the derivation of predictions for hard scattering processes which is analogous to the operator product expansion and renormalization group techniques of the previous lectures. Unfortunately we do not have the time to discuss it in detail here.

4.2 Diagrammatic approach to deep inelastic lepton-hadron scattering

In Lectures 2 and 3 we have seen how the use of the operator product expansion and the renormalization group enables us to make predictions for the violation of Bjorken scaling in deep inelastic lepton-hadron scattering. In this subsection we will study the same process with a different approach, one which is less rigorous but is on the other hand also applicable to processes which are not light-cone dominated. The basic assumption is that asymptotically hard scattering cross-sections can be written as a convolution of soft hadronic wave functions which are process-independent, with the cross-section for the "hard subprocess" which involves only quarks and gluons and can be studied perturbatively. Thus, for example, in the case of deep inelastic scattering we assume that we can write (see Fig. 25).

$$F(x, q^2) = \sum_{\text{partons}} \int_x^1 d\bar{x} \int d^2 \underline{k}_\perp dk^2 f(\bar{x}, \underline{k}_\perp, k^2) \times \sigma_{\text{parton}}\left(\frac{x}{\bar{x}}, k^2, q^2\right) \quad (4.1)$$

where f is related to the square of the soft wave function and σ_{parton} is the cross-section for deep inelastic scattering with the parton as the target. Within perturbation theory this assumption can be justified (see below). We will now study σ_{parton} in perturbation theory.

When computing σ_{parton} in perturbation theory, we will find terms of the type $g^2 \log Q^2/\mu^2$ and $g^2 \log Q^2/p^2$ (Q^2 , p^2 and μ^2 are defined as in Lectures 1 and 2). We will see below that with each power of g^2 there is at most one logarithm.

4.2.1 Equivalence of the light-cone result and the summation of leading logarithms

We would like to demonstrate that the results of Lecture 2 can be obtained by summing the leading logarithms of perturbation theory. For simplicity, we will consider a non-singlet combination of structure functions; the extension to general combinations of structure functions complicates the technical details, but does not alter the essential features.

We start with Eq. (2.56) for the moment M_N of a structure function

$$M_N(q^2) = \bar{F}^N(1, g^2(q^2)) \exp \int_{g(\mu^2)}^{g(q^2)} \frac{\gamma_{ON}(g')}{\beta(g')} dg' \bar{O}^N \quad (4.2)$$

where $\bar{O}^N = \langle p | O^N | p \rangle$; $|p\rangle$ is e.g. a quark state of moment p . We now expand the right-hand side of Eq. (4.2) in terms of a power series in $g^2(\mu^2)$; this we can do because we know how to relate $g(q^2)$ to $g(\mu^2)$. This relation is given by the solution of the equation

$$\frac{\partial g(\mu^2)}{\partial \ln \mu} = \beta(g) \quad (4.3)$$

In each order of perturbation theory we keep only the terms which are of the type $g^{2n} \log^n X$, where X is a ratio of two of the three mass scales Q^2 , p^2 , and μ^2 , i.e. we work in the leading logarithm approximation. In the leading logarithm approximation the solution of (4.3) is given (1.56) by

$$g^2(q^2) = g^2(\mu^2) \frac{1}{1 + \frac{\beta_0}{16\pi^2} g^2(\mu^2) \ln q^2/\mu^2} \quad (4.4)$$

We now look at the three factors of the right-hand side of Eq. (4.2) in turn.

a) $\bar{F}^N$: $\bar{F}^N$ is a power series in $g^2(q^2)$ with no logarithms. Therefore, in the leading logarithm approximation, we need to keep only the first term which is 1 *).

*) For the moments of F_L , the first term in $\bar{F}^N$ is equal to g^2 times a constant.

b) The exponential factors: here we expand γ_{0N} and β and use Eq. (4.4) to obtain

$$\exp \left[\int \frac{g(q^2)}{g(\mu^2)} \frac{\gamma_{0N}(g')}{\beta(g')} dg' \right] \underset{\substack{\text{leading} \\ \text{logarithm} \\ \text{approximation}}}{\approx} \left[1 + \frac{\beta_0 g^2(\mu^2)}{16\pi^2} \ln \frac{q^2}{\mu^2} \right]^{-\gamma_0^N/2\beta_0} \quad (4.5)$$

c) $\bar{O}^N$: this can either be calculated directly, or we can use the fact that M_N is μ independent so that the μ dependence of (4.5) must be cancelled by that of $\bar{O}^N$. Since $\bar{O}^N$ is independent of q , we find

$$\bar{O}^N = \left\{ 1 + \frac{\beta_0 g^2(\mu^2)}{16\pi^2} \ln \frac{\mu^2}{\mu^2} \right\}^{\gamma_0^N/2\beta_0} \quad (4.6)$$

Thus, we find in the leading logarithm approximation that*)

$$M_N(q^2) \sim \left[\frac{1 + \beta_0 \frac{g^2(\mu^2)}{16\pi^2} \ln \mu^2/\mu^2}{1 + \beta_0 \frac{g^2(\mu^2)}{16\pi^2} \ln q^2/\mu^2} \right]^{\gamma_0^N/2\beta_0} \quad (4.7)$$

If we did not know anything about the light-cone operator product expansion and renormalization group techniques of Lecture 2, but had instead calculated Feynman diagrams and kept only the leading logarithms, we would have discovered the series generated by (4.7), and thus obtained the result

$$\frac{M_N(q^2)}{M_N(q_0^2)} = \left[\frac{g^2(q^2)}{g^2(q_0^2)} \right]^{\gamma_0^N/2\beta_0} \quad (4.8)$$

which is the result obtained in Lecture 2. In this sense, the sum of the leading logarithms in perturbation theory is equivalent to the result obtained by light-cone techniques.

We will now study some Feynman diagrams to see if we can understand what regions of phase space give the leading logarithms, and how to obtain (4.7) more directly.

4.2.2 Which regions of phase space give the leading logarithms?

We choose to write all logarithms as $\log q^2/p^2$ and $\log q^2/\mu^2$, and look for possible sources of $\log \mu^2$ and $\log p^2$. The $\log \mu^2$ terms arise in any renormalizable

*) If we differentiate the right-hand side of (4.7) with respect to $\log \mu$, we find we get 0 in the leading logarithm approximation, i.e. all terms of the form $g^{2n} \log^{n-1} \mu^2$ cancel.

field theory after renormalization has been performed. In the minimal subtraction (MS) scheme they appear because, although the coupling constant in QCD is dimensionless in the physical number of dimensions, it is not dimensionless in $4 - \epsilon$ ($\epsilon \neq 0$) dimensions, and hence a mass scale has to be introduced. We will return to the question of the $\log \mu^2$ terms below.

As far as the $\log p^2$ terms go, we can use the machinery which exists for the behaviour of physical processes in field theory in the limit where one or more of the masses vanish. In particular, if the appropriate Feynman integrals are finite in this limit, then there can be no $\log p^2$ term which scales (p^2 is a mass which is being taken to zero). Thus, since we are looking for possible sources of $\log p^2$ terms, we are interested in finding the regions of phase space which yield a divergence as one or more masses vanish. We will discuss the two types of divergence which are often called infrared divergences and mass singularities. Both of these have been thoroughly studied in field theory.

Infrared divergences^{61,62} arise from the presence (in a frame in which the external particles are not at rest) of a soft, real or virtual, massless particle. For example, if we evaluate the diagram of Fig. 26 for the process $e^+e^- \rightarrow \mu^+\mu^-$ (or $q\bar{q}$), we find that there is a Feynman integral of the form

$$\int d^4k \frac{1}{k^2 [(p_1+k)^2 - m^2] [(k-k')^2 - m^2]} \sim_{k \rightarrow 0} \int \frac{d^4k}{k^4} = \infty \quad (4.9)$$

We see in Fig. 26 that when k is soft three propagators simultaneously become close to their poles. The theorem of Bloch and Nordsieck^{61,62} states that in inclusive cross-sections these infrared divergences cancel. In QED, for example, they cancel between diagrams with real photons and those with virtual photons as in Fig. 27. All physical cross-sections are in fact inclusive ones, since in any experiment the energy resolution is not perfect and hence there may be any number of undetected soft photons. Hence, when calculating predictions for measurable cross-sections, we have to sum over states with these additional photons and we then arrive at a finite result.

This cancellation of infrared divergences also occurs in deep inelastic scattering. The only slightly novel feature occurs at the point $x = 1$. For example, the diagram of Fig. 28a (evaluated in the Feynman gauge), has a behaviour as $x \rightarrow 1$ of the form

$$\text{Diag. (28a)} \sim_{x \rightarrow 1} \frac{1}{(1-x)^{1+\epsilon}} \quad (4.10a)$$

where we have chosen to use dimensional regularization to regulate the infrared divergences^{63,64}). The infrared divergences occur, of course, at $x = 1$. The corresponding diagram with a virtual gluon is that of Fig. 28b and has a behaviour

$$\text{Diag. (28b)} \sim \delta(x-1) \frac{1}{\epsilon} \quad (4.10b)$$

The right-hand sides of (4.10a) and (4.10b) are equal and opposite in the sense of distributions⁴²). If we smear $1/(1-x)^{1+\epsilon}$ with any smooth function of x regular at $x = 1$ by taking the integral from 0 to 1, then we get the opposite result by smearing $\delta(x-1)1/\epsilon$ with the same function. Thus, if we obtain a term of the type $1/(1-x)$, we will be able to rewrite it in terms of $1/(1-x)_+$, which is defined as a distribution through

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} \equiv \int_0^1 dz \frac{f(z) - f(1)}{(1-z)} \quad (4.11)$$

where the smearing function $f(z)$ is regular at $z = 1$.

It is true then that infrared divergences always cancel in inclusive cross-sections, such as the ones which will be discussed in this section. Hence, the region of phase space where the gluons (or quarks) are soft does not provide us with the $\log p^2$ terms we are seeking.

We come now to mass singularities. These occur in theories with coupled massless particles, and are due to the simple kinematical fact that two massless particles (with momenta k_1, k_2 , say) which are moving parallel to each other have a combined invariant mass equal to zero

$$k^2 \equiv (k_1 + k_2)^2 = \left[\omega_1 (1, 0, 0, 1) + \omega_2 (1, 0, 0, 1) \right]^2 = 0 \quad (4.12)$$

Thus, when considering the divergences of Fig. 29, we have to evaluate the contribution of the region of phase space where k_1 is parallel to k_2 , as well as that where k_1, k_2 are soft.

There is also a theorem, the Kinoshita⁶⁵) -- Lee-Nauenberg⁶⁶) (KLN) theorem, which ensures that for inclusive enough cross-sections, the mass singularities also cancel. For example, the mass singularities of Fig. 27a (in the limit where the mass of the muon is zero) cancel those of Fig. 27b, and so for the process $e^+e^- \rightarrow \mu^+\mu^-X$ we can set both the μ and photon masses to zero and still get a finite result. The interpretation of this result is similar to that for infrared divergences, namely that in a physically measurable process the angular resolution is never perfect and therefore we should sum over all indistinguishable states, i.e. all states in which there are some collinear particles. For the exact statement

of the KLN theorem we refer the reader to the original papers^{65,66}). Here we will just note that it assures us that all mass singularities coming from final-state undetected particles moving parallel to each other cancel.

In deep inelastic scattering, however, we still have mass singularities left over; those coming from regions of phase space in which internal particles have momenta parallel to the momentum of one of the incoming particles. For example, in Fig. 30 we have a factor of $\log p^2$ from the region where k is parallel to p . We will see this explicitly below.

Whereas one never gets infrared divergences if all external particles are neutral, this is not true for mass singularities. For example, if we take deep inelastic scattering on a photon, where the lowest order diagrams are shown in Fig. 31, we still get a factor of $\log p^2$.

Thus we see that the required $\log p^2$ terms come from regions of phase space in which at least some of the internal particles have momenta parallel to the incoming ones, or possibly to the trigger particle in an inclusive cross-section.

4.2.3 Some low-order diagrams

Let us look at some low-order diagrams. In this way we shall see explicitly the origin of the $O(g^2)$ term in the series generated by (4.7), and perhaps get some insight into how the higher orders should work. We consider deep inelastic scattering on a quark and start by looking at the interference term shown in Fig. 32a, in the Feynman gauge. Evaluating this integral, we readily find that it contains an integral of the form

$$I = \int \frac{d^4 k}{(p-k)^2} f(k, p, q) \quad (4.13)$$

Evaluating $(p-k)^2$ in terms of the components of the momenta of p and k , we find

$$(p-k)^2 = -2\omega (E - |p| \cos \theta) \quad (4.14)$$

where $p = (E, 0, 0, |p|)$ in the centre-of-mass frame for example, ω is the energy of k , and θ is the angle between p and k . Thus

$$I \propto \int_{-1}^1 \frac{d(\cos \theta)}{(1 - \frac{|p|}{E} \cos \theta)} \sim \ln q^2/p^2 \quad (4.15)$$

where we have used the fact that in our approximation, $E \simeq p$, $(E+p)^2 \simeq q^2$, $E^2 - p^2 = p^2$. Thus we have obtained the promised $\log q^2/p^2$. Notice that we could have performed the integral from $1 - \epsilon$ to 1 for a fixed small ϵ (so that

$\log 1/\epsilon \ll \log q^2$) and still have obtained the same result. Thus, the dominant region of integration is that corresponding to small angles as stated above. In terms of k_T , the component of $\underline{k}$ orthogonal to $\underline{p}$, this dominant region corresponds to $m^2/\delta < k^2 < \epsilon q^2$, where δ and ϵ are small fixed numbers.

We would like now to concentrate on another feature of the calculation of the diagram of Fig. 32a. When evaluating the Dirac algebra of this diagram, we will find a term

$$(\not{p}-\not{k}) \gamma^\mu \not{p} \quad (4.16)$$

where μ is the Lorentz index of the gluon. From the form of the integral in (4.15) we see that if we are only interested in the $\log q^2/p^2$ term, we can set $\cos \theta = 1$ everywhere in the numerator, and, in particular, we can write $k = (1 - \rho)p$ (in the numerator we can also consider p to be a null vector). Then in this approximation

$$(\not{p}-\not{k}) \gamma^\mu \not{p} = \frac{\rho}{(1-\rho)^2} \not{k} \gamma^\mu \not{k} = \frac{2\rho}{(1-\rho)^2} k^\mu \not{k} \quad (4.17)$$

If we now choose a gauge in which we sum only over the physical transverse polarizations diagram by diagram, so that

$$k^\mu \sum_{\text{polarizations}} \epsilon^\mu \epsilon^\nu = 0 \quad (4.18)$$

we will get no term with a factor of $\log q^2/p^2$ from the diagram of Fig. 32a. Popular choices for the gauge are axial gauges of the type $n \cdot a = 0$ ⁶⁷⁾, where n is some four vector, sometimes chosen to be q ²⁾.

If, instead of setting $\theta = 0$ and writing $k = (1 - \rho)p$, we would have kept θ , we would have found that at each vertex in which the gluon momentum is almost parallel to the incoming quark momentum there is a suppression factor of θ . For small θ ,

$$\begin{aligned} (p-k)^2 &\sim 2\omega E (1 - \cos \theta) \\ &\sim \omega E \theta^2 \end{aligned}$$

and phase space is proportional to $d(\cos \theta) \approx \theta d\theta$. Thus, in a transverse gauge the angular integration for $\theta \sim 0$ is of the form

$$\text{Diag. (32a)} \sim \int \theta d\theta \frac{\theta}{\theta^2} = \text{finite} \quad (4.19)$$

as we have already discovered.

For the diagram of Fig. 32b, however, there are two vertices with a suppression factor θ at each, and two factors of $1/(p-k)^2$, so that the angular integration is now of the form

$$\text{Diag. (32b)} \sim \int \theta d\theta \frac{\theta^2}{(\theta^2)^2} = \int \frac{d\theta}{\theta} = \infty \quad (4.20)$$

Thus the diagram of Fig. 32b has a mass singularity, even in a transverse gauge. In fact, it is the only diagram with a real gluon to have a mass singularity.

