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APPROACHING THE CHIRAL LIMIT IN QCD *)

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We use a version of Rayleigh-Schrödinger methods to evaluate the leading non-analytic corrections (LNAC) to first order chiral perturbation theory. We work out, in the framework of QCD, all LNAC to the mass of pions and give the corresponding expressions for bosons and fermions of any spin. We show that our formalism naturally extends to the evaluation of non-analytic corrections to other amplitudes.

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1. INTRODUCTION

(1)
QCD is presently the leading candidate for a complete description of the strong interactions in terms of only a few parameters: The running coupling constant $\alpha_{st}(Q^2)$ and the parameters of the quark mass matrix M . Rules have been developed for the expansion of physical quantities in powers of $\alpha_{st}(Q^2)$. These calculations dominantly cover processes at large momentum transfer, and the light (current) quark masses m_u , m_d and m_s are either set equal to zero or their effect must be considered as an unobservable 'renormalization' of bare quantities which become observable after 'renormalization' only⁽²⁾.

Nonvanishing current quark masses produce observable non-negligible effects e.g. in the hadron mass spectrum. Although no one has solved the confinement problem, we can calculate the influence of the symmetry breaking term $\bar{q}Mq$ on the hadron masses, once the mass spectrum in the chiral limit $M = 0$ is known. First order (in M) corrections to the hadron spectrum have been given already in the early days of SU(3). It has also been pointed out⁽³⁾ that the next to leading corrections must be nonanalytic in the parameters M due to the Nambu-Goldstone realization of the symmetric theory. These leading nonanalytic corrections (LNAC) have been evaluated for the pseudoscalar octet⁽⁴⁾, whereas the corresponding calculation for the baryons has not yet been done⁽⁵⁾, let alone the LNAC for any hadron of any spin.

We develop in the present article a concise tool to do perturbation theory (in M) around the chiral symmetric world. We observe that once the confinement problem is assumed to be solved, chiral perturbation theory is reduced to a problem whose solution is well known from nonrelativistic quantum mechanics: The Hamiltonian which describes the world (we consider here only QCD) may be written as

$$H = H_{QCD}^0 + \int \bar{q} M q d^3x$$

with a (supposedly) known spectrum of the symmetric part H_{QCD}^0 : The symmetry is thought to be realized à la Nambu-Goldstone with an octet of pseudoscalar Goldstone bosons. Rayleigh-Schrödinger perturbation methods⁽⁶⁾ would allow us to calculate the corrections due to the symmetry breaking term $\bar{q}Mq$ were it not that nonanalyticity in M forbids such an expansion. A possibility to cope with this difficulty is embodied in Feynman's formula⁽⁷⁾. It delivers the Taylor series expansion around $M \neq 0$. Nonanalytic terms are then discovered as derivatives

$\frac{\partial^n M}{\partial \lambda^n}$ which become singular as $M \rightarrow 0$, and the correct nonanalytic behaviour is regained by quadrature⁽⁸⁾.

We aim at the evaluation of the leading nonanalytic corrections for the meson- and baryon-octet, taking into account contributions from electromagnetic interactions. Such a calculation will allow a better determination of the mass ratios of the light quarks m_u , m_d and m_s ⁽⁹⁾ or, if coupled with U(6) relations⁽¹⁰⁾, a better absolute determination of these values (valid at some mass scale μ which will presumably be of the order of 1 GeV). These calculations are underway⁽¹¹⁾. In this article we present the basic tools which allow the evaluation of LNAC in the expansion of the mass of a particle of any spin.

The rest of this paper is organized as follows. In section 2 we evaluate LNAC in the nonlinear σ -model. This section is mostly pedagogical and may be skipped by those who are familiar with chiral perturbation theory. We only mention here that this nonlinear σ -model, well known to reproduce the current-algebra $\pi\pi$ -scattering results in tree approximation, leads at the one-loop level to the correct LNAC for the pions. In section 3 we adapt Feynman's formula to Quantum Field Theory. We end up with a set of formulae for the second derivative $\frac{\partial^2 M}{\partial \lambda^2}$ of the mass M of any hadron with respect to a symmetry breaking parameter λ (eqs. (3.5) - (3.9)). In section 4 we relate the parameter λ to the current quark masses and discuss the ultraviolet convergence properties of these mass formulae. In the next two sections we apply the formalism to calculate LNAC for Goldstone bosons, massive bosons (eqs. (5.8) and (5.11)) and massive fermions (eq. (6.5)). In the last section we summarize our results and present an outlook on future work yet to be done.