We now outline the calculation of the diagram of Fig. 32b in the gauge $n \cdot a = 0$. In this gauge the sum over polarizations is given by

$$- \left[g^{\mu\nu} - \frac{k^\mu n^\nu + k^\nu n^\mu}{n \cdot k} + \frac{n^2 k^\mu k^\nu}{(n \cdot k)^2} \right] \quad (4.21)$$

We label the three terms in (4.21) by (a), (b), and (c). It is convenient to use Sudakov variables⁶⁸⁾. We first define q' by $q' \equiv q + xp$ which has the property $q'^2 \approx 0$, and the Sudakov variables ρ , β , and k are then defined by

$$k = \rho p + \beta q' + k_T \quad (4.22)$$

where k_T is orthogonal to p and q . The Jacobian is given by

$$d^4k = (p \cdot q) d\rho d\beta d^2k_T \quad (4.23)$$

and in terms of these variables $(p-k)^2 \approx p^2 - 2\beta p \cdot q$. Performing the Dirac algebra we find that each of the terms (a) and (b) in (4.21) gives one power of $(p-k)^2$, whereas term (c) gives a factor of $(p-k)^4$ and therefore does not contribute to the $\log q^2/p^2$ term. The relative contributions of (a) and (b) are

$$(a) \quad C_F (1-\rho) \quad (4.24a)$$

$$(b) \quad C_F \frac{2\rho}{(1-\rho)} \quad (4.24b)$$

where $C_F = 4/3$ is the usual colour factor. Adding these two contributions together we find that we get

$$C_F \left(\frac{1+\rho^2}{1-\rho} \right) \quad (4.25)$$

which is the Altarelli-Parisi⁴²⁾ function $P_{q \rightarrow q}(\rho)$ for $\rho \neq 1$. If we add the diagrams with the virtual gluons, we would find also the terms proportional to $\delta(\rho - 1)$ in $P_{q \rightarrow q}(\rho)$.

The appropriate integral now is of the form

$$I \propto (p \cdot q) \int d\rho \left(\frac{1+\rho^2}{1-\rho} \right) \int \frac{d\beta d^2 k_T}{(p^2 - 2\beta p \cdot q)} \times$$

$$\times \delta(k^2) \delta((p+q-k)^2) T_r [\gamma^\mu (\rho \not{p} + \not{q}) \gamma^\nu \not{p}] \quad (4.26)$$

where the two arguments of the δ functions are given by

$$k^2 = 2\rho\beta(p \cdot q) - \underline{k}_T^2 \quad (4.27a)$$

and

$$(p+q-k)^2 = 2(p \cdot q)(1-\rho) \quad (4.27b)$$

Any structure function (F_1 , F_2 , or F_3) is now given by

$$F \propto g^2 \int d\rho \frac{1}{\rho} P_{q \rightarrow q}(x) \delta(p-x) \int \frac{d^2 k_T}{[p^2 + \underline{k}_T^2/\rho]} \quad (4.28)$$

$$\propto g^2 \int d\rho P_{q \rightarrow q}(x) \delta(p-x) \int_{p^2}^{q^2} \frac{dt}{t}$$

Although (4.28) can obviously be simplified, we leave it in this form because the higher order diagrams will give us contributions which are simple generalizations of this form. Since the moments of the Altarelli-Parisi function $P_{q \rightarrow q}$ are proportional (2.65) to the anomalous dimensions of the twist-two non-singlet operators, we see that (4.28) agrees with the N dependence of the g^2 term in (4.7).

4.2.4 Higher order diagrams and the asymptotic result

Similar features to those in the one-loop diagrams also appear in the general case^{*)}. The general feature is that the dominant diagrams in the transverse gauges are the generalized ladder diagrams (Fig. 33), i.e. ladder diagrams with vertex and self-energy insertions (we denote all these insertions by a cross at the vertex). Gribov and Lipatov⁷¹⁾ have taught us how to evaluate such diagrams and that the dominant region of integration is where

$$p^2 \ll k_1^2 \ll k_2^2 \ll \dots \ll k_n^2 \ll q^2 \quad (4.29)$$

*) For comprehensive discussions of this general case, see for example Refs. 2, 69, 70, and references therein.

One of the effects of the vertex and self-energy insertions is that at each vertex we should put $g(t_i)$, where $t_i = k_i^2$ is the largest four momentum squared at the vertex. This may seem surprising in view of the fact that the momenta in the different legs are not equal (see Fig. 34). We can choose, however, to associate each self-energy insertion of the fermion lines with the vertex below the insertion, instead of the usual association of half the self-energy insertion with the vertex above and half with the vertex below. In this way we see that we should write $g(t_i)$ at each vertex.

We now outline the calculation of such a set of diagrams. We start by doing the transverse momentum integrals, which we can rewrite in terms of the t_i analogous to (4.28). The only difference is now that for each t_i we have a factor of $\alpha(t_i) \sim 1/2\beta_0 \log t_i/\Lambda^2$, so that we have

$$I_T = \int_{p^2}^{q^2} \frac{dt_n}{t_n} \alpha_s(t_n) \dots \int_{p^2}^{t_3} \frac{dt_2}{t_2} \alpha_s(t_2) \int_{p^2}^{t_2} \frac{dt_1}{t_1} \alpha_s(t_1) \quad (4.30)$$

Thus, from the transverse momentum integrals we find a term proportional to*)

$$I_T \propto \frac{1}{n!} \left(\ln \ln q^2/\Lambda^2 \right)^n \left(\frac{1}{2\beta_0} \right)^n \quad (4.31)$$

Now we have to calculate the integral over the longitudinal component of momentum. This takes the form

$$\begin{aligned} & \int_x^1 dx_1 P_{q \rightarrow q}(x_1) \int_x^{x_1} \frac{dx_2}{x_1} P_{q \rightarrow q}\left(\frac{x_2}{x_1}\right) \dots \int_x^{x_{n-2}} \frac{dx_{n-1}}{x_{n-2}} \times \\ & \times P_{q \rightarrow q}\left(\frac{x_{n-1}}{x_{n-2}}\right) \int \frac{dx_n}{x_{n-1}} P_{q \rightarrow q}\left(\frac{x_n}{x_{n-1}}\right) \delta(x_n - x) \end{aligned} \quad (4.32)$$

The fraction of momentum p carried by quark i is x_i , and that by quark $i + 1$ is x_{i+1} , so that the fraction of the momentum of quark i which is carried by quark $i + 1$ is x_{i+1}/x_i . This is the reason for the arguments in the $P_{q \rightarrow q}$ functions in (4.32). Since (4.32) is a multiple convolution we take moments,

*) This result is obtained by performing the integrals in (4.30) and keeping the top limit of the integration range. Below we will come back to the terms obtained by keeping one or more of the bottom limits.

$$\int_0^1 x^{N-1} I_L(x) dx = \left[\int_0^1 dx P_{q \rightarrow q}(x) x^{N-1} \right]^n \propto (-\gamma_0^N)^n \quad (4.33)$$

Thus we find that for the Nth moment of the diagram of Fig. 33 the asymptotic behaviour is proportional to

$$\frac{1}{n!} \left(\ln \ln q^2/\Lambda^2 \right)^n \left(-\frac{\gamma_0^N}{2\beta_0} \right)^n \quad (4.34)$$

whence on summing over n we reproduce the standard deep inelastic result

$$M_N(q^2) \propto \left(\ln q^2/\Lambda^2 \right)^{-\gamma_0^N/2\beta_0} \quad (4.35)$$

We would like to make a few comments concerning this calculation. First of all, we notice that the corrections to (4.31), obtained by keeping a term which involves at least one of the lower limits, are of the form

$$\frac{1}{(n-1)!} \left(\ln \ln q^2/\Lambda^2 \right)^{n-1} + \dots \quad (4.36)$$

and it may therefore seem that the corrections to (4.35) should be suppressed by one or more powers of $\log \log q^2/\Lambda^2$, rather than powers of $\log q^2/\Lambda^2$ which we know to be correct (see Lecture 3). But this argument is not valid, since if we keep the lower limit in one of the integrals of (4.30), say the t_i integral, then we keep a term from the first j integrals which is q^2 independent. If for $j \leq i \leq n$, we now keep the upper limit in the t_i integrations, we generate a term proportional to

$$\frac{1}{(n-j)!} \left(\ln \ln q^2/\Lambda^2 \right)^{n-j}$$

whence, summing over n from j to infinity after having performed the integrals over the longitudinal components of momenta, we obtain a new contribution of the form (4.35). This just means that we are in effect including the part of the diagram in which the transverse momenta are finite in the soft wave function f and not in σ_{parton} . Of course we are not able to calculate the constant of proportionality in (4.35); this is analogous to not being able to calculate the operator matrix elements.

Another relevant question is: Why are we entitled to substitute the asymptotic form for $\alpha(t_1)$ in (4.30), when we perform the integral from a finite mass p^2 ? This is so because in the leading logarithm approximation, we could equally well have taken the integrals from Xp^2 , where X is a large fixed number, such that $\alpha(Xp^2)$ is well approximated by its asymptotic form.

Finally, we would like to comment about the fact that, although the dominant region of phase space is where the transverse momenta k_i^2 are $\ll q^2$, nevertheless, the k_i do grow with q^2 . For example, the average transverse momentum of a quark jet is predicted to behave like

$$\langle \underline{k}_T^2 \rangle \sim \alpha_s(q^2) \cdot q^2 \quad (4.37)$$

So we have to abandon the notion of fixed k_T jets, not only in deep inelastic lepton-hadron scattering, but also in other processes. Perhaps the most conclusive evidence that this is so experimentally comes from e^+e^- annihilation at PETRA¹³⁾.

We are now in a position to understand the interesting result of Witten⁷²⁾ about the structure function of a photon. In the transverse gauges the dominant diagrams are again the generalized ladder diagrams^{73,74)} (see Fig. 35). The only difference is at the bottom of the diagram, where we now have two electromagnetic vertices. These vertices are hard vertices, so that we can have large values of t_1 , but we have no factor of $\alpha_s(t_1)$. Thus, the transverse momentum integral analogous to (4.30) is now

$$I_T^\gamma = \int_{p^2}^{q^2} \frac{dt_n}{t_n} \alpha_s(t_n) \dots \int_{p^2}^{t_3} \frac{dt_2}{t_2} \alpha_s(t_2) \int_{p^2}^{t_2} \frac{dt_1}{t_1} \quad (4.38)$$

which can be evaluated to give

$$I_T^\gamma \propto \ln Q^2 \left(\frac{1}{2\beta_0} \right)^n \quad (4.39)$$

We obtain this result when we keep the upper limit in all the integrals of (4.38). If somewhere we keep a lower limit, we get terms of the form which we encountered in the hadronic case, i.e. (4.31). The N th moment of the integral over longitudinal momenta is now equal to [compare with (4.33)]

$$\left(\gamma_0^N \right)^{n-1} f^N \quad (4.40)$$

where f^N is the Nth moment of $P_{\gamma \rightarrow q}(x)$ ⁷⁵⁾, the probability density of finding a quark in a photon with a fraction x of its momentum. We now sum over n , as in the hadronic case, but now we have a geometric series which we sum to find that the Nth moment of the structure function of a photon is given by

$$\ln q^2 \rightarrow \frac{f^N}{1 + \frac{\delta_0^N}{2\beta_0}} \quad (4.41)$$

Witten first obtained this result by using the operator product expansion ⁷²⁾. In addition to the usual twist-two operators, we now have operators based on $F^{\mu\nu}$, the electromagnetic field-strength tensor. It is these operators which lead to the behaviour (4.41). These predictions will be tested in future e^+e^- experiments at PEP and PETRA.

In a fixed point theory with a small ultraviolet stable fixed point, the structure functions of hadrons are still dominated by the diagrams of Fig. 33, but now in the analogous integral to (4.30) we have no $\alpha_s(t_i)$ factors. This means that we can immediately deduce the result; we just make the substitution $\log \log q^2/\Lambda^2 \rightarrow \log q^2/\Lambda^2$ and hence the Nth moment of a structure function in such a theory has a q^2 dependence given by

$$(q^2)^{-\delta_0^N/2\beta_0} \quad (4.42)$$

Thus, in such theories the violations of Bjorken scaling are given by powers of q^2 .

4.3 Asymptotic predictions for other hard scattering processes

Once we understand the ideas and calculations of Section 4.2 for the deep inelastic case, it is conceptually fairly straightforward to generalize the ideas to other hard scattering processes. We start by giving a few examples.

4.3.1 The Drell-Yan process

We now consider the inclusive production of a pair of leptons with large invariant mass in hadronic collisions, the Drell-Yan process ⁷⁶⁾. Sample diagrams which contribute to this process are shown in Fig. 36. Let us denote the lowest order contributions to the cross-section for the subprocess $q\bar{q} \rightarrow \ell^+\ell^-x$ (that corresponding to Fig. 36a) by σ_0 . Then when all the radiative corrections of order g^2 are calculated (including those of Fig. 36b and c) in the leading logarithm approximation, one finds ⁷⁷⁾ that these corrections are equal to

$$a(\tau) g^2 \left[\ln \frac{Q^2}{p_1^2} + \ln \frac{Q^2}{p_2^2} \right] \quad (4.43)$$

where $\tau \equiv Q^2/s$. It turns out that $a(\tau)$ is a very significant function; it is exactly the same function which appears in the order g^2 corrections to deep inelastic scattering on a quark. The structure function of a quark is proportional to

$$\delta(x-1) + a(x) g^2 \ln Q^2/p^2 + O(g^4 \ln^2 Q^2/p^2) \quad (4.44)$$

Thus, to this order, at least, it seems that the deviations from the naive Drell-Yan picture are intimately related to the violations of Bjorken scaling in deep inelastic scattering. In other words, the diagrams which spoil the simple Drell-Yan picture (such as those of Figs. 36b and c) are related to those which are responsible for the violation of Bjorken scaling (such as those of Figs. 34a and b). In particular, to this order we can write a modified Drell-Yan formula

$$\begin{aligned} \frac{d\sigma}{dQ^2} &= \frac{4\pi\alpha^2}{9Q^4} \sum_{\substack{\text{quark} \\ \text{flavours } a}} e_a^2 \int_0^1 dx_1 dx_2 x_1 x_2 \delta(x_1 x_2 - \tau) \times \\ &\times \left[G_{q a/h_1}(x_1, Q^2) G_{\bar{q} a/h_2}(x_2, Q^2) + G_{\bar{q} a/h_1}(x_1, Q^2) G_{q a/h_2}(x_2, Q^2) \right] \end{aligned} \quad (4.45)$$

where the G distribution functions are the appropriate linear combinations of experimentally determined deep inelastic structure functions.

This result is easy to understand in transverse gauges, because in these gauges it is again the generalized ladder diagrams (Fig. 38) which dominate. In particular, in the planar gauge²⁾ $(p_1 + \alpha p_2) \cdot A = 0$, and in the leading logarithm approximation, the top of the diagram decouples from the bottom, and so we get (4.45) naturally. In other axial gauges we also have to consider the mass singularities in the vertex corrections at the electromagnetic vertex. Equation (4.45) is true to all orders of perturbation theory in the leading logarithm approximation^{2,78,79)}. The proof is analogous to the derivation of (4.35). Thus the large logarithms which appear in the Drell-Yan process turn out to be the same as those in deep inelastic scattering. So once we have "measured" the sum of these large logarithms in deep inelastic scattering we can use these measurements to make predictions for lepton-pair production.

We now turn our attention to other subprocesses which contribute to massive lepton pair production. As an example, let us consider $q\bar{q}$ scattering for which some lowest order diagrams are shown in Figs. 36d and e. When one calculates the contribution to the cross-section for the subprocess from the diagrams of Figs. 36d and e one finds⁸⁰⁾ that the result is proportional to

$$g^2 (1 - 2\tau(1-\tau)) \ln Q^2/p^2 \sigma_0 \quad (4.46)$$

Again the coefficient of $g^2\sigma_0$ is highly significant. It is the probability of finding an antiquark in a gluon with a fraction τ of its longitudinal momentum^{*)}. When the distribution functions are measured in deep inelastic scattering, they include the effects of gluons through diagrams such as those of Figs. 37c and d. Equation (4.46) implies that the contribution to the distribution functions from these diagrams is related to the contribution to the cross-section for lepton pair production from diagrams such as those of Figs. 36d and e. Thus, even these diagrams are included in the right-hand side of Eq. (4.45). In other words, the dominant contribution from the subprocess $qg \rightarrow \ell^+\ell^-x$ can be expressed as the probability of finding an antiquark in the gluon times the cross-section for this antiquark to annihilate the incident quark.

A similar feature has been found in all other subprocesses which have been studied (e.g. $qq \rightarrow \ell^+\ell^-x$, $gg \rightarrow \ell^+\ell^-x$ ⁸⁰⁾). In each case the leading logarithmic terms can be absorbed into the distribution functions of the incident hadrons, so that the asymptotic behaviour of the cross-section is given by (4.45), where the G 's are extracted from linear combinations of experimentally measured deep inelastic structure functions. Thus the dominant contribution can be represented by Fig. 39, where the shaded blocks represent these non-scaling distribution functions.