2. CHIRAL PERTURBATION THEORY IN THE NONLINEAR SIGMA MODEL

The tree graphs of the nonlinear σ -model are well known to reproduce the current algebra results for $\pi\pi$ -scattering. The model has been studied beyond the tree approximation (in its $SU(2) \otimes SU(2)$ symmetric version) e.g. in refs. (12,13). There it was found that the pole of the pion propagator stays at $p^2 = 0$ also if loop corrections are taken into account (at least in low orders of perturbation theory). Of course the pion will pick up a mass M_π if we add chiral symmetry breaking terms to the original symmetric Lagrangian. As has been shown quite generally⁽³⁾, M_π will be a nonanalytic function of the symmetry breaking parameter. We shall determine in this section the leading nonanalytic piece in M_π , together with the analogous LNAC to the pion decay constant F_π .

We add two remarks concerning this calculation:

1) Although the evaluation of M_π and F_π involves loop graphs, the final answer depends only on the on-shell behaviour of the Green functions. This will become clear in sections 3 and 4, where we give a representation of the mass shift in terms of on-shell quantities only. As a first argument in this direction and for pedagogical reasons, we show at the end of this section that a redefinition of the pion field, $\varphi^a = \phi^a - \frac{\alpha}{2F_\pi^2} \phi^a \phi^b \phi^b$, which results in a different off-shell behaviour of Green functions, nevertheless gives the same value for M_π and F_π as before.

2) It is amusing to see that the same Lagrangian which leads to the soft pion theorems of current algebra also gives the correct expression for the leading nonanalytic piece in M_π and F_π . In fact our final results (2.8a) and (2.8b) agree with the expressions obtained by Pagels and Langacker⁽⁸⁾ along different lines.

A. The model

The Lagrangian for the nonlinear σ -model is given by

$$\begin{aligned} \mathcal{L}(x) &= \mathcal{L}_{\text{sym}}(x) + \mu^2 c \mathcal{L}_{\text{sb}}(x) \\ \mathcal{L}_{\text{sym}}(x) &= \frac{1}{2} \partial_\mu \varphi^a \partial^\mu \varphi^a + \frac{1}{2c^2} \partial_\mu \Sigma \partial^\mu \Sigma + O(c^{-4}) \\ \mathcal{L}_{\text{sb}}(x) &= (c^2 - 2\Sigma)^{1/2} = c - \frac{1}{c} \Sigma - \frac{1}{2c^3} \Sigma^2 + O(c^{-5}) \end{aligned}$$

where

$$\Sigma = \frac{1}{2} \varphi^a \varphi^a \quad (2.1)$$

The 3 pseudoscalar fields φ^a , $a = 1, 2, 3$ form an isovector under $SU(2)$, and the symmetry breaking term $\mathcal{L}_{\text{sb}}$ transforms as $(\frac{1}{2}, \frac{1}{2})$ under $SU(2) \otimes SU(2)$. The model involves 3 parameters: The coupling constant $\frac{1}{c^2}$ (to be identified with $\frac{1}{F_\pi^2}$ up to loop corrections, see below), the mass parameter μ^2 and a cutoff Λ^2 which is used for regularization. (Extending loop integrals only up to momenta Λ is a chiral invariant regularization⁽¹²⁾.)

With our choice of the pion field, the 3 associated partially conserved axial-vector currents A_μ^a read

$$A_\mu^a = c \partial_\mu \varphi^a + \frac{1}{2c} \left\{ \partial_\mu \Sigma \varphi^a + \varphi^a \partial_\mu \Sigma - \Sigma \partial_\mu \varphi^a - \partial_\mu \varphi^a \Sigma \right\} + O(c^{-3}) \quad (2.2)$$

They have the formal property

$$\partial^\mu A_\mu^a = -\mu^2 c \varphi^a$$

(In the following no use will be made of this relation.)

It is clear that we cannot do a perturbation expansion in both parameters $\frac{1}{c^2}$ and μ^2 because of infrared divergences. Therefore calculations are done to all orders in μ^2 , but only to lowest order in $\frac{1}{c^2}$. This amounts to a partially summed perturbation series. The Feynman rules for a Lagrangian of the type (2.1) may be found e.g. in ref. (12).