4.3.2 Inclusive production of particles and jets with large transverse momenta

Deep inelastic scattering and massive lepton pair production both involve an off-shell photon. Here we study a purely strong interaction process, the production of particles and jets with large transverse momenta in hadronic collisions. The mechanism usually believed to be responsible for this process can be summarized by Fig. 40, which can be expressed mathematically as

$$\begin{aligned}
 E_c \frac{d\sigma}{d^3p_c} (A+B \rightarrow C+X) &= \sum_{a,b;c,d} \int_0^1 dx_a \int_0^1 dx_b \frac{dx_c}{x_c^2} \times \\
 &\times G_{a/A}(x_a) G_{b/B}(x_b) \tilde{G}_{c/c}(x_c) \delta(s'+t'+u') \times \\
 &\times \left. \frac{s'}{\pi} \frac{d\sigma^{(B)}}{dt'} (a+b \rightarrow c+d) \right|_{\substack{s' = x_a x_b s \\ t' = x_a/x_c t ; u' = x_b/x_c u}}
 \end{aligned} \tag{4.47}$$

*) The moments of $1 - 2\tau(1 - \tau)$ with respect to τ are thus proportional to one of the off-diagonal elements in the anomalous dimension matrix of the lowest-twist quark and gluon operators.

where $\tilde{G}_{C/c}(x_c)$ is the probability of constituent c to fragment into particle C , C carrying a fraction x_c of the longitudinal momentum of c . To see what, if anything, QCD can teach us about this process, let us study a concrete example: quark (A) + quark (B) $\rightarrow$ quark (C) + anything, up to order g^6 in the cross-section and, as before, in the leading logarithm approximation⁸¹⁾. For simplicity we start with the case $x < 1$, where $x = (2E_c/\sqrt{s})$, in which case only the inelastic diagrams of Fig. 41 contribute. We notice that even in this order there are diagrams with a three-gluon vertex.

There are a number of kinematic regions which contribute to the leading logarithmic behaviour. All these contributions can be interpreted in an elegant way. From the region where k is parallel to $p_A(p_B)$ we obtain a contribution to $E_c(d\sigma/d^3p_c)$ of the form $\log s/p_A^2 (\log s/p_B^2)$, which can be interpreted as being the convolution of the probability of finding a quark in quark A(B), and the Born term for quark-quark elastic fixed angle scattering (Figs. 42a and b). In other words, the coefficient of these logarithms is just the function "a" of Eqs. (4.43) and (4.44) (in spite of the presence of the three-gluon vertex). From the region where k is parallel to p_C , we obtain a contribution which can be interpreted as the convolution of the Born term for qq elastic fixed angle scattering, and the probability for one of the resulting quarks to fragment into the observed one + anything (Fig. 42c). The fragmentation function is just that which would be calculated from $e^+e^- \rightarrow qX$ and is related to the distribution function (in this order, at least) by the Gribov-Lipatov reciprocity relation⁸²⁾. The final dominant contribution comes from the region in which k balances the transverse momentum of C , and this can be interpreted as the probability of finding a gluon in quark B with a fraction of the longitudinal momentum of B^{*)} convoluted with the Born term for gq elastic fixed angle scattering (Figs. 42d, e, and f).

At $x \approx 1$, in addition to the contributions from the diagrams of Fig. 41, one must also include the diagrams which contribute to the elastic qq scattering amplitude. The leading $\log s/p^2$ contributions can also be absorbed into the initial distribution and final fragmentation functions. In addition to these $\log s/p^2$ terms, after renormalization there will also be $g^6(\mu^2) \log s/\mu^2$ terms. These terms are exactly those which, when combined with the $g^4(\mu^2)$ contribution from the Born term give the first two terms in the expansion of $g^4(s)$, the running coupling constant.

Thus we conclude that the cross-section for $qq \rightarrow qX$ to order g^6 in the cross-section and in the leading logarithmic approximation can be written in the form (Fig. 43)

*) The moments of this probability are, of course, proportional to an off-diagonal element in the anomalous dimension matrix for the lowest-twist quark and gluon operators.

$$\begin{aligned}
 E_c \frac{d\sigma}{d^3p_c} (A+B \rightarrow C+X) \\
 = \sum_{a,b,c,d} \int_0^1 dx_a dx_b \frac{dx_c}{x_c^2} G_{a/A}(x_a, Q^2) G_{b/B}(x_b, Q^2) \tilde{G}_{c/C}(x_c, Q^2) \times \\
 \times \delta(s' + t' + u') \frac{s'}{\pi} \frac{d\sigma^{(B)}}{dt'} (a+b \rightarrow c+d) \Bigg|_{\substack{s' = x_a x_b s \\ t' = x_a/x_c t \\ u' = x_b/x_c u}}
 \end{aligned} \tag{4.48}$$

where $d\sigma^{(B)}/dt'$ is the Born term contribution to the elastic cross-section for qg or qq scattering calculated using the running coupling constant, and Q^2 represents a typical large invariant mass squared (we assume $s \sim t \sim u \sim Q^2$). In the leading logarithm approximation we cannot distinguish between $\log s$, $\log t$ and $\log u$. It is interesting to see how the different subprocesses are interrelated: starting with the quark-quark scattering we have found that we also need to consider quark-gluon scattering. If we had started off by looking at radiative corrections to quark-gluon scattering, we would have found that we also need to consider the third subprocess, gluon-gluon scattering. Again, general arguments such as those in Ref. 78 ensure the validity of (4.48) to all orders of perturbation theory in the leading logarithm approximation.

Studies, such as the two examples we have discussed above, show that for all hard scattering processes of the type considered here (i.e. those with the three mass scales Q^2 , p^2 , and μ^2), the ansatz for calculating the asymptotic behaviour of hard scattering cross-sections as $Q^2 \rightarrow \infty$ is:

- a) take the parton model hard scattering formula;
- b) replace the scaling distribution and fragmentation functions by the appropriate non-scaling ones;
- c) keep only the contributions of lowest order in the coupling constant for the hard scattering subprocesses, calculated using the running coupling constant.

Equations (4.45) and (4.47) are specific examples of this ansatz. For other examples, see Ref. 83 and the references therein.

We now understand how to calculate the leading asymptotic prediction for these hard scattering processes. In the next lecture we will see that all the logarithmic corrections are also calculable.

5. BEYOND LEADING LOGARITHMS

5.1 Introduction

In Lecture 3 we saw how to calculate logarithmic corrections to the predictions for the scaling violations in deep inelastic structure functions. In this lecture we will outline how one would approach the problem of calculating these logarithmic corrections to lowest order formulae [such as Eqs. (4.45) and (4.49)] for other processes, processes in which we relate the cross-section in question to deep inelastic structure functions or e^+e^- fragmentation functions.

5.2 A simple example: The longitudinal structure function

As a simple example of a hard scattering process, let us take the longitudinal structure function F_L . In the parton model (with spin $\frac{1}{2}$ quarks) this structure function decreases with q^2 like a power of q^2 ; e.g. under certain assumptions the ratio F_L/F_T is given by⁸⁴⁾

$$F_L/F_T = \frac{4 \langle k_T^2 \rangle}{(-q^2)} \quad (5.1)$$

where $\langle k_T^2 \rangle$ is the average transverse momentum squared of the quarks in the hadron. To see that $F_L \sim 1/q^2$ in the parton model one has to calculate the contribution from Fig. 44a.

In QCD, however, there are contributions from higher order diagrams, such as those of Figs. 46b and c, which scale to within logarithmic corrections. Thus, using the ansatz at the end of Lecture 4 we would write for the leading term asymptotically

$$F_L(q^2, x) = \sum_{f=q, \bar{q}, g} \int_x^1 \frac{dy}{y} f(y, q^2) F_{L,f}^{(0)}(q^2, \frac{x}{y}, g(q^2)) \quad (5.2)$$

where f runs over all quarks and gluons and is defined by means of, for example, F_T and $F_{L,f}^{(0)}$ is the lowest order contribution to the longitudinal structure function of f , evaluated using the running coupling constant. This can easily be checked to be the same answer as would be obtained by the light-cone techniques of Lecture 2. Even though in QCD F_L/F_T and $\langle k_T^2 \rangle/q^2$ are both logarithmically suppressed, relation (5.1) is no longer valid.

We would like to calculate the next-to-leading corrections to Eq. (5.2), and to do this we start by calculating both F_L and F_T in perturbation theory^{*)}. We start by calculating F_L , and for simplicity we will restrict ourselves to the

*) For a light-cone derivation of the relation between F_L and F_T beyond leading logarithms see Ref. 83.

non-singlet quantity $F_L^{\text{up}} - F_L^{\text{un}}$, the generalization to F_L^{up} itself being straightforward. The lowest order diagram is shown in Fig. 45a and its contribution can be written

$$\alpha_s \int dk^2 d^2 \underline{k}_T \int_x^1 \frac{dy}{y} f(y, k^2, \underline{k}_T^2) F_L^{(0)}\left(\frac{x}{y}\right) \quad (5.3)$$

where f is defined as in Lecture 4 and $F_L^{(0)}$ is the contribution from the lowest order diagram (Fig. 45a) to the longitudinal structure function of a quark. We now have to calculate two-loop diagrams such as that of Fig. 45b. The extra loop integration will give us a logarithmic term of the type $\log q^2/k^2$ and a constant term. The contribution from the two-loop diagrams can be written in the form

$$\begin{aligned} & \frac{\alpha_s^2}{2\pi} \int dk^2 d^2 \underline{k}_T \int_x^1 \frac{dy}{y} f(y, k^2, \underline{k}_T^2) \int_x^y \frac{dz}{z} \times \\ & \times \left[P_{q \rightarrow q}\left(\frac{z}{y}\right) F_L^{(0)}\left(\frac{x}{z}\right) \ln \frac{q^2}{k^2} \right] \\ & + \frac{\alpha_s^2}{2\pi} \int dk^2 d^2 \underline{k}_T \int_x^1 \frac{dy}{y} f(y, k^2, \underline{k}_T^2) F_L^{(1)}\left(\frac{x}{y}\right) \end{aligned} \quad (5.4)$$

where $F_L^{(1)}$ is the constant term (the term without the $\log q^2$) in the contribution to the longitudinal structure function of a quark from all the two-loop diagrams such as that of Fig. 44d. The first term on the right-hand side of (5.4) has, of course, precisely the parton-model-like interpretation of the leading logarithms discussed in Lecture 3.

We now have to calculate F_T to the same accuracy. We start by calculating the lowest order diagram, that of Fig. 46a and we find for the contribution to F_T :

$$\int dk^2 d^2 \underline{k}_T f(x, k^2, \underline{k}_T^2) \quad (5.5)$$

From the one-loop diagrams, such as those of Fig. 46b we find for the contribution to F_T :

$$\begin{aligned} & \frac{\alpha_s}{2\pi} \int dk^2 d^2 \underline{k}_T \int_x^1 \frac{dy}{y} f(y, k^2, \underline{k}_T^2) P_{q \rightarrow q}\left(\frac{x}{y}\right) \ln \frac{q^2}{k^2} \\ & + \frac{\alpha_s}{2\pi} \int dk^2 d^2 \underline{k}_T \int_x^1 \frac{dy}{y} f(y, k^2, \underline{k}_T^2) F_T^{(1)}\left(\frac{x}{y}\right) \end{aligned} \quad (5.6)$$

where $F_T^{(1)}$ is the constant term in the contribution from the one-loop diagrams to the transverse structure function of a quark (such as that of Fig. 44b).

Since Eqs. (5.3) to (5.6) contain convolutions it is convenient to take the moments with respect to x , and we denote the N th moment of the longitudinal and transverse structure functions by F_L^N and F_T^N , respectively. It is possible now to eliminate f (of which we are ignorant) and we find

$$\frac{F_L^N(q^2)}{F_T^N(q^2)} = \alpha_s(q^2) F_L^{N(0)}(q^2) + \frac{\alpha_s^2(q^2)}{2\pi} F_L^{N(1)}(q^2) - \frac{\alpha_s^2(q^2)}{2\pi} F_L^{N(0)}(q^2) F_T^{N(1)}(q^2) \quad (5.7)$$

which is the final result. It can also be obtained by light-cone techniques⁸³).

5.3 General discussion

For (5.7) to be valid, the "next-to-leading" logarithms in all orders of perturbation theory must arrange themselves in a particular way, again generating the series (4.7) in both F_L and F_T . That this is the case for general hard scattering processes has been proved in (for example) Refs. 70, 78, and 79. In view of the discussion of Lecture 4 this is not so surprising, because whereas we saw that the leading logarithms in the axial gauge come from "ladder" diagrams, the next-to-leading logarithms come from modified ladder diagrams in which two of the rungs have been crossed (which are common to all hard processes and therefore do not change the leading order formulae) or from non-"collinear" regions of integration in the innermost loop. This last effect is exactly taken into account by calculating the constant term (i.e. the term independent of $\log q^2$) in the lowest-but-one order of perturbation theory, in the way discussed for F_L in Section 5.2. Thus for a general hard scattering process, in order to obtain the expression for the cross-section σ to next-to-leading order, one must calculate the constant terms in the lowest order but one for the cross-section for the hard scattering subprocesses ($\Delta\tilde{\sigma}$), and also the corrections to the appropriate quark and gluon distributions (Δf , $f = q, g, \bar{q}$). The correction is then the appropriate difference of terms involving the $\Delta\tilde{\sigma}$ and Δf [analogously to (5.7), except that in general there are more than one $\Delta\tilde{\sigma}$ and Δf] at the level of moments. Notice that because the quarks and gluons are not on mass-shell $\Delta\tilde{\sigma}$ and Δf separately are not gauge invariant, it is only the difference which is physically meaningful.

In addition to including non-leading terms from dominant subprocesses, for consistency one must also include subprocesses which start in the next-to-leading order in α_s . The discussion of Sections 5.2 and 5.3 can be easily generalized to still less leading logarithms.

Recently the $O(\alpha_s)$ corrections to the Drell-Yan process have been calculated⁸⁵⁻⁸⁹), and they turn out to be very significant. There are several sources of renormalization of the basic cross-section (4.45):

$$\begin{aligned}
 \frac{d\sigma}{dQ^2} = & \frac{4\pi\alpha^2}{3.3.Q^4} \sum_{\substack{\text{quark} \\ \text{flavours } a}} e_a^2 \int_0^1 dx_1 dx_2 x_1 x_2 \delta(x_1 x_2 - \tau) \times \\
 & \times \left[G_{q^a/h_1}(x_1, Q^2) G_{\bar{q}^a/h_2}(x_2, Q^2) + G_{\bar{q}^a/h_1}(x_1, Q^2) G_{q^a/h_2}(x_2, Q^2) \right] \\
 & \times \left[1 + \frac{\alpha_s(Q^2)}{2\pi} \frac{4}{3} \left(1 + \frac{4\pi^2}{3} \right) \right] \\
 & + \sum_a \left[\int_{x_1}^1 dy_1 \int_{x_2}^1 dy_2 G_{q^a/h_1}(y_1, Q^2) G_{\bar{q}^a/h_2}(y_2, Q^2) \sigma(\alpha_s(Q^2)) + (1 \leftrightarrow 2) \right]^{(5.8)} \\
 & + \sum_a \left[\int_{x_1}^1 dy_1 \int_{x_2}^1 dy_2 \left\{ G_{q^a/h_1}(y_1, Q^2) G_{g/h_2}(y_2, Q^2) " \sigma(\alpha_s(Q^2)) " \right. \right. \\
 & \quad \left. \left. + (q^a \leftrightarrow \bar{q}^a) \right\} + (1 \leftrightarrow 2) \right] \\
 & + O((\alpha_s(Q^2))^2)
 \end{aligned}$$

Notice that in the first line of Eq. (5.8) the factor of $1/3$ from colour has been exhibited explicitly. The factor of $[1 + (\alpha_s/2\pi)(4/3)(1 + 4\pi^2/3)]$ is the renormalization of the $q\bar{q}$ annihilation diagram of Fig. 39 due to vertex corrections and soft gluon corrections as in Figs. 36b and c. The part $\propto \pi^2$ comes directly from the continuation from space-like to time-like Q^2 of $[\ln(Q^2/p^2)]^2$ terms when one crosses from deep inelastic scattering to the Drell-Yan process. This correction factor is very large. For example, if we take $\alpha_s = 0.37$, a not unreasonable value in the presently accessible range of Q^2 , then this correction is $1 + (1.1)$. Since the first-order radiative correction is so large at present Q^2 , we cannot legitimately expect that the higher order radiative corrections are negligible. Also we do not really know what is the best value of the effective α_s to put into formula (5.8) until higher order corrections are computed and tell us whether there is a suitable effective momentum scale $\tilde{Q}^2 \neq Q^2$ at which to evaluate α_s . It is clearly very important to try to get some handle on these higher order terms. In particular, it may be that all or some of the large π^2 terms exponentiate. [We may note facetiously that $e^{1.1} \approx 3$, neatly cancelling the colour factor in the first line of (5.8).] The next term in (5.8) comes from $q\bar{q}$ scattering with hard

gluon bremsstrahlung as in Fig. 36b, where one or the other of the incoming q and $\bar{q}$ starts with a longitudinal momentum fraction $y > x$. The magnitude of this term relative to the lowest order diagram depends on the Q^2 and longitudinal momentum fraction of the observed lepton pair, as well as on the projectile and target used. It can be of comparable magnitude to the lowest order term in certain kinematic situations⁸⁶⁾. The gluon-quark term in (5.8) comes from graphs like that in Figs. 36d and e, where the leading logarithm has been subtracted in line with the discussion in Section 3.1. After this subtraction the "cross-section" may be negative, which accounts for the quotation marks in Eq. (5.8). Calculations⁸⁶⁾ suggest that this term is not of large magnitude compared to the lowest Drell-Yan cross-section. Finally, buried in the $O(\alpha_s)^2$ term in (5.8) are terms such as that due to quark-quark scattering as in Fig. 41. Unfortunately, a meaningful calculation of the qq contribution is difficult unless all diagrams of $O(\alpha_s)^2$ are computed, and this is a difficult task.

We have seen that in QCD one expects the lowest order Drell-Yan cross-section (4.45) represented in Fig. 39 to be strongly renormalized, probably in the upwards direction. Recent experiments indeed seem to find a larger cross-section than expected from the naive cross-section (4.45). In view of the theoretical uncertainties due to yet higher orders of perturbation theory, it is not clear that these observations can be related quantitatively to the next-to-leading order QCD calculations.

5.4 Interactions involving "spectator" partons⁹⁰⁾

In the previous sections (with the exception of Section 2.3), we calculated predictions for hard scattering processes by evaluating low-order Feynman diagrams for quark and gluon subprocesses. We have not considered diagrams such as those of Fig. 47, which involve the interactions of spectator partons. By assuming that the hadronic wave function f is soft, one can argue that these diagrams are logarithmically suppressed, because the softness of the wave function prevents the loop integral diverging so that there is no mass singularity logarithm, compared to the leading ones discussed in Lecture 4. But that is all -- if we want to calculate the logarithmic corrections we must study these diagrams some more. Moreover, the contributions of these diagrams are gauge dependent. In the case of deep inelastic scattering the light-cone analysis guarantees that Eq. (2.56) is correct, irrespective of the contributions of diagrams such as Fig. 47a. In a general gauge, the effects of such diagrams for deep inelastic scattering are included in the operator matrix element, and the higher order radiative corrections again generate the series (4.7). However, *a priori* we have no such guarantee that diagrams involving spectator quarks for other hard scattering processes, such as Fig. 47b, are negligible [i.e. $O(m^2/Q^2)$]. We shall show in this section that this is nevertheless the case.

For definiteness let us take a simple ϕ^3 model for the wave function. We take an interaction term of the type $\lambda\phi\chi^+\chi$, where ϕ and χ are scalar fields, with ϕ being both electrically neutral and a colour singlet, whereas χ is a charged and coloured field belonging to the fundamental representation of colour SU(3). Thus ϕ is our model for the "hadron" and the χ 's are our "quarks". The quarks interact in the usual way with the coloured gluons, which belong to the adjoint representation. λ has dimensions of mass and therefore the $\phi\chi^+\chi$ vertex is soft. We believe that this is the only relevant assumption, the further details of the model (such as whether the quarks have spin 0, as in this case, or spin $\frac{1}{2}$) are not relevant. The advantage of working in a specific simple model is that we can obtain explicit answers for the Feynman diagrams. In particular, we can see how gauge invariance is restored between diagrams involving only the "participating" quarks and those which also involve "spectator" quarks.

In order to try and get some insight into the problem, we start by calculating some low-order diagrams. For the moment we will work in the Feynman gauge. The lowest order diagram is that of Fig. 48a and gives an exactly scaling contribution. Some sample diagrams contributing to the next order are shown in Figs. 48b, c, and d. The dominant contribution from Fig. 48b has a $\log q^2/p^2$ term (analogously to the diagram of Fig. 30), which comes from the usual "collinear" region of integration corresponding to $p^2 \ll k_T^2 < \epsilon q^2$. The transverse momentum is allowed to get as large as ϵq because in the loop marked with the arrow in Fig. 48b there are no soft vertices. This is not the case in Figs. 48c and d, and we find that for these diagrams we get a scaling contribution, i.e. no logarithms. This is because there is no loop in which the transverse momentum could get large without flowing through a soft vertex. So, in this order of perturbation theory, the contributions from diagrams which include interactions involving spectator quarks are suppressed by only one logarithm.

Let us now study the diagram of Fig. 48c in some detail. An analysis of the dominant momentum flow in the diagram (using, for example, the techniques of Refs. 91 and 92) shows that, as is marked on the diagram, the momentum flowing in the vertical propagators is proportional to p , whereas the two top horizontal propagators are hard. Comparing this to the lowest order scaling diagram of Fig. 48a we now see that we have:

- a) 1 extra hard propagator giving a factor $\sim 1/p \cdot q$;
- b) 1 factor from the numerator which is $\sim p \cdot q$.

Thus the diagram of Fig. 48c gives a scaling contribution. We notice that to obtain the factor of $p \cdot q$ in the numerator, we have had to take a factor of p^λ from the convection current corresponding to the gluon vertex at the bottom of the diagram (the only large momentum available at the bottom is proportional to p)

and therefore a q^λ from the convection current corresponding to the gluon vertex at the top of the diagram. If we now choose to work in the gauge

$$q \cdot A = 0 \quad (5.9)$$

[or indeed $(q + \alpha p) \cdot A = 0$, where α is an arbitrary constant], it is easy to see that we do not get the power of $p \cdot q$ in the numerator. Thus the diagrams involving spectator quark interactions are gauge-dependent and in some gauges contribute to the "leading-twist" terms and in other gauges do not.

Before trying to understand this feature, let us look at the Drell-Yan process, $h_1 h_2 \rightarrow \ell^+ \ell^- X$ and study some of the diagrams involving interactions of spectator quarks, e.g. those of Figs. 49b and c. We leave the diagram of Fig. 49c till later, and start by looking at that of Fig. 49b, which has many similar features to that of Fig. 48c in the deep inelastic case. In the dominant region of phase space, in the Feynman gauge this diagram also has

- i) one more hard propagator and hence a factor of $\sim 1/s$; and
- ii) one extra numerator and hence a factor of $\sim s$

compared to the lowest order diagram, that of Fig. 49a. So in the Feynman gauge the diagram of Fig. 49b gives a Drell-Yan scaling contribution, and must be considered. However, also in this Drell-Yan case, we can define an axial gauge

$$(p_1 + \beta p_2) \cdot A = 0 \quad (5.10)$$

where β is an arbitrary constant, in which the diagram of Fig. 49b is suppressed by a factor of s .

We notice that in both the deep inelastic and Drell-Yan cases, the gauge in which the diagrams involving spectator parton interactions are negligible (e.g. Figs. 48c and 49b) depends on the external momenta. Can we find a single gauge in which both the diagrams of Figs. 48c and 49b are negligible? To this end we define

$$q_1 = \frac{1}{x_1} y_1 (p_2 - x_1 p_1) \quad (5.11a)$$

and

$$q_2 = \frac{1}{x_2} y_2 (p_1 - x_2 p_2) \quad (5.11b)$$

so that

$$\frac{-q_1^2}{2p_1 \cdot q_1} = x_1, \quad \frac{-q_2^2}{2p_2 \cdot q_2} = x_2 \quad (5.12)$$

and y_1, y_2 are arbitrary constants. Now, if we calculate the deep inelastic structure functions in the cases $q = q_1, p = p_1$ and $q = q_2, p = p_2$ in the gauge (5.9), we see that there is no contribution from the diagram of Fig. 48c. Hence in the gauge (5.10) we can simultaneously eliminate the "unwanted" diagrams of Fig. 48c in the deep inelastic scattering case, and that of Fig. 49b in the Drell-Yan case. We also notice from (5.11) that although $q_1^2(q_2^2)$ is proportional to s , the constant of proportionality depends on $y_1(y_2)$ and hence is not determined. This is a statement that these diagrams do not really contribute to the next-to-leading logarithms.

In the Feynman gauge, as we have seen above, both the diagrams of Figs. 48c and b contribute to the next-to-leading logarithms. These diagrams have many similar features and closer examination reveals that they are related by the Drell-Yan formula. Thus, when we write down this formula and insert the experimentally extracted parton distribution functions, we are including the contributions from diagrams such as Fig. 48c for these distribution functions, and hence the contributions of diagrams such as Fig. 49b for the Drell-Yan cross-section.

We are not finished yet; there is a class of diagrams which has not been included, such as that in Fig. 49c. This diagram has no analogue in the deep inelastic case. The dominant region of phase space is that in which k is orthogonal to p_1 and p_2 and is therefore space-like. This means that if we evaluate the contribution to the lepton-pair cross-section from this diagram by taking all the unitarity cuts, we can neglect any cuts through the gluon. This diagram is very reminiscent of that studied by De Tar et al.⁹³⁾, in which the gluon is replaced by a Pomeron. The conclusion of Ref. 93 was that the Pomeron diagram was suppressed by a factor $\sim 1/s$. When evaluating the diagram, since we are dealing with an inclusive cross-section, we have to take care with the $i\epsilon$'s in the propagators -- we are taking the Mueller discontinuity⁹⁴⁾. Having done this, it turns out that there is always an integration which can be shown to be zero by using Cauchy's theorem. In our case we have an extra singularity due to the pole in the gluon propagator; however, as stated above, the contribution from this pole is also suppressed by a factor of s . It is amusing to see how one obtains this answer of zero by taking unitarity cuts. In the Pomeron case, the Pomeron is imaginary and there are three cuts (see Fig. 50a) giving the relative contributions 1, -2, 1. In the gluon case, the gluon is real so there are just two cuts (see Fig. 50b) (remember the cut through the gluon is zero) giving the relative contributions 1, -1.

Thus we find that in lowest order perturbation theory, when we follow the prescription for calculating hard scattering cross-sections outlined in Lectures 3 and 4 above, there is no additional contribution from diagrams involving spectator partons. These diagrams are suppressed by a power of Q^2 , and this feature can be

generalized to higher orders of perturbation theory⁹⁰). It is straightforward to see that (in the Feynman gauge, for instance), there will be many contributions to the Drell-Yan cross-sections which have corresponding contributions to the deep inelastic structure functions and (somewhat less straightforwardly) it can be shown that interactions between "spectators" of different hadrons cancel out. We have used the Drell-Yan process as an example, but the above argument can easily be generalized to other hard scattering processes.

The arguments given in this Lecture lead us to believe that we now know how to calculate, not only the asymptotically leading contribution to the cross-section for hard scattering processes, but also all the logarithmic corrections.

6. EXCLUSIVE PROCESSES

6.1 Introduction

All the discussion of the previous sections has concerned inclusive reactions; but in the past few months there has been some very interesting progress in the application of QCD to exclusive processes⁹⁵⁾, and in this lecture we will briefly review these developments. Before doing this, however, let us recall how one approached exclusive processes within the context of the parton model.

It is popular, within the parton model, to think of hard exclusive processes in terms of "dimensional counting rules"^{96,97)}. For example, for the spin averaged form factor of a hadron H, these give

$$F_H(t) \sim \frac{1}{t^{N-1}} \quad (6.1)$$

where $t = q^2$ and N is the number of fundamental constituents in H. Thus the form factor of a pion is predicted to behave asymptotically as $1/t$, and that of a proton as $1/t^2$, both of which agree with the data, which exist for $|t|$ up to 4 (30) GeV^2 for the pion (proton). We can see heuristically that (6.1) is correct by counting the "hard fermion propagators" (see Fig. 51). We iterate the hadronic wave function, so that the large transverse momentum is routed through the wiggly lines, which represent gluons. In deriving the dimensional rules, it is assumed that the four-quark interaction is scale invariant. We see now that in the meson form factor there is at least one hard fermion propagator ($\sim 1/t$), and in the proton case there are at least two such propagators ($\sim 1/t^2$). The generalization to arbitrary N is trivial.

For elastic scattering at fixed angle the dimensional counting rules are

$$\frac{d\sigma}{dt} (A+B \rightarrow C+D) \sim \frac{1}{t^{N-2}} f(\theta) \quad (6.2)$$

where N is the total number of constituents in A, B, C, and D, and θ is the scattering angle. Thus $d\sigma/dt$ is predicted asymptotically to behave like $1/t^6$, $1/t^8$, and $1/t^{10}$ for meson-meson, meson-baryon and baryon-baryon scattering, respectively. These predictions seem to be consistent with the experimental data. To see that (6.2) is reasonable we repeat the "hard fermion" counting arguments which we used in the form factor case. For example, in Fig. 52 we present some sample diagrams which contribute to meson-meson scattering at large angle. The lines marked with a cross represent hard quark propagators, and we see that in each diagram there are two such propagators. Thus the amplitude $\sim 1/t^2$ and hence $d\sigma/dt \sim 1/t^2 (1/t^2)^2 \sim 1/t^6$ which satisfies (6.2). The generalization to higher N is straightforward.

There is one further complication. Landshoff⁹⁸⁾ proposed that instead of diagrams such as those in Fig. 52, in which there are "hard fermion" propagators, the relevant mechanism for fixed angle scattering may contain no such propagators at all. For example, he suggested that meson-meson scattering may be dominated by two separate scatterings of the two valence quarks through the same angle (see Fig. 53a). Although there are no hard fermion propagators to suppress the amplitude, there is little phase space for both quarks to scatter through the same angle. Calculation of the diagram of Fig. 53 reveals that it contributes to the amplitude $M_{\pi\pi}$ a term

$$M_{\pi\pi} \sim \frac{1}{t^{3/2}} \quad (6.3a)$$

which leads to

$$\left. \frac{d\sigma}{dt} \right|_{\pi\pi} \sim \frac{1}{t^5} \quad (6.3b)$$

which should dominate asymptotically over the dimensional counting contribution which behaves like $1/t^6$. Similarly the analogous "multiple scattering" contribution in the proton-proton case gives a contribution

$$\left. \frac{d\sigma}{dt} \right|_{pp} \sim \frac{1}{t^8} \quad (6.4)$$

which dominates over the dimensional counting contribution which gives $1/t^{10}$. The meson-baryon case is not so clear⁹⁹⁾.

In any renormalizable field theory we expect there to be at least some logarithmic modifications to Eqs. (6.1) to (6.4). We shall see that this is indeed the case in QCD.

6.2 Pion form factor in QCD

In this section we review the approach and results of Brodsky and Lepage^{95,100)} concerning the pion form factor in QCD, which is the most complete discussion of the problem. Other contributions to this problem can be found in Refs. 101 to 104. We will study the form factor in time-ordered perturbation theory in the infinite momentum frame. The meson Fock state ψ can be represented as a column vector with infinitely many components, corresponding to $q\bar{q}$, $q\bar{q}g$, $q\bar{q}q\bar{q}$, etc., states. It satisfies the bound state equation

$$\psi = SK\psi \quad (6.5)$$

which is represented in Fig. 54: S is a diagonal, infinitely dimensional, matrix whose elements are the multiparticle propagators, while K is the irreducible kernel.

The first step is to separate the hard and soft components in the wave function. To this end we introduce a mass cut-off λ , and define S_λ by

$$S_\lambda = \begin{cases} 0 & \text{if } |M^2 - \sum \frac{k_T^2 + m^2}{x}| > \lambda^2 \\ S & \text{otherwise} \end{cases} \quad (6.6)$$

where

$$M^2 = \sum_{i=1}^n \frac{k_{T,i}^2 + m_i^2}{x}$$

is the inverse n-particle propagator. $S_\lambda = 0$ for virtual states which are far off the energy shell, and is in particular zero if any of the constituents has transverse momentum $> \lambda$. Defining the soft component in the wave function ψ_λ by

$$\psi_\lambda \equiv S_\lambda K \psi \quad (6.7)$$

and

$$\Delta S \equiv S - S_\lambda \quad (6.8)$$

we find

$$\begin{aligned} \psi_\lambda &= S K \psi \\ &= S_\lambda K \psi + (S - S_\lambda) K \psi \\ &= \psi_\lambda + \Delta S K \psi \\ &= \psi_\lambda + \Delta S K \psi_\lambda + \Delta S K \Delta S K \psi_\lambda + \dots \end{aligned} \quad (6.9)$$

so that we can obtain the full wave function by perturbation theory from the soft wave function ψ_λ . We notice also that only off-shell propagation occurs. From now on we need only consider ψ_λ in Feynman diagrams. The form factor can now be represented by the set of diagrams symbolized in Fig. 55.

We would like to get some sort of handle on which components in the Fock space we need to consider. In the dimensional counting arguments in Section 6.1 we assumed we could restrict ourselves to the $q\bar{q}$ component. It turns out that components with more than two fermions (e.g. $q\bar{q}q\bar{q}$) are suppressed by at least one power of Q^2 (see, for example, Fig. 56a). In general, components with $q\bar{q}$ + several gluons (e.g. Fig. 56b) are not suppressed, but in the light-cone gauge $(p_0 + p_3) \cdot A = 0$, these components are also down by powers of Q^2 . Thus in this gauge we need only to consider the $q\bar{q}$ component in the Fock space, i.e. the diagrams of Fig. 57.