B. Evaluation of M_π and F_π

We start with the calculation of the pion mass. Up to order $\frac{1}{c^2}$, we have to evaluate the contribution from the two graphs shown in Fig. 1.

One finds for the inverse pion propagator

$$\delta_b^a [\Delta_c'(p)]^{-1} = \delta_b^a \left(\mu^2 - p^2 - \frac{1}{c^2} \left(p^2 - \frac{12}{8} \mu^2 \right) \frac{1}{i} \Delta_c(0, \mu^2) \right)$$

where

$$\Delta_c(0, \mu^2) = (2\pi)^{-4} \int \frac{d^4 p}{\mu^2 - p^2} \quad (2.3)$$

Therefore,

$$M_\pi^2 = \mu^2 (1 + (2c^2 i)^{-1} \Delta_c(0, \mu^2)) \quad (2.4)$$

We see that M_π^2 vanishes as $\mu^2 \rightarrow 0$, in agreement with the remarks made at the beginning of this section.

The evaluation of F_π goes as follows. Let $|\pi^a(p)\rangle$ be the exact one-pion

eigenstate of the total Hamiltonian (including symmetry breaking terms), and let $|0\rangle$ be the corresponding vacuum. Then we define F_π through

$$\langle 0 | A_\mu^a(x) | \pi^b(p) \rangle = i p^\mu \delta_a^b F_\pi \exp[-i p x] \quad (2.5)$$

In order to expand F_π in powers of $\frac{1}{c^2}$, we must carry out mass renormalization. This is easily achieved: We write, according to eq. (2.4),

$$\mu^2 = M_\pi^2 (1 - (2c^2)^{-1} \Delta_c(0, M_\pi^2))$$

and obtain in the interaction picture

$$\langle 0 | A_\mu^a(x) | \pi^b(p) \rangle = \langle 0 | T[\tilde{A}_\mu^a(x) S] | \pi^b(p) \rangle \cdot (\langle 0 | S | 0 \rangle)^{-1}$$

where

$$S = T \exp[-i \int \mathcal{H}_{int}(x) d^4x]$$

$$\mathcal{H}_{int}(x) = -\frac{1}{2c^2} \left\{ \partial_\mu \Sigma \partial^\mu \Sigma - M_\pi^2 \Sigma^2 + M_\pi^2 \frac{1}{i} \Delta_c(0, M_\pi^2) \Sigma \right\} + O(c^{-4})$$

The fields in this equation are free fields of mass M_π . $|\pi^b(p)\rangle$ is a free one-pion state of mass M_π , and $|0\rangle$ denotes the vacuum. The current $\tilde{A}_\mu^a$ is obtained from eq. (2.2) by inserting free fields on the right-hand side of eq. (2.2).

Fig. 2 shows the 5 graphs which contribute to the matrix element (2.5) up to $O(\frac{1}{c^2})$.

One finds

$$F_\pi = -c (1 - (ic^2)^{-1} \Delta_c(0, M_\pi^2))$$

In order to compare our result with that given in ref. (8), we first replace the parameter $\frac{1}{c^2}$ with F_π^0 , the pion decay constant in the symmetry limit:

$$F_\pi^0 \doteq F_\pi(\mu^2=0) = -c (1 - (ic^2)^{-1} \Delta_c(0, 0)) \quad (2.6)$$

Then we use the mass parameter m_0 instead of μ ,

$$\delta_b^a m_o^2 \doteq \langle \pi^a(p) | \mathcal{H}_{sb}(0) | \pi^b(p) \rangle_{o, \text{connected}} \quad (2.7a)$$

where

$$\mathcal{H}_{sb} = \mu^2 \Sigma + \frac{\mu^2}{2c^2} \Sigma^2 + O(c^{-4})$$

and $|\pi^a(p)\rangle_o$ is an eigenstate of the total symmetric Hamiltonian. One easily finds

$$m_o^2 = \mu^2 (1 + (2c^2)^{-1} \Delta_c(0,0)) \quad (2.7b)$$

If we express c and μ^2 in terms of F_π^0 and m_o^2 , we find

$$F_\pi = F_\pi^0 (1 - (16\pi^2 F_\pi^{02})^{-1} M_\pi^2 \ln \frac{M_\pi^2}{\Lambda^2}) \quad (2.8a)$$

$$M_\pi^2 = m_o^2 (1 + (32\pi^2 F_\pi^{02})^{-1} m_o^2 \ln \frac{m_o^2}{\Lambda^2}) \quad (2.8b)$$

This result agrees with the calculation done in ref. (8), see Appendix A.