We now turn our attention to the leading logarithms. Analogously to the deep inelastic structure functions we find that in the light-cone gauge the dominant diagrams are the generalized ladder diagrams of Fig. 58, where the dots represent vertex and self-energy corrections. The dominant region of integration is (again analogously to the deep inelastic case):

$$\underline{k}_{1,T}^2 \ll \underline{k}_{2,T}^2 \ll \underline{k}_{3,T}^2 \ll \dots \ll Q^2 \quad (6.10)$$

Because of the structure of Fig. 58 we can now write

$$F_{\pi}(Q^2) = \int_0^1 dx \int_0^1 dz \phi^+(z, Q^2) T_B(z, x, Q^2) \phi(x, Q^2) \quad (6.11)$$

where T_B can be readily evaluated and one finds^{*)}

$$T_B = \frac{16\pi \alpha_s(Q^2) C_F}{Q^2} \frac{1}{(1+x)(1+z)} \quad (6.12)$$

and $\phi(x, Q^2)$ is defined by

$$\phi(x, Q^2) \equiv \int_0^{Q^2} \frac{d\underline{k}_T^2}{16\pi^2} \psi(x, \underline{k}_T^2) \left(\ln \frac{Q^2}{\Lambda^2} \right)^{-C_F/\beta_0} \quad (6.13)$$

Explicit evaluation of T_B for a vector meson shows that the behaviour (6.12) comes from the case where the helicities of the quark and antiquark are opposite. If we calculate the contribution from the equal helicity case (the form factor of a transversely polarized ρ meson, for example), we find it is suppressed by one additional power of Q^2 .

Because of the ladder-like structure of the dominant diagrams in the wave function (Fig. 58) we can write down an integral equation for the function $\tilde{\phi}$, where

$$\tilde{\phi} \equiv \phi \left(\ln \frac{Q^2}{\Lambda^2} \right)^{-C_F/\beta_0} \quad (6.14)$$

which is

$$\tilde{\phi}(x, Q^2) = \tilde{\phi}(x, \Lambda^2) + \int_{\Lambda^2}^{Q^2} \frac{d\underline{k}_T^2}{\underline{k}_T^2} \frac{C_F \alpha_s(\underline{k}_T^2)}{4\pi} \int_0^1 dy V(x, y) \tilde{\phi}(y, \underline{k}_T^2) \quad (6.15)$$

*) Strictly speaking, because of the way T_B is defined in Fig. 58, there should be a factor of $(\log Q^2/\Lambda^2)^{-2C_F/\beta_0}$ on the right-hand side of (6.12) from the vertex and self-energy corrections. We include this factor in the ϕ^+ and ϕ .

where the kernel $V(x,y)$ is the one-gluon potential

$$V(x,y) = \frac{1-x}{1-y} \left(1 + \frac{2}{(x-y)_+}\right) \Theta(x-y) + \frac{1+x}{1+y} \left(1 - \frac{2}{(y-x)_+}\right) \Theta(y-x) \quad (6.16)$$

with Δ defined by

$$\Delta \tilde{\phi} = \tilde{\phi}(y, Q^2) - \tilde{\phi}(x, Q^2) \quad (6.17)$$

Equation (6.15) can be solved explicitly and the solution is given by

$$\phi(x, Q^2) \underset{Q^2 \rightarrow \infty}{\approx} (1-x^2) \sum_{\substack{\text{even} \\ N}} a_N C_N^{3/2}(x) (\ln Q^2/\Lambda^2)^{-d_N} \quad (6.18)$$

where the d_N are the familiar anomalous dimensions of the twist-two non-singlet operators (see Lecture 2), $C_N^{3/2}$ are the Gegenbauer polynomials [complete orthonormal polynomials on measure $\int_{-1}^1 dx (1-x^2)$], and the a_N are in general uncalculable constants. Since the d_N are monotonically increasing ($d_0 = 0$, $d_2 = 0.62$, $d_4 = 0.90$, ...) the asymptotic behaviour of the pion form factor is given by

$$F_\pi(Q^2) \underset{Q^2 \rightarrow \infty}{\sim} \frac{\alpha_s(Q^2)}{Q^2} \quad (6.19)$$

We can go even further and calculate the normalization factor in (6.19), since the same wave function appears in the decay $\pi \rightarrow \mu\nu$. We find

$$F_\pi(Q^2) \underset{Q^2 \rightarrow \infty}{\rightarrow} \frac{36\pi \alpha_s(Q^2/4)}{Q^2} \frac{f_\pi^2 C_F}{3} \quad (6.20)$$

which is the final answer. The pion decay constant f_π is equal to 0.94 MeV. The result (6.20) is much smaller than the values of F_π measured in the range $2 < Q^2 < 4 \text{ GeV}^2$ ⁹⁵). However, Eq. (6.20) is an asymptotic prediction, and the expected value at finite Q^2 depends substantially on the input form of the driving term in (6.15), i.e. $\tilde{\phi}(x, \lambda^2)$. A reasonable assumption which is made in the above derivation is that $\phi(x, \lambda^2)$ falls at least like $(1-x^2)^\varepsilon$ at $x^2 \rightarrow 1$ (where $\varepsilon > 0$). In this case the singularity in T_B as $x \rightarrow \pm 1$ is integrable, so that up to logarithmic corrections (which in this case happen to be $\propto (\ln Q^2)^{-1}$ asymptotically because $d_0 = 0$) the dimensional counting rules are valid.

The results above (6.18), (6.20) can be understood in terms of the operator product expansion and the renormalization group¹⁰⁴). In the wave functions ψ in Fig. 58, the outermost legs are hard, so that we want

$$\langle 0 | \psi(z_1) \bar{\psi}(z_2) | \pi \rangle \quad (6.21)$$

in the limit $z_1^+ = z_2^+$. We expand $\psi\bar{\psi}$ in terms of local operators (we are still working in the light-cone gauge and in this gauge the path-ordered exponential of the gauge potential just gives a factor 1, so we can neglect it), and again the non-singlet twist-two operators are the dominant ones. There is a slight difference however, since we now have to consider operators of the type

$$\partial_{\mu_{k+1}} \dots \partial_{\mu_m} \bar{\psi} \gamma_{\mu} D_{\mu_1} \dots D_{\mu_k} \psi \quad (6.22)$$

In the deep inelastic case we take the forward matrix element of the operators, and in that case it is only the operator with $k = m$ which has a non-zero matrix element. Now, for each value of m we have m operators, and therefore in principle m eigenvalues of an anomalous-dimension matrix. However, the anomalous-dimension matrix is a triangular one; renormalization of the operator with an $m - k$ external derivative gives a linear combination of the operators (6.22) with $m - k$, $m - k + 1$, ..., etc., external derivatives. Thus the eigenvalues of this matrix are just the diagonal elements, and these are just the γ_0^N for $N < m$. In this way we understand Eq. (6.12).

6.3 Form factors of other hadrons

In the previous section we saw that QCD prediction for the asymptotic behaviour of the form factor was very similar to the dimensional counting prediction, the only difference being an extra factor of $\alpha_s(Q^2)$ in the QCD case. In general, however, the lowest anomalous dimension is not zero, and there are additional logarithmic modifications to Eq. (6.1).

Perhaps the simplest example is the form factor of the vector meson. For the helicity zero form factor of the ρ meson, for example, the arguments of the previous section go through exactly so that

$$F_{\rho}(Q^2, \lambda_{\rho}=0) \sim \frac{\alpha(Q^2)}{Q^2} \quad (6.23)$$

In this case the normalization constant can also be determined; it is the same as for the pion (6.20) with f_{π} replaced by the ρ decay constant g_{ρ} defined by

$$\langle 0 | \bar{\psi} \gamma_{\alpha} (\frac{1}{2} \tau^+) \psi | \rho^+ \rangle = \epsilon_{\alpha} g_{\rho} m_{\rho} \quad (6.24)$$

In fact $g_\rho \approx f_\pi$ reflecting the KSFR relation¹⁰⁵⁾, so that $F_\pi \approx F_\rho$ asymptotically. For the transverse ρ (helicity = ± 1) there are two differences:

- i) T_B is suppressed by an additional power of Q^2 and
- ii) the structure of the potential is different.

The bound state equation can be solved¹⁰⁰⁾ giving

$$F_\rho(Q^2, \lambda_\rho = \pm 1) \sim \frac{m^2(Q^2)}{Q^2} \frac{\alpha_s(Q^2)}{Q^2} [\alpha_s(Q^2)]^{2C_F/\beta_0} \quad (6.25)$$

The exponent $2C_F/\beta_0$ is the anomalous dimension of

$$\bar{\psi} \sigma_{\alpha\beta} \left(\frac{i}{2} \tau^+ \right) \psi \quad (6.26)$$

The subdominant contributions to this form factor have exponents which are just the anomalous dimensions of the tower of operators based on (6.26).

For the baryons we can carry out the same procedure as for mesons¹⁰⁶⁾. The integral equation for the wave function is more complicated; nevertheless it can be solved (at least numerically). For example, for the magnetic form factor of a nucleon, the lowest anomalous dimension is not zero, and one finds

$$G_M(Q^2) \sim \left(\frac{\alpha_s(Q^2)}{Q^2} \right)^2 [\alpha_s(Q^2)]^{C_F/\beta_0} \quad (6.27)$$

so that the nucleon form factor satisfies the dimensional counting rules, but with logarithmic modifications. As in the meson case, form factors in which the baryon's helicity is changed, or in which its helicity is > 1 , are suppressed by powers of Q^2 .

C_F/β_0 is just the anomalous dimension of the fermion field ($\equiv \gamma_F$) in the Feynman gauge. One can readily see why the exponents in (6.25) and (6.27) are $2\gamma_F$ and γ_F , respectively. The lowest anomalous dimension of the familiar non-singlet twist-two operator, $\bar{\psi} \gamma_\mu \psi$ is proportional to the ultraviolet divergence in the diagrams of Figs. 59a and b. These cancel by the Ward identity $Z_1 = Z_2$; thus there is zero anomalous dimension in (6.9). If we next take the operator (6.26), and calculate the same diagrams, we find (by simple Dirac algebra) that the diagram of Fig. 59a vanishes in the Feynman gauge, leaving us with a contribution equal to $2\gamma_F$ from the two diagrams of Fig. 59b. The anomalous dimension in the case where the form factor of a helicity $\frac{1}{2}$ baryon can be deduced from the helicity 0 and 1 meson form factors is discussed above. When we calculate the lowest anomalous dimension of the three-quark operator (Fig. 60) we find the following contributions:

- i) $3\gamma_F$ from the self-energy diagrams;
- ii) $-2\gamma_F$ from the two vertex diagrams where the gluon links quarks of opposite helicity; and
- iii) 0 from the vertex where the gluon links quarks of the same helicity.

Hence we end up with an anomalous dimension equal to γ_F in (6.27). Similarly, for a helicity 3/2 baryon we deduce that the relevant anomalous dimension is equal to $3\gamma_F$ [only the first contribution (i) above is not zero].

Before proceeding with a discussion of elastic scattering at large angle, we review very briefly the properties of the Sudakov form factor; this form factor plays an important role for these large-angle processes.

6.4 Sudakov form factor¹⁰⁷⁾

It is well known that in QED the form factor of the electron (of momentum p) at large momentum transfer (q) in the leading logarithmic approximation falls very rapidly with q^2 , in fact like

$$\exp \left[- (\text{const}) e^2 \ln^2 q^2/p^2 \right] \quad (6.28)$$

We notice that in this case every loop gives two logarithms, in contrast to the inclusive processes where there was only one logarithm per loop, corresponding to the mass singularities. The second logarithm corresponds to an infrared divergence: these divergences now survive, since there are no diagrams with real photons to cancel them.

Equation (6.28) is interpreted as signifying that it is difficult to bend a charged particle through a finite angle, without it radiating many photons. Since the result of summing leading logarithms is much smaller than the neglected non-leading terms, it is relevant to ask whether these terms also sum to something small. Calculations by Korthales-Altes and de Rafael¹⁰⁸⁾ seemed to indicate that this may be so, and recently Mueller¹⁰⁹⁾ has argued that all the non-leading logarithms in massive QED, sum up to give an asymptotic form

$$\exp \left[- c(q^2) \ln^2 q^2/M^2 \right] \quad (6.29)$$

where M is a finite mass. It is still just possible that the terms which are suppressed by powers of q^2 in perturbation theory sum up to give something larger than (6.29), but this seems implausible.

Similarly, when one studies electron-electron scattering through a fixed angle in perturbation theory, we find factors such as (6.28) appearing. For example, to order α^2 in the amplitude the dominant diagrams (in the Feynman gauge) are those in Fig. 61. The relative contributions of Figs. 61a, b, and c are $\log^2 t$,

$-\log^2 t$, and $-y^2 t$, respectively (in our approximation we do not distinguish $\log s$ from $\log u$ or $\log t$). We interpret this result as being that the diagrams of Figs. 61a and b cancel, leaving us with the diagrams of Fig. 61c, which is just the Born diagram modified by the Sudakov form factor. This is a general feature to all orders of perturbation theory⁹²).

Although the calculations are more complicated in QCD, it seems that the same results hold (up to a trivial colour factor), when we consider the form factors of coloured fields or fixed angle scattering of coloured fields^{2,110,111}).

6.5 Elastic scattering at fixed angle

The problem of elastic scattering at fixed angle has not been studied very thoroughly yet in QCD, but certain features seem to have emerged⁹⁵). In particular, in time-ordered perturbation theory in the light-cone gauge the dominant radiative corrections to diagrams such as those of Fig. 52, which obey the dimensional counting rules, lead to the ladder-like structure of Fig. 62, in which each rung joins together two quarks moving almost parallel to each other. These radiative corrections are identical to those in the form factors (Sections 6.2 and 6.3), so that we can write

$$\frac{d\sigma}{dt} (A+B \rightarrow C+D) = \frac{\alpha_s^2(t)}{t} F_A(t) F_B(t) F_C(t) F_D(t) f(\theta) \quad (6.30)$$

In principle, the function $f(\theta)$ and the normalization are also calculable, but quite clearly this is a horrendous task. Thus we see that the diagrams which obeyed the dimensional counting rules are in QCD only modified by calculable logarithmic corrections.

What about the Landshoff diagrams, e.g. those of Fig. 53, which were the dominant ones in the parton model? When we evaluate the radiative corrections to these diagrams, we find that the leading contributions are exactly those corresponding to the Sudakov form factor; thus each quark-quark scattering amplitude should be multiplied by $F_q^2(t) \alpha(t)$ (where F_q is the Sudakov form factor). Since asymptotically the Sudakov form factor falls faster than any power of t , the "multiple scattering" diagrams will be suppressed relative to those obeying the dimensional counting rules. Moreover, even at low values of t [~ 0 (10 GeV²)] they can be estimated numerically to be less important.

Let us briefly recall why there was no sign of the Sudakov form factor in the pion form factor (Section 6.2) or in other processes for diagrams obeying the dimensional counting rules (e.g. those of Fig. 62). As an example, let us take the pion form factor in the Feynman gauge and consider the diagram of Fig. 63a.

Clearly, this has a double logarithmic (i.e. $\log^2 t$) contribution, exactly corresponding to the contribution of Fig. 61c. However, we should also consider the contribution of Fig. 63b which also has a $\log^2 t$ term, which cancels that of Fig. 63a. It is crucial that the external particle is a colour singlet; otherwise the colour factors of the two diagrams of Figs. 63a and b would not be equal and the cancellation would not take place. This cancellation corresponds to the cancellation of the infrared divergences when we consider radiation from a colour neutral particle. This cancellation occurs in all diagrams which do not have nearly on-shell partons scattering through a finite angle.

Why does such a cancellation not take place in the multiple scattering diagrams? In these diagrams it might seem natural that there should be a cancellation of the $\log^2 t$ terms within various sets of diagrams, such as the pair in Fig. 64. However, this is not the case, since in the diagram of Fig. 64b, k_T is bounded above by λ , as it has to be routed through at least one of the soft hadronic vertices. Notice that this is not the case in Fig. 64a, nor in either of the diagrams of Fig. 63. This means that the integral over k_T , which in the Sudakov case is

$$\int_{p^2}^t \frac{d\vec{k}_T^2}{k_T^2} \sim \ln(t/p^2) \quad (6.31)$$

is in the case of diagram Fig. 64b independent of t . Thus we lose a factor of $\log t$ and no cancellation of the $\log^2 t$ factor occurs. There seems to be no way of avoiding a Sudakov suppression of the multiple scattering diagrams¹¹¹).

From the above discussion we see that we are at least beginning to understand elastic scattering processes in QCD.

7. JETS

7.1 Introduction

We were led to study QCD because of the tentative indications discussed in Section 1.1 that it had qualitative features desirable in a theory of quark dynamics. One of the most dramatic pieces of evidence for quarks was the jets observed¹¹²⁾ in e^+e^- annihilation into hadrons, with their angular distribution

$$\frac{d\sigma(2\text{jets})}{d(\cos\theta)} \propto (1 + \cos^2\theta)$$

characteristic of spin $\frac{1}{2}$ particles. We have seen in the intervening lectures how to use perturbative QCD to make asymptotic predictions for inclusive hard processes. Since jet production is an archetypal inclusive process and of considerable topical¹¹³⁾ interest we would like to study how one can describe jets in perturbative QCD. Our hope would be that in addition to understanding the apparently dominant two-jet processes in e^+e^- annihilation and elsewhere, we might also be able to predict detectable (small?) deviations from the naive jet predictions of the parton model. For example, perhaps $e^+e^- \rightarrow$ three jets¹⁶⁾ of hadrons sometimes? One can go further, and hope that QCD jetology will help us probe aspects of theory which are generally somewhat inaccessible, such as the three-gluon vertex. This possibility is particularly enticing in view of the suggestive data on jets which are now¹¹³⁾ emerging from DESY.

7.2 Jets in QCD perturbation theory

In a naive scaling parton model for deep inelastic processes, the total cross-section and the final states are "understood" in terms of the lowest order diagram of Fig. 8. We have seen in Lecture 2 how the scaling prediction for the structure function gets modified by logarithmic factors (2.57), and we saw in Lecture 4 how the origin of these logarithms can be seen diagrammatically. It was shown there, following the Bloch-Nordsieck theorem⁶¹⁾, that all divergences associated with soft gluons cancel when one computes a physical inclusive cross-section with an appropriate finite (imperfect) energy resolution. However, logarithms are obtained from mass singularities which occur when one has two or more massless particles moving parallel to each other. Their analysis is simplified⁶⁹⁾ if one uses a gauge in which one sums only over the physical transverse polarizations of the gluons. Then, as discussed in Lecture 4, one finds that interference diagrams like Fig. 32c do not have leading singularities, whereas embryonic ladder diagrams like Fig. 32b do contribute in leading logarithmic order⁶⁹⁾. The physical structure of the dominant diagrams for deep inelastic scattering is shown in Fig. 33. The dominant regions in the phase-space integrations were shown [Eq. (4.29)] to be

$$p^2 \ll k_1^2 \ll k_2^2 \ll \dots \ll k_n^2 \ll q^2 \quad (7.1)$$

with the transverse components k_{Ti}^2 of the momenta always obeying

$$m^2/\delta < k_{Ti}^2 < \epsilon q^2 \quad (7.2)$$

where δ and ϵ are fixed small numbers. When asymptotic freedom is taken into account, each of the diagrams indicated by Fig. 33 makes a contribution $\propto (\ln \ln q^2/\Lambda^2)^n$, and their sum exponentiates to yield the asymptotically leading QCD prediction (2.57).

In attempting to reconstruct the form of the hadronic final state from Fig. 33 we should remember the Kinoshita-Lee-Nauenberg theorem^{65,66}, which tells us that mass singularities are cancelled when one sums over all indistinguishable final states, i.e. those with all combinations of collinear particles. Hence each of the final-state quarks and gluons in Fig. 33 should be understood as a sum over configurations with parallel-moving quanta. For example, the quark at the top of Fig. 33 corresponds (in a physical gauge) to sums over diagrams of the type shown in Fig. 65, where the invariant masses s_i are nested so that

$$q^2 \gg s_n \gg s_{n-1} \gg \dots \gg s_1 \gg p^2 \quad (7.3)$$

analogously to the ordering (4.29), (7.1), and the transverse momenta again obey a condition analogous to (7.2).

We can now see that the structure of the final state at leading logarithmic level consists of two jets of hadrons (quarks and gluons at the moment). One jet consists of quanta moving collinearly with the incoming parton (cf. Fig. 33) -- the target jet, and one set of quanta moving collinearly with the parton after it is struck (cf. Fig. 65) -- the current jet. A similar structure can be seen in e^+e^- annihilation. According to the KLN theorem^{65,66}, the final-state $q\bar{q}$ pair in the diagram of Fig. 66 which corresponds to the leading part of the total e^+e^- cross-section should each be understood as a sum over collinear particles like Fig. 65 with some transverse momentum cut-off of the type (7.2). Thus the dominant part of the cross-section corresponds to two-jet configurations, as long as we assume that the mysterious non-perturbative mechanism whereby quarks and gluons metamorphose into hadrons does not involve large momentum transfers which distort the momentum space picture of the final state.

As a more explicit example how this analysis works for a simple low-order diagram, consider e^+e^- annihilation in order g^2 . The diagrams in Fig. 67 give¹⁶⁾ the following cross-section for $e^+e^- \rightarrow q\bar{q}g$:

$$\frac{1}{\sigma_{\text{tot}}} \frac{d^2\sigma}{dx_q dx_{\bar{q}}} = \frac{2\alpha_s(Q^2)}{3\pi} \frac{x_q^2 + x_{\bar{q}}^2}{(1-x_q)(1-x_{\bar{q}})} \quad (7.4)$$

where

$$x_q \equiv 2p_q/Q \quad ; \quad x_{\bar{q}} \equiv 2p_{\bar{q}}/Q \quad (7.5)$$

For convenience, we renormalize the theory at Q^2 when we compute (7.4), so that the cross-section $\propto \alpha_s(Q^2) \sim 1/\ln Q^2$. We see immediately that the formula (7.4) has a singularity when x_q (or $x_{\bar{q}}$) $\rightarrow 1$. These limits clearly correspond to collinear configurations where the gluon is emitted parallel to the $\bar{q}$ (or q) respectively, as illustrated in Fig. 68. When we integrate the cross-section (7.4) over the momentum fraction x_q , we see that the region

$$\int_{1-\epsilon}^{1-O(m^2/Q^2)} \frac{dx_q}{1-x_q} \sim \ln Q^2 \quad (7.6)$$

which logarithm cancels out the $1/\ln Q^2$ coming from the explicit factor of $\alpha_s(Q^2)$ in Eq. (7.4). The integration region (7.6) is a prototype of the collinear configurations (7.2). It illustrates the remark of the previous paragraph that the logarithmically leading final states consist of two jets of collinear particles. The KLN theorem^{65,66)} guarantees that the sum over all such configurations will be free from mass singularities.

What happens in Eq. (7.4) when we look at the region where x_q and $x_{\bar{q}}$ are both strictly less than 1? It is easy to see that they correspond to kinematic configurations where the q , $\bar{q}$, and g are separated by finite angles, with no pair collinear. In these configurations the pairwise invariant masses

$$\left. \begin{aligned} S_q &\equiv (p_q + p_g)^2 = Q^2(1-x_{\bar{q}}) \\ S_{\bar{q}} &\equiv (p_{\bar{q}} + p_g)^2 = Q^2(1-x_q) \end{aligned} \right\} \quad (7.7)$$

are of order Q^2 , and the cross-section is explicitly seen from (7.4) to be $O(1/\ln Q^2)$. One might expect that a judicious application of the KLN theorem would tell us that if one interprets each of the individual quanta q , $\bar{q}$, and g in Fig. 67 as a sum over distinct sets of collinear particles, then Eq. (7.4) would become¹⁶⁾ the cross-section for $e^+e^- \rightarrow$ three hadronic jets.

This is indeed the case, and in an attempt to make this more plausible let us return to the diagrammatic analysis⁶⁹⁾ of deep inelastic scattering given in Lecture 4 and summarized earlier in this section. In view of the ordering (7.1) of momenta, let us consider the region where

$$\epsilon q^2 < k_n^2 < q^2 \quad (7.8)$$

and all the other k_i^2 obey the previous restrictions (7.2). Then the leading behaviour of the integral over k_n^2 takes the form

$$\int_{\epsilon q^2}^{q^2} \frac{dk_n^2}{k_n^2} \alpha_s(k_n^2) \approx \alpha_s(q^2) \ln\left(\frac{1}{\epsilon}\right) \quad (7.9)$$

whereas the other integrals over k_i^2 , where $i \leq n-1$, give the familiar factors of $(\ln \ln q^2)^{n-1}$. When one sums over all n one gets a contribution to the total cross-section which is of order $\alpha_s(q^2) \sim 1/\ln q^2$ as in (7.9), compared with the leading logarithmic behaviour obtained from the integration region (7.1), (7.2). It is easy to satisfy oneself that a similar phenomenon occurs when one considers the integration region

$$\epsilon q^2 < s_n < q^2 \quad (7.10)$$

and $\epsilon q^2 > s_{n-1} \gg s_{n-2} \gg \dots s_1 \gg p^2$. It is also easy to convince oneself that this and the previous kinematic region (7.8) correspond to kinematic configurations where there are two widely separated particles (jets according to KLN) in the current fragmentation region of the final state, instead of the one particle (jet) found in the previous integration region (7.1), (7.2), (7.3). A similar diagrammatic analysis can also be made to render more plausible the suggestion that Eq. (7.4) can be interpreted as an e^+e^- three-jet cross-section when $x_q, x_{\bar{q}} < 1$.

One is then led to the general prescription that in deep inelastic scattering and e^+e^- annihilation there will be subdominant cross-sections for multijet configurations which are suppressed by powers of $\alpha_s(Q^2)$:

$$\frac{\sigma(n \text{ jet})}{\sigma(2 \text{ jet})} = O(\alpha_s^{n-2}) \quad (7.11)$$

and that, as described at the end of Lecture 4, these may be calculated using the simple lowest order Born diagrams (cf. Fig. 67 or Fig. 69). This prescription and its natural extension to other hard scattering processes have indeed been rigorously justified in QCD perturbation theory^{2,69,70}). We will not discuss its derivation here in any more detail: rather we will turn to some of its phenomenological applications.

7.3 Widths of QCD jets

In the previous section we motivated the idea that multijet cross-sections should be suppressed (7.11) by powers of $\alpha_s(Q^2)$. To make this statement more precise we should make a more careful definition of a jet, so as to specify how we distinguish configurations with different numbers of jets. The plausibility of (7.11) rests on the absence of extra logarithms of Q^2 which would invalidate the simple expansion in powers of $\alpha_s(Q^2) \sim 1/\ln Q^2$. The absence of such logarithms was guaranteed by appeal to two theorems.

Infrared singularities due to the emission of soft gluons are cancelled according to Bloch and Nordsieck⁶¹) if one sums over all configurations using a finite energy resolution. This means that in a general hard scattering process with n jets one should always allow a finite fraction, ϵ say, of the total

centre-of-mass energy to lie outside the kinematic regions of the observed jets. According to Kinoshita, Lee and Nauenberg^{65,66}, mass singularities due to collinear configurations are cancelled if one sums over all indistinguishable configurations with any number of collinear particles. As in QED, one may do this by defining each jet by a finite opening angle, δ say. Combining these two constraints on the specification of jets, we may define¹¹⁴) a generic n-jet cross-section such that:

all except a small finite fraction ϵ of the total centre-of-mass energy lies within n cones of finite opening angles δ_i .

This prescription and the types of configuration it includes are illustrated in Fig. 70.

Explicit computations to $O[\alpha_s(Q^2)]$ of the two-jet cross-sections defined by this prescription have been made^{114,115}) for $e^+e^- \rightarrow q \text{ jet} + \bar{q} \text{ jet}$, and the production of two gluonic jets by a source of the type $F_{\mu\nu}^a F_a^{\mu\nu}$ or $\epsilon_{\mu\nu\alpha\beta} F_a^{\mu\nu} F_a^{\alpha\beta}$ (116). In the case of e^+e^- annihilation one has^{114,115})

$$F_q \equiv \frac{\sigma(2 \text{ jet})}{\sigma_{\text{total}}} = 1 - \frac{\alpha_s(Q^2)}{\pi} \left[4 \left\{ \ln(2\epsilon) + 3 \right\} \frac{4}{3} \ln \delta + (\text{terms with no logs}) \right] \quad (7.12)$$

The terms in (7.12) proportional to $\ln \epsilon$ and $\ln \delta$ are the relics of the infrared and mass singularities which have been cancelled by our definition of a two-jet cross-section. We see explicitly that neither the soft gluon cut-off ϵ nor the jet angular cut-off δ can be removed if one wants a finite cross-section. The finite terms with no logarithms have been computed in deep inelastic scattering¹¹⁵) as well as e^+e^- annihilation. In the order g^2 to which we work, $1 - F_q$ is the three-jet cross-section. In the case of gluonic jets one has¹¹⁶)

$$F_g = 1 - \frac{\alpha_s(Q^2)}{\pi} \left[\left\{ 12 \ln(2\epsilon) - \left(11 - \frac{3}{3} f \right) \right\} \ln \delta + (\text{terms with no logs}) \right] \quad (7.13)$$

We see that the leading coefficient of $\ln \epsilon \ln \delta$ is much larger than in the case of quark jets (7.12), suggesting that the perturbative width of gluon jets is much wider than that of quark jets. Does this mean that presently observable gluon jets should be wider than the commonly observed quark jets? Not necessarily, for on the perturbation-theoretic side, the finite terms in (7.13) are very process-dependent, and it is easy to convince oneself¹¹⁵) that the leading logarithms of ϵ and δ cannot dominate the two-jet cross-section formulae at present values of Q^2 . Secondly, on the phenomenological side, one knows that the presently observed quark jets have almost constant $\langle p_T \rangle$ rather than the constant angle predicted asymptotically by QCD. This suggests that the widths of present jets are greatly dominated

by non-perturbative effects. Explicit calculations¹¹⁷⁾ do indeed suggest that for quark jets of energy ~ 5 GeV

$$\Delta \text{Width}|_{\text{pert.}} \approx \Delta \text{Width}|_{\text{non-pert.}} \quad (7.14)$$

and recent data from PETRA¹¹³⁾ indicate a modest increase in $\langle p_T \rangle$ for more energetic quark jets. Data¹¹⁸⁾ from DORIS on T decays suggest that

$$\text{Width}|_{\text{gluon jet}} \approx \text{Width}|_{\text{quark jet}} \quad (7.15)$$

for quark and gluon jets with energies ~ 3 to 5 GeV. The two observations (7.14) and (7.15) suggest that gluon jets with energies somewhat above 5 GeV may indeed be wider than quark jets of corresponding energies, though it is not¹¹⁹⁾ to be expected that the leading logarithms in (7.12) and (7.13) will show up before much higher energies are achieved.

One curiosity of the perturbative gluon jet width (7.13) may be worth noticing: it is that the coefficient of the $\ln \delta$ term is exactly the same as the coefficient of the $O(g^3)$ term in the Callan-Symanzik β -function (1.54b). Is this just a coincidence, or does it mean something profound?

7.4 Looking for QCD-jet effects

How can one search for specifically QCD effects in jets, and are any present in existing data? The deviations from a naive two finite- p_T -jet picture that one expects to see are:

- a growth in $\langle p_T \rangle$, like $Q\alpha_s(Q^2) \sim Q/\ln Q^2$ asymptotically;
- an increasing asymmetry in the $\langle p_T \rangle$ in the two jets in e^+e^- annihilation, arising from the perturbative nature of the hard bremsstrahlung phenomenon which generates large p_T by radiating mainly from one of the initial $q, \bar{q}$;
- an eventual three-jet structure in some fraction of the final states¹⁶⁾, corresponding to a broadening of the "wider" jet beyond the typical $\langle p_T \rangle \sim 300$ MeV of a traditional non-perturbative jet. The cross-section for such events should be given by Eq. (7.4).

To help in looking for these effects, a variety of global measures of jet broadening have been proposed. A key element in their construction has been the desire to have a quantity which is reliably computable in QCD perturbation theory and unlikely to have its value substantially altered by the ill-understood non-perturbative hadronization process. For quantities to be reliably computable in perturbative QCD they should be free of infrared and mass singularities. Both these criteria are met by quantities which are linear in the particle momenta -- clearly any such quantity would have identical values for the three different configurations in Fig. 71. Examples of proposed QCD variables are:

$$\text{Thrust}^{120):} \quad T \equiv \max_{\underline{n}} \frac{\sum_i |\underline{p}_i \cdot \underline{n}|}{\sum_i |\underline{p}_i|} \quad (7.16)$$

where $\underline{n}$ is an arbitrary unit vector, and the $\underline{p}_i$ are the particle momenta. $T = 1$ for an ideal two-jet event, and $T = \frac{1}{2}$ for a perfectly spherical event.

$$\text{Spherocity}^{121):} \quad S \equiv \left(\frac{4}{\pi}\right)^2 \min_{\underline{n}} \left(\frac{\sum_i |\underline{p}_{Ti}|^2}{\sum_i |\underline{p}_i|} \right)^2 \quad (7.17)$$

where the $\underline{p}_{Ti}$ are measured transverse to the unit vector $\underline{n}$. $S = 0$ for an ideal two-jet event, and $S = 1$ for a perfectly spherical event.

$$\text{Acoplanarity}^{117):} \quad A \equiv 4 \min_{\underline{n}} \left(\frac{\sum_i |\underline{p}_{out,i}|^2}{\sum_i |\underline{p}_i|} \right)^2 \quad (7.18)$$

where the $\underline{p}_{out,i}$ are measured transverse to a plane with normal $\underline{n}$. $A = 0$ for idealized two- and three-jet events, and $A = 1$ for a perfectly spherical event.