C. Transforming the pion field

As promised at the beginning of this section, we now show that the above results for M_π and F_π are independent of the definition of the pion field used in the Lagrangian (2.1). We define a new triplet ϕ^a through

$$\varphi^a = \phi^a - \frac{d}{2c^2} \phi^a \phi^b \phi^b, \quad d \in \mathbb{R}$$

The Lagrangian (2.1) becomes

$$\mathcal{L}(x) = \mathcal{L}_{sym}(x) + \mu^2 c \mathcal{L}_{sb}(x)$$

$$\mathcal{L}_{sym} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a + \frac{1}{2c^2} (1-2d) \partial_\mu \Sigma \partial^\mu \Sigma - \frac{d}{c^2} \partial_\mu \phi^a \partial^\mu \phi^a \Sigma + O(c^{-4})$$

$$\mathcal{L}_{sb} = c - \frac{1}{f} \Sigma + \frac{1}{2c^3} (4d-1) \Sigma^2 + O(c^{-5})$$

with

$$\Sigma = \frac{1}{2} \phi^a \phi^a$$

The new axial currents $A_\mu^a(x)_\alpha$ are

$$\begin{aligned} A_\mu^a(x)_\alpha = & c \partial_\mu \phi^a + \frac{1}{2c} (1-d) (\phi^a \partial_\mu \Sigma + \partial_\mu \Sigma \phi^a) \\ & - \frac{1}{2c} (1+d) (\Sigma \partial_\mu \phi^a + \partial_\mu \phi^a \Sigma) \end{aligned} \quad (2.9)$$

Carrying out exactly the same calculations as above, we find

$$\delta_b^a [\Delta'_C(p)_\alpha]^{-1} = \delta_b^a \left\{ \mu^2 - p^2 - \frac{1}{c^2} \left[(p^2 - \frac{3}{2} \mu^2) + 5d(\mu^2 - p^2) \right] \frac{1}{i} \Delta_C(0, \mu^2) \right\} \quad (2.10)$$

Obviously the value of $\Delta'_C(p)_\alpha$ at $p^2 = \mu^2$ remains unchanged, whereas its off-shell behaviour is α -dependent. In particular, eq. (2.10) shows that the pion wave function renormalization constant is α -dependent ($|\pi^b(p)\rangle_\alpha$, $|0\rangle_\alpha$ are eigenstates of the full Hamiltonian with $\alpha \neq 0$):

$$\langle 0 | \phi^a(0) | \pi^b(p) \rangle_\alpha = \delta_b^a \left[1 - (2c^2 i)^{-1} \Delta_C(0, \mu^2) (1 - 5d) \right]$$

The axial currents (2.9) lead to the same value of F_π as above, i.e.,

$${}_d \langle 0 | A_\alpha^\mu(x)_\alpha | \pi^b(p) \rangle_\alpha = i p^\mu \delta_a^b F_\pi \exp[-ipx]$$

3. MASS SHIFTS IN TERMS OF ON-SHELL QUANTITIES

We now develop a procedure to evaluate LNAC in the expansion of meson- and baryon masses around $m_u = m_d = m_s = 0$. This procedure is not restricted to QCD, but applies to the following general problem:

Given a Hamiltonian H_0 (not necessarily describing free particles) and its mass spectrum, add new terms to H_0

$$H = H_0 + \lambda H_I \quad (3.1)$$

and determine the behaviour of the spectrum as $\lambda \rightarrow 0$. We show in this and the following section

i) How to express the mass shift $M^2(\lambda) - M^2(0)$ (of a particle of any spin) in terms of on-shell matrix elements of H_I in the "symmetric" theory where $\lambda = 0$, provided $M^2(\lambda)$ is analytic at $\lambda = 0$.

If $M^2(\lambda)$ depends nonanalytically upon λ , we show

ii) How to calculate the leading nonanalytic piece of $M^2(\lambda)$ in terms of on-shell matrix elements of H_I in the limit $\lambda = 0$, provided these matrix elements are nonsingular.