The standard example of a variable which does not meet the perturbative QCD criteria of cancelling infrared and mass singularities is sphericity:

$$S' \equiv \min_{\underline{n}} \left(\frac{\sum_i \underline{p}_{Ti}^2}{\sum_i \underline{p}_i^2} \right) \quad (7.19)$$

where the transverse momenta $\underline{p}_{Ti}$ are measured relative to a jet axis $\underline{n}$.

For practical purposes, the variables S and T give very similar information about the final state. Using one or the other of them in combination with A one has a reasonably complete set of variables for categorizing hadronic final states, as illustrated in Fig. 72. The events of immediate interest are those on the horizontal axis with $A = 0$ and $T < 1$. Do such events exist, and do they have a three-jet structure as expected in QCD perturbation theory? The following strategy for searching for such QCD effects has been suggested¹¹⁷⁾.

* First measure $\langle 1 - T \rangle$. According to QCD perturbation theory, this should be larger than that expected for finite p_T -jets when one goes to very large Q :

$$\left. \begin{aligned} \langle 1 - T \rangle &\approx 1.57 \left(\frac{2\alpha_s}{3\pi} \right) \sim 1/\ln Q \quad (\text{pert.}) \\ \text{cf } \frac{1}{2} \langle n(Q) \rangle &\frac{\langle p_T \rangle}{Q} \sim \ln Q / Q \quad (\text{non-pert.}) \end{aligned} \right\} \quad (7.20)$$

Calculations¹¹⁷⁾ suggest that at $Q \sim 10$ GeV

$$\langle 1-T \rangle_{\text{pert.}} \approx \langle 1-T \rangle_{\text{non-pert.}} \quad (7.21)$$

and if one adds the two estimates together the result compares very well with e^+e^- data throughout the PETRA range, as illustrated in Fig. 73. Included in this figure is the expected effect of the $b\bar{b}$ threshold at about $10\frac{1}{2}$ GeV. One expects a heavy quark, such as the $t(b)$, to decay according to the chain

$$\begin{aligned} t &\rightarrow b q \bar{q} \\ &\quad \hookrightarrow c q \bar{q} \\ &\quad \quad \hookrightarrow s q \bar{q} \end{aligned} \quad (7.22)$$

which gives 14 (10) quarks and antiquarks in the final state in the case of $e^+e^- \rightarrow t\bar{t}$ ($b\bar{b}$). Even if one cannot distinguish three jets in $c \rightarrow sq\bar{q}$, (7.22) gives rise to 10 (6) potentially distinguishable final-state jets. Such events will not in general be planar, and would indeed tend to look essentially isotropic ($\langle T \rangle \sim \frac{1}{2}$, $\langle S \rangle \sim 1$, $\langle A \rangle \sim 1$) at threshold. Far above threshold one expects the heavy quarks to be Lorentz boosted, so that the transverse momenta of their decay products are unaffected. We therefore expect¹¹⁷⁾ a jump in $\langle 1-T \rangle$, $\langle S \rangle$ and $\langle A \rangle$ at threshold followed by a relatively long-tailed fall-off:

$$2\langle 1-T \rangle \approx \langle S \rangle \approx \langle A \rangle \approx \frac{4m_{t,b}^2}{Q^2} \quad (7.23)$$

The long tail (7.23) is manifest in Fig. 73. It means that one may use coarse scanning in Q to look for a new quark threshold through its effects on the variables [(7.16), (7.17), (7.18)]. A particularly useful variable may be the acoplanarity variable A which should only get a small contribution $O(\alpha_s/\pi)^2$ from QCD perturbation theory. Let us return to the search for three-jet events.

* Select events with low T and low A . The A cut selects planar events, the T cut ensures that they cannot have a narrow two-jet structure. According to perturbative QCD the cross-section for such low- T events is expected¹¹⁷⁾ to be

$$\frac{1}{\sigma_{\text{total}}} \frac{d\sigma}{dT} = \left(\frac{2\alpha_s}{3\pi} \right) \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{(1-T)} \right] + O\left(\frac{\alpha_s}{\pi}\right)^2 \quad (7.24)$$

which should be noticeable above the non-perturbative background at the higher PETRA and PEP energies.

* For such events, plot the angular distribution of the hadronic energy $(1/Q)(dE/d\theta)$ in the event plane. The angular variable θ can be specified by

setting $\theta = 0$ to be the thrust axis in the hemisphere with smaller $\sum_i |\underline{p}_{T_i}|$, and fixing $0 < \theta < \pi$ to be the half-plane with more hadronic energy. This definition resolves all mirror ambiguities. Event by event this antenna pattern or "Pointing vector" plot should reveal two other peaks (jets) in addition to one at $\theta = 0$. An estimate of the form of the integrated version of such a plot is shown in Fig. 74a. Recent experimental data¹²²⁾ seem to have a similar form. Unfortunately, distributions from a phase-space Monte Carlo tend¹¹⁸⁾ to have a similar shape after all the cuts and other procedures suggested above are used. This makes the interpretation of Pointing vector plots a murky business. However, no phase-space model can reproduce the expected non-trivial angular distributions of three-jet events.

* The normal event plane should¹²³⁾ have an angular distribution

$$\frac{dN}{d(\cos\theta)} \propto (3 - \cos^2\theta) \quad (7.25)$$

relative to the e^+e^- beam axes.

* Look at the angular distribution within the event plane by making¹²⁴⁾ a "jet boost". First exclude particles in the $\theta \approx 0$ thrust jet, and then boost the remaining particles to their centre of mass as in Fig. 75. In an ideal world, this boost ζ is given uniquely by the value of thrust T :

$$\zeta : \sinh \zeta = \frac{T}{2\sqrt{1-T}}, \quad \cosh \zeta = \frac{2-T}{2\sqrt{1-T}} \quad (7.26)$$

In the boosted reference frame one should find two back-to-back finite- p_T jets which have a non-trivial dependence on their angle $\tilde{\theta}$ relative to the thrust axis (see Fig. 76). This angular dependence is expected to be:

$$\frac{dN}{d(\cos\tilde{\theta})} \propto \left\{ \begin{array}{ll} \sim 1 + 2\cos^2\tilde{\theta} & \text{for } q\bar{q} + \text{vector gluon} \\ \sim 1 + 0.2\cos^2\tilde{\theta} & \text{for } q\bar{q} + \text{scalar gluon} \\ \sim 1 - 0.1\cos^2\tilde{\theta} & \text{for } Q\bar{Q} \text{onium} \rightarrow 3 \text{ gluons} \end{array} \right\} \quad (7.27)$$

The difference between the angular distributions for vector and scalar gluons in Eq. (7.27) gives a test of the gluon spin which is analogous to, and in principle equally sensitive as, the classical $(1 + \cos^2\theta)$ angular distribution of $e^+e^- \rightarrow$ two-jet events as a test of the quark spin $\frac{1}{2}$.

Beautiful evidence has recently been presented by the TASSO group (Fig. 65) and¹¹³⁾ other experiments at PETRA for the existence of three-jet events in e^+e^- annihilation at $Q \sim 30$ GeV. It remains to be seen whether these will turn out to have been QCD $e^+e^- \rightarrow q\bar{q}$ gluon events.

7.5 Jets in resonance decays

One of the most rewarding places to look for QCD jet effects is probably in the decays of heavy quarkonia [the T ($b\bar{b}$) system, the topsilon ($t\bar{t}$) system, etc.]. It is believed that the basic dynamics of the decays are very different from those in the continuum. In the latter case one has a $q\bar{q}$ pair flying apart, whereas in onium decays first we must dispose of the heavy $Q\bar{Q}$ pair by annihilating them, essentially at a point. Then, just as the dominant heavy products of positronium and muonium are photons, so we expect the dominant decay products of quarkonia to be gluons. These should manifest themselves as jets of hadrons. The number of gluons (jets) in the final state, and their angular distributions, then depend on the internal quantum numbers of the initial onium state. We now examine in turn some interesting cases.

$1^{--} \rightarrow 3 \text{ gluons}$

The orthoquarkonium 3S_1 state is forbidden by colour to decay into one gluon, and two-gluon decays are forbidden by charge conjugation. Decays into $q\bar{q}$ must either proceed through a virtual photon, or are of higher order in α_s . The dominant decay is therefore expected to be $^3S_1 \rightarrow$ three gluons. Decays into four or more gluons are conventionally assumed to be small, because they would be of higher order in α_s which is assumed to be small for heavy onium decays. However, these and other QCD radiative corrections to 3S_1 decay have not been computed, whereas the QCD radiative corrections to paraquarkonium 1S_0 decay have been computed¹²⁵⁾ and turn out to be large

$$\frac{\Gamma(^1S_0 \rightarrow \text{hadrons})}{\Gamma(^1S_0 \rightarrow \gamma\gamma)} = \frac{2}{9e_q^2} \left(\frac{\alpha_s(4m_Q^2)}{\alpha} \right) \left[1 + 16 \frac{\alpha_s(4m_Q^2)}{\pi} + \dots \right] \quad (7.28)$$

if the $\bar{M}S$ renormalization scheme⁴⁸⁾ is used to define $\alpha_s(Q^2)$. It may be that these QCD radiative corrections mess up the leading order jet predictions (in the case of paraquarkonium, $^1S_0 \rightarrow 2g \rightarrow 2 \text{ jets}$), but this is not yet known. If we ignore this problem, and take the jet predictions for 3S_1 decay from the lowest order in QCD perturbation theory, we expect¹²³⁾ the following differential distribution for the gluon jet energies ($x_i \equiv 2E_{g_i}/m_{^3S_1}$):

$$\frac{1}{\Gamma_{\text{had}}} \frac{d^2\Gamma}{dx_1 dx_2} \approx \left(\frac{1}{\pi^2 - q} \right) \left\{ \frac{(1-x_1)^2}{x_1^2 x_3^2} + \frac{(1-x_2)^2}{x_1^2 x_3^2} + \frac{(1-x_3)^2}{x_1^2 x_2^2} \right\} \quad (7.29)$$

In addition to these three-gluon decays, we also expect some $q\bar{q}$ decays via virtual photons, but one still expects the fraction of three-jet events in orthoquarkonium

decays to be much larger than that in the continuum. They may be searched for and analysed in a similar way to the continuum $q\bar{q}g$ events. One first selects low-thrust events -- which should¹¹⁷⁾ have a distribution

$$\frac{1}{\Gamma_{had}} \frac{d\Gamma}{dT} \approx \left(\frac{3}{\pi^2 - 9} \right) \left\{ \frac{4(1-T)}{T^2(2-T)^3} (5T^2 - 12T + 8) \ln\left(\frac{2-2T}{T}\right) + \frac{2(3T-2)(2-T^2)}{T^3(2-T)^2} \right\} \quad (7.30)$$

with modifications due to finite (non-perturbative) jet widths, which get progressively less significant for higher onium masses. One should then check that the low- T events are roughly planar, as would be expected if three-gluon (jet) decays dominate. If so, then there is a specific QCD prediction¹²³⁾ for the angular distribution of the normal to the event plane relative to the (unpolarized) e^+e^- beam directions:

$$\frac{dN}{d(\cos\theta)} \propto (3 - \cos^2\theta) \quad (7.31)$$

There is also a characteristic prediction¹¹⁷⁾ for the angular distribution of hadronic energy within the event plane as shown in Fig. 74b. Another way to check the decay dynamics is to make the jet boost (7.26) introduced earlier, remove the most energetic jet, and plot the angular distribution of the remaining dijet system $dN/d(\cos\tilde{\theta})$, for which one has the prediction¹²⁴⁾ given in Eq. (7.27).

$$\underline{1^{--} \rightarrow 2g + \gamma}$$

This decay may be a substantial fraction of the $\Gamma(^3S_1 \rightarrow 3g)$ decay, being suppressed by a factor of order $(\alpha/\alpha_s)^2$ which need not be small because of the small value of $\alpha_s(4m_Q^2)$ for heavy onia. There are characteristic predictions¹²⁶⁾ for the energy and angular distributions of the photons and gluons which give interesting tests of QCD, but we do not have time to discuss them here.

$$\underline{0^\pm \rightarrow 2g}$$

Charge conjugation permits these decays, and they are expected to be the dominant decay modes of the spinless onia. As remarked above, the QCD radiative corrections to the total 0^{--} decay rate have been computed¹²⁵⁾, but not those for the $0^{++} \rightarrow 3P_0 \rightarrow Q\bar{Q}$ state. It is not clear to what extent either of them can be regarded as decaying via a pure $F_{\mu\nu}F^{\mu\nu}$ or $\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}$ point source, for which the perturbative width of the gluon jet has been computed¹¹⁶⁾.

$$\underline{1^{++} \rightarrow g\bar{g}}$$

A 1^{++} state is forbidden by a theorem of Yang to decay into two identical spin 1 particles. Therefore the 3P_1 decay into two real gluons is forbidden, but

decays into one virtual and one real gluon are allowed, as are decays into three gluons. It has been found¹²⁷⁾ that the dominant decays are those indicated in Fig. 77, where the virtual gluon takes almost all the mass of the 3P_1 bound state and subsequently decays into light $q\bar{q}$ pairs, while the real gluon is very soft. This is because of a singular dependence of the differential decay rate on the binding energy E of the heavy $Q\bar{Q}$ pair, favouring real gluon energies of order E [$\approx 0(1)$ GeV for T and toponium?]. One therefore expects the dominant final states to feature two jets, and these would consist of essentially equal numbers of jets from each of the light ($m_q \ll m_Q$) quarks q . Hence one would expect

$$\Gamma(^3P_1(b\bar{b}) \rightarrow c\bar{c}g) \approx \Gamma(^3P_1(b\bar{b}) \rightarrow (u\bar{u} \text{ or } d\bar{d} \text{ or } s\bar{s})g) \quad (7.32)$$

and

$$\Gamma(^3P_1(t\bar{t}) \rightarrow b\bar{b}g) \approx \Gamma(^3P_1(t\bar{t}) \rightarrow (c\bar{c} \text{ or } \dots)g)$$

These should be experimentally accessible and non-trivial tests of the decay mechanism, which is dependent on rather subtle aspects of QCD.

$$\underline{2^{++} \rightarrow 2g}$$

There is no charge conjugation argument or theorem of Yang to forbid the two-gluon decay of 3P_2 onium, and this is expected to give predominantly two-jet final states.

Because of the non-trivial spin of the 2^{++} and 1^{++} states, angular correlations are possible in cascade decays such as

$$\gamma' \rightarrow \gamma + ^3P_{1,2} \rightarrow 2 \text{ jets}$$

The differential distributions and correlations in the angles between the beams, photons, and hadronic jets afford sensitive tests^{128,117)} of the assumed spin of the P-wave state, its presumed decay mode, and the spin of the gluon. In fact, some non-trivial tests of the gluon spin have already been done with T decays. In an (unmotivated) model in which the T decays into three scalar gluons it was found¹²⁹⁾ that the angular distribution of the thrust axis was of the form

$$\frac{dN}{d(\cos\theta)} \propto (1 - 0.995 \cos^2\theta) \quad (7.33)$$

whereas QCD predicts¹³⁰⁾

$$\frac{dN}{d(\cos\theta)} \propto (1 + 0.39 \cos^2\theta) \quad (7.34)$$

and experiment finds¹¹⁸⁾ $1 + (0.83 \pm 0.23) \cos^2 \theta$. Other aspects of the distributions in T decays also¹¹⁸⁾ disagree with the scalar gluon decay model, whereas no significant discrepancy with the QCD three-vector gluon decay expectations has yet been found.

7.6 QCD jets in other processes

As discussed in Section 7.2, the occurrence of extra jets due to gluon bremsstrahlung or $q\bar{q}$ pair creation, in addition to the naive parton model jets, is expected to be a general feature of deep inelastic or "hard" scattering processes. Two examples are

7.6.1 Deep inelastic eN, μ N, or ν N scattering

Here one expects^{131,132)} extra jets from the processes illustrated in Fig. 69. Their existence can be probed in a similar way to that proposed earlier for e^+e^- annihilation. It turns out¹³³⁾ that the normalized thrust distribution $(1/\sigma)(d\sigma/dT)$ is almost independent of Q^2 in the kinematic range presently accessible, but depends rather more sharply on the invariant hadronic mass W . Figure 78 shows the Q^2 dependence at fixed $W = 10$ GeV, while Fig. 79 shows the dependence on W integrated over Q^2 . We see in Fig. 79a that at $W = 10$ GeV the perturbative contribution to the thrust distribution is expected to be submerged by non-perturbative effects due to the finite widths of two-jet events, and that the BEBC-Gargamelle data¹³⁴⁾ agree with these expectations. When W is increased to 20 GeV, there is perhaps the beginning of a detectable perturbative tail which should contain three-jet events, and this is quite significant at $W = 40$ GeV. It therefore seems that perturbative QCD effects may be seen in the final states of present SPS and FNAL deep inelastic experiments, though this will not be easy. In principle, there should be interesting angular correlations between the planes of the scattered lepton and the final-state hadrons, but these are probably swamped by non-perturbative effects at present values of Q^2 and W^2 .

7.6.2 Hadron-hadron collisions at large p_T

The usual naive parton model for these processes is indicated in Fig. 43. It leads one to expect events with two large- p_T jets which come out back to back in the azimuthal angle ϕ around the beam directions, though they will not in general be back to back in a plane containing the beams. In QCD, one expects a plethora of mechanisms, such as gluon bremsstrahlung, to yield events with three large- p_T jets. In this case, one no longer expects the large- p_T jets to emerge back to back in azimuth. Some events of this type have been seen¹³⁵⁾, but there is as yet no clear evidence that these events are due to specifically perturbative QCD effects, or that they have a three-jet structure.

7.7 Inside QCD jets

Not only does perturbative QCD make predictions for the generation of extra jets in deep inelastic processes, but it also makes rather specific predictions for the distributions of hadrons within jets, and how they evolve with increasing Q^2 . These predictions may be divided into two aspects.

7.7.1 Longitudinal momentum distributions

We saw in Lecture 2 that deep inelastic structure functions evolve with increasing Q^2 , and in subsequent lectures we saw how to interpret this as a result of "momentum migration" due to the splitting of a gluon or quark into two sub-constituents at higher Q^2 , as indicated in Fig. 17. It is apparent from the structure of hard scattering cross-sections in QCD, discussed in earlier lectures, that one expects the same splitting functions⁴²⁾

$$P_{q \rightarrow q}(x), \quad P_{q \rightarrow g}(x), \quad P_{g \rightarrow q}(x) \quad \text{and} \quad P_{g \rightarrow g}(x)$$

to generate scaling violations in longitudinal momentum distributions. Defining Q^2 -dependent fragmentation functions for some type of hadron -- $D_q(z, Q^2)$, $D_g(z, Q^2)$ -- one expects¹³⁶⁾ them to obey the evolution equations

$$\begin{aligned} Q^2 \frac{\partial}{\partial Q^2} D_q(z, Q^2) &= \frac{\alpha_s(Q^2)}{2\pi} \int_z^1 \frac{dy}{y} \left[D_q(y, Q^2) P_{q \rightarrow q}\left(\frac{z}{y}\right) + D_g(y, Q^2) P_{g \rightarrow q}\left(\frac{z}{y}\right) \right] \\ Q^2 \frac{\partial}{\partial Q^2} D_g(z, Q^2) &= \frac{\alpha_s(Q^2)}{2\pi} \int_z^1 \frac{dy}{y} \left[D_q(y, Q^2) P_{q \rightarrow g}\left(\frac{z}{y}\right) + D_g(y, Q^2) P_{g \rightarrow g}\left(\frac{z}{y}\right) \right] \end{aligned} \quad (7.35)$$

Note an important difference from the Altarelli-Parisi equations for structure functions: the transposition of the off-diagonal terms, whose origin can easily be seen diagrammatically.

There is an important phenomenological aspect of Eq. (7.35): because of the large colour charge of the gluon, as manifested in the SU(3) Casimir C_A and the relatively large magnitude of $P_{g \rightarrow g}(z)$, gluon jets evolve faster with Q^2 than do quark jets. This can be seen most clearly by taking the moments of the hadron distributions inside singlet quark and gluon jets. Shown¹³⁷⁾ in Fig. 80 are the ratios

$$D_g(N, Q^2) / D_g(N, 100 \text{ GeV}^2) \quad \text{and} \quad D_q(N, Q^2) / D_q(N, 100 \text{ GeV}^2)$$

for the moments

$$\left. \begin{aligned} D_g(N, Q^2) &\equiv \int_0^1 dz z^{N-1} D_g(z, Q^2) \\ D_q(N, Q^2) &\equiv \frac{1}{2f} \sum_{q, \bar{q}} \int_0^1 dz z^{N-1} D_q(z, Q^2) \end{aligned} \right\} \quad (7.36)$$

We see from Fig. 80 that one expects much larger scaling violations to be seen when comparing gluon jets at different Q^2 , for example in the decays of successively more massive quarkonia: J/ψ , T , $(t\bar{t})$, There are indeed some indications from PLUTO data¹¹⁸⁾ that scaling violations are larger between J/ψ and T decays than they are in the continuum at nearby energies. It is not clear whether this means anything, though, because we do not know whether the three-gluon decay model is valid for either, both or neither of the J/ψ and T resonances.

Is there any other evidence for the validity of the equations (7.35)? There is some experimental information on the "non-singlet" quark fragmentation functions from a BEBC neutrino experiment¹³⁸⁾. If one takes the moments of the difference in the distributions of positively and negatively charged hadrons

$$D^{NS}(N, Q^2) \equiv (D_q^+ - D_q^-)(N, Q^2) \equiv \int_0^1 dz z^{N-1} (D_q^+ - D_q^-)(z, Q^2) \quad (7.37)$$

then the gluon terms in (7.35) do not contribute and one finds the familiar Q^2 dependence

$$\left. \begin{aligned} D^{NS}(N, Q^2) &\approx (\ln Q^2/\Lambda^2)^{-d_N} \\ d_N &= \left(\frac{4}{33-2f}\right) \left[1 - \frac{2}{N(N+1)} + 4 \sum_{j=2}^N \frac{1}{j} \right] \end{aligned} \right\} \quad (7.38)$$

Figure 81 shows that there is some evidence for the prediction (7.38) for the "non-singlet" moments. On the other hand, other combinations of h^+ and h^- fragmentation functions definitely do not exhibit the behaviour (7.38). Perturbative QCD gives us no reason to expect that they should! Indeed, in order to fit the Q^2 dependence of the ratio of h^+ and h^- in neutrino collisions, it is necessary¹³⁹⁾ to introduce the gluon fragmentation function $D_g(z, Q^2)$ with moments which are comparable to those of singlet combinations of quarks [cf. Eq. (7.36)] at $Q^2 = 20 \text{ GeV}^2$. PLUTO data¹¹⁸⁾ on T decays are also consistent with gluon and quark singlet fragmentation functions being comparable.

We conclude that the QCD predictions (7.35) for longitudinal momentum distributions within jets are not incompatible with experiment, and may even be experimentally supported.

7.7.2 Transverse momentum distributions

The same processes of splitting which give rise to longitudinal scaling violations also cause the transverse momentum within a jet to increase with Q^2 . We saw earlier that one can establish a jet perturbation theory for finite-angle jets

(δ fixed), and saw that the fraction $f(Q^2)$ of e^+e^- annihilation events with $\geq 1 - \epsilon$ of the total energy inside a cone of opening angle δ is given by

$$f = 1 - \frac{\alpha_s(Q^2)}{\pi} \bar{f} \ln \epsilon \ln \delta + \dots \quad (7.39)$$

Instead of keeping δ fixed and watching $f(Q^2)$ approach 1:

$$f = 1 - O\left(\frac{1}{\ln Q^2}\right) \quad (7.40)$$

we can ask¹⁴⁰⁾ for a Q^2 -dependent opening angle $\delta(Q^2)$ such that a fixed fraction f of events have $\geq 1 - \epsilon$ of their energy inside its ambit. It is clear from (7.39) that δ must decrease as a power of Q^2 which depends on f and ϵ :

$$\delta(Q^2) \approx \left(\frac{Q^2}{\Lambda^2}\right)^{-d(f,\epsilon)} \quad (7.41)$$

Values of d for gluon and quark singlet jets, for representative values of f and ϵ , are shown in the following table¹⁴⁰⁾:

$\epsilon \backslash f$	0.5	0.7	0.9	
0.1	0.28	0.19	0.08	← quark jet
	0.15	0.09	0.03	← gluon jet

Once again, we see that the perturbative width of a gluon jet is asymptotically wider than that of a quark jet, because its powers $d(f,\epsilon)$ are smaller. No evidence yet exists to support this expectation; presumably we need higher Q^2 gluon jets to test it. We await with interest the discovery of the topsilon -- if it exists -- for the moral of this lecture seems clear:

IF YOU WANT TO SNIFF GLUE, JOIN THE JET SET!

7.8 How can one really test QCD?

It seems appropriate and necessary to try to answer this question at the end of these lectures. Probably there will never be any single devastatingly convincing and conclusive test of QCD, in the same way that $(g - 2)$ has been a proving ground for QED. The reason is that in the range of Q^2 accessible to foreseeable experiments $\alpha_s(Q^2)$ is always relatively large (≥ 0.1) by comparison with the fine structure constant. This means that higher order effects will always be more important than in QED. To reduce the error on a leading order prediction to less than 1% requires a three-loop calculation in general, but these are generally more complicated than in QED, and require vast stamina. So for the foreseeable

future we will have to content ourselves with a statistically significant sample of qualitative or 10% accurate tests of QCD. These may be provided by some combination of moments of deep inelastic structure functions, of T decays, of $e^+e^- \rightarrow$ three-jet events and/or of other tests.

Faced with this somewhat unsatisfactorily qualitative state of affairs, what are the really crucial aspects of the theory that we would like to pin down, and how may we approach them?

7.8.1 Vector gluons

In order to distinguish vector from scalar gluons, one should continue the by now traditional measurements of valence deep inelastic structure functions, and of their close cousins the non-singlet fragmentation functions^{34,138}), whose moments should behave in specific ways [(2.57), (7.38)] if we have vector gluons with a small coupling constant. Another promising set of tests -- which also already yield some positive results¹¹⁸) -- are those involving the angular distributions of jets in heavy resonance decays^{129,130}) and the e^+e^- continuum¹²⁴). Probably we will soon be quite confident of the spin of the gluon.

7.8.2 Three-gluon vertex

This is expected to give relatively rapid violations of scaling in the distribution functions of gluons within hadrons, and of hadrons within gluon jets¹³⁷). It is also expected to make gluon jets wider than quark jets at very high energies¹¹⁶). It is unfortunately difficult to measure directly the distributions of gluons inside hadrons and see if they are Q^2 dependent, because gluons do not couple directly to lepton probes. Some indirect methods have been proposed¹⁴¹), but it seems that longitudinal and transverse momentum distributions in gluon jets -- believed to be copiously produced in heavy resonance decays -- will provide the best laboratory for searching for the three-gluon vertex. The gluon self-coupling probably causes the multiplicity to rise rapidly¹⁴²) at high Q^2 , but we may have to wait a while to test this.

7.8.3 Asymptotic freedom

It arises from the existence of the three-gluon coupling -- only non-Abelian gauge theories can be asymptotically free -- but it is obviously important to test independently. One method would be to check that the behaviour of deep inelastic moments is really logarithmic, $(\ln Q^2)^{-d_N}$, and exclude conclusively a power $(Q^2)^{-\delta_N}$ of Q^2 . Another possibility is to check the increasing validity of the jet perturbation theory at very high Q^2 , e.g. by comparing $e^+e^- \rightarrow q\bar{q}g$ events at PETRA and LEP. Alternatively, one may study the systematics of quarkonium properties as the mass of the heavy quark is increased: does $e_Q^2 \Gamma(\text{had})/\Gamma(e^+e^-)$ decrease for successive onia? Does the spectroscopy look increasingly Coulombic at large masses, and does the Coulomb potential seem to be softened by a power of $1/\ln Q^2$?

We can expect that answers to these and our other questions will be available eventually, and that QCD will at some time be either proved or disproved conclusively. But this may take a long while, and we should wait patiently. Or perhaps we will just be seduced by the beauty of QCD, and seek to embed it in a wider synthesis?

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$$\frac{a}{\mu} \xrightarrow[k]{b} \frac{v}{\nu} = -i\delta^{ab} \left[\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) / k^2 + \alpha \frac{k_\mu k_\nu}{k^4} \right]$$

$$\frac{a}{\mu} \xrightarrow{b} \frac{v}{\nu} = i\delta^{ab} / k^2$$

$$i \xrightarrow{j} = -i\delta^{ij} / \kappa$$

$$\begin{array}{c} a \xrightarrow{\lambda} \\ \mu \quad p \\ \swarrow \quad \searrow \\ q \quad r \\ \mu \quad \nu \\ b \quad c \end{array} = g_f^{abc} \left[(p-q)_\nu g_{\lambda\mu} + (q-r)_\lambda g_{\mu\nu} + (r-p)_\mu g_{\nu\lambda} \right]$$

$$\begin{array}{c} a \xrightarrow{\lambda} \quad d \xrightarrow{\sigma} \\ \mu \quad \nu \\ \swarrow \quad \searrow \\ b \quad c \\ \mu \quad \nu \end{array} = \begin{array}{l} -ig^2 f^{abe} f^{cde} (g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) \\ -ig^2 f^{ace} f^{bde} (g_{\lambda\mu} g_{\nu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) \\ -ig^2 f^{ade} f^{bce} (g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\mu} g_{\sigma\nu}) \end{array}$$

$$\begin{array}{c} a \xrightarrow{\mu} \\ i \quad j \end{array} = -ig\gamma^\mu (T^a)_{ij}$$

$$\begin{array}{c} b \xrightarrow{\mu} \\ a \quad p \quad c \end{array} = g_f^{abc} p^\mu$$

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

$$W \equiv \text{diagram with two circles} = \frac{1}{2\pi} \text{Im} T : T \equiv \text{diagram with one oval}$$

Fig. 10

Fig. 11

Fig. 12

$$\begin{aligned}
 & \text{Diagram 1} = \not{A} (\Delta \cdot k)^{n-1} \\
 & \text{Diagram 2} = -g \frac{\lambda^a}{2} \Delta_\mu \not{A} \sum_{j=1}^{n-1} (\Delta \cdot k_j)^{j-1} (\Delta \cdot k_2)^{n-1-j}
 \end{aligned}$$

Fig. 13

(a)

(b)

Fig. 14

Fig. 15

Fig. 16

Fig. 17

Fig. 18

Fig. 19

Fig. 20

$\Lambda = 0.3$
 $\Lambda = 0.5$
 $\Lambda = 0.7$

$$\left[\frac{M^N(q^2)}{M^N(q_0^2)} \right] \frac{1}{d_N}$$

$q^2 = 45 \text{ GeV}^2$
 $q_0^2 = 6.5 \text{ GeV}^2$

(a)

$$\left[\frac{M^N(q^2)}{M^N(q_0^2)} \right] \frac{1}{d_N}$$

$q^2 = 45 \text{ GeV}^2$
 $q_0^2 = 10^2 \text{ GeV}^2$

(b)

Fig. 21

Fig. 22

Fig. 23

$$\begin{pmatrix} \gamma_{ff} & \gamma_{gf} \\ \gamma_{fg} & \gamma_{gg} \end{pmatrix} = \begin{pmatrix} \text{diagram 1} & + \dots & \text{diagram 2} & + \dots \\ \text{diagram 3} & + \dots & \text{diagram 4} & + \dots \end{pmatrix}$$

The equation shows a matrix of vertex functions γ_{ij} on the left, equal to a sum of four diagrams in parentheses on the right. The diagrams represent various loop topologies involving quarks and gluons. The first two diagrams in the top row have a quark loop, and the first two in the bottom row have a gluon loop. Each diagram is followed by "+ ...", indicating a series of higher-order corrections.

Fig. 24

Fig. 25

Fig. 26

Fig. 27

Fig. 28

Fig. 29

Fig. 30

Fig. 31

(a)

(b)

Fig. 32

Fig. 33

Fig. 34

Fig. 35

Fig. 36

Fig. 37

Fig. 38

Fig. 39

Fig. 40

Fig. 41

(a)

(b)

(c)

(d)

(e)

(f)

Fig. 42

$$E_c \frac{d\sigma}{d^3p_c} (A+B \rightarrow CX) = \sum_{\text{quarks gluons}}$$

Fig. 43

(a)

(b)

(c)

(d)

Fig. 44

Fig. 45

Fig. 46

Fig. 47

Fig. 48

Fig. 49

Fig. 50

Fig. 51

Fig. 52

Fig. 53

$$\Psi = S K \Psi$$

Fig. 54

Fig. 55

Fig. 56

Fig. 57

Fig. 58

Fig. 59

Fig. 60

Fig. 61

Fig. 62

Fig. 63

Fig. 64

Fig. 65

Fig. 66

Fig. 67

Fig. 68

Fig. 69

Fig. 70

Fig. 71

Fig. 72

Fig. 73

Fig. 74

Fig. 75

Fig. 76

Fig. 77

Fig. 78

Fig. 79

Fig. 80

Fig. 81