We postpone the exact meaning of the parameter λ in Quantum Field Theory and the occurrence of ultraviolet divergences to section 4.

A. Basic mass formula

Let $M(\lambda)$ be the mass which characterizes a one-particle eigenstate of the total Hamiltonian $H = H_0 + \lambda H_I$. We can uncover nonanalyticity of $M(\lambda)$ at $\lambda = 0$ by differentiating M sufficiently often at $\lambda = 0$: Nonexistence of $\frac{\partial^n M}{\partial \lambda^n}$ at $\lambda = 0$ signals nonanalyticity (not vice versa). In fact it is enough to find the lowest derivative which is singular to get the LNAC. Consider e.g. $M_\pi^2(m_0^2)$ as given in eq. (2.8b). It suffices to know

$$\frac{\partial}{\partial m_0^2} \frac{\partial}{\partial m_0^2} M_\pi^2(m_0^2) \sim (16\pi^2 F_\pi^2)^{-1} \ln \frac{m_0^2}{\Lambda^2}$$

for the determination of the LNAC in M_π^2 . In general, given the singular derivative of lowest order, one finds the LNAC by simple quadrature.

In nonrelativistic quantum mechanics, these derivatives may be calculated using Feynman's trick⁽⁷⁾. Let $H|i\rangle = E_i(\lambda)|i\rangle$, $\langle i|j\rangle = \delta_{ij}$, $H = H_0 + \lambda V$. Then

$$\frac{\partial E_i}{\partial \lambda} = \langle i|V|i\rangle$$

and

$$\frac{\partial^2 E_i}{\partial \lambda^2} = 2 \sum_{E_j \neq E_i} |\langle j | V | i \rangle|^2 (E_i - E_j)^{-1}$$

(These formulae are also valid in the case of degenerate eigenvalues.)

In order not to interrupt the line of reasoning, we postpone the proof of the analogous formulae in Quantum Field Theory to Appendix B. They read

$$\begin{aligned} \frac{\partial M^2(\gamma)}{\partial \lambda} &= \langle \underline{p}, \gamma | \mathcal{H}_I(0) | \underline{p}, \gamma \rangle \\ \frac{\partial^2 M^2(\gamma)}{\partial \lambda^2} &= 2 \langle \underline{p}, \gamma | \mathcal{H}_I(0) \frac{1 - \mathcal{P}_0(\gamma)}{\mathcal{P}_0(\gamma) - H} H_I | \underline{p}, \gamma \rangle_{\text{connected}} \\ &\quad + \frac{1}{2 \mathcal{P}_0(\gamma)^2} |\langle \underline{p}, \gamma | \mathcal{H}_I(0) | \underline{p}, \gamma \rangle|^2 \end{aligned} \quad (3.2)$$

The notation is as follows: $|\underline{p}, \gamma\rangle$ is a one-particle eigenstate of H with quantum numbers γ . Its normalization is

$$\langle \underline{p}', \gamma' | \underline{p}, \gamma \rangle = 2(2\pi)^3 \mathcal{P}_0(\gamma) \delta^3(\underline{p}' - \underline{p}) \delta_{\gamma' \gamma} \quad ; \quad \mathcal{P}_0(\gamma) = (M^2(\gamma) + \underline{p}^2)^{1/2}$$

$\mathcal{P}_0(\gamma)$ is the projection operator

$$\mathcal{P}_0(\gamma) = \sum_{\gamma'}' \int |\underline{q}, \gamma'\rangle \langle \gamma', \underline{q}| d\mu(q) \quad ; \quad d\mu(q) = (2(2\pi)^3 q^0)^{-1} d^3q$$

where $\sum_{\gamma'}'$ denotes a sum over all one-particle states with $M_{\gamma'} = M_{\gamma}$. A spin average is understood on the right-hand side of eq. (3.2). The connection between H_I and

$\mathcal{H}_I(0)$ is as usual, $H_I = \int \mathcal{H}_I(x) d^3x$. Below we shall also use the notation $\mathcal{H}_I(x) = e^{iHx^0} \mathcal{H}_I(x) e^{-iHx^0}$. From here on we omit the label γ whenever no con-

fusion can occur. Furthermore we shall assume that there is no mixing,

$\langle \underline{p}', \gamma' | H_I | \underline{p}, \gamma \rangle \sim \delta^3(\underline{p}' - \underline{p}) \delta_{\gamma' \gamma}$. The projection operator $\mathcal{P}_0(\gamma)$ can then be replaced by $\mathcal{P}_0(\gamma) = \int |\underline{q}, \gamma\rangle \langle \underline{q}, \gamma| d\mu(q)$.

In order to prove the statements made under point i) above, we assume that $M^2(\lambda)$ is analytic in λ . Then we may put $\lambda = 0$ in eq. (3.2) and in the corresponding expressions for the higher derivative $\frac{\partial^n M^2}{\partial \lambda^n}$. In this way we obtain the co-

efficients of a Taylor series expansion of $M^2(\lambda)$ around $\lambda = 0$. These coefficients are given, up to numerical constants, by matrix elements of H_I in the symmetry limit $\lambda = 0$.

In QCD, it will turn out that $\frac{\partial^2 M^2}{\partial \lambda^2}$ is singular as $\lambda \rightarrow 0$. Such a singular behaviour can only occur from vanishing energy denominators in eq. (3.2). (We assume that $\langle p, \gamma | \mathcal{H}_I(0) | \alpha \rangle$ is finite as $\lambda \rightarrow 0$.) These denominators will vanish whenever the intermediate states $|\alpha\rangle$ contain Goldstone bosons such that $E_p \sim E_\alpha$. Given this singular second derivative, we get the LNAC by quadrature, as was mentioned above. The coefficients of this LNAC are determined by the matrix elements of $\mathcal{H}_I(0)$ in the chiral limit $\lambda = 0$. This is what we asserted in point ii).

Remark: This general result is not yet of great help in the evaluation of quark mass ratios. It is just a statement about a generalization of usual perturbation theory to nonanalytic interactions. We shall see that in fact a much stronger result is valid: Leading nonanalytic corrections do only depend on the matrix elements of $\mathcal{H}_I(0)$ between single-particle states. These matrix elements are also present in first order perturbation theory. Therefore LNAC do not introduce new parameters. (This is strictly true for fermions, see eq. (6.5). For bosons see eqs. (5.8), (5.11) and remark after eq. (5.6b)).

Eq. (3.2) constitutes our basic formula from which we could in principle evaluate all the necessary mass shifts. However, it turns out that we may rewrite (3.2) in a much more compact way which avoids e.g. the discussion of disconnected graphs. This improved representation of $\frac{\partial^2 M^2}{\partial \lambda^2}$ renders the evaluation of LNAC much easier, and the discussion of ultraviolet divergences in section 4 will be very simple.

B. Connection with commutator function

The connection of eq. (3.2) with the usual formulas for mass shifts is not quite straightforward, since in our case H_I will have nonvanishing matrix elements between single-particle states. Were this not the case, we would have

$$\frac{\partial^2 M^2}{\partial \lambda^2} \sim \int \langle p | T[\mathcal{H}_I(x) \mathcal{H}_I(0)] | p \rangle d^4 x \quad (3.3)$$

The right-hand side of this equation of course does not make sense if

$\langle p | \mathcal{H}_I(0) | p \rangle \neq 0$. The recipe to consider only 1PI contributions in eq. (3.3) is not adequate: One-particle reducible contributions in eq. (3.3) do not coincide with one-particle reducible contributions in eq. (3.2).

We show now that the correct expression for $\frac{\partial^2 M^2}{\partial \lambda^2}$ may be obtained from the retarded commutator, viz,

$$R(p, q) = i \int d^4x \exp[iq \cdot x] \langle p | \Theta(x^0) [\mathcal{H}_I(\frac{x}{2}), \mathcal{H}_I(-\frac{x}{2})] | p \rangle_{\text{connected}} \quad (3.4)$$

where "connected" means that we disregard all vacuum bubbles.

To see the connection with eq. (3.2), we insert a complete set of intermediate states in eq. (3.4) and consider the contributions from one-particle intermediate states separately:

$$R^{1P} = R^+ + R^-$$

$$R^\pm = [(2E^\pm)(E^\pm - p^0 \mp q^0)]^{-1} |\langle p | \mathcal{H}_I(0) | p \pm q \rangle|^2; \quad E^\pm = (M^2 + (p \pm q)^2)^{1/2}$$

This part R^{1P} of the retarded commutator (3.4) does not in general have a decent limit as $q^\mu \rightarrow 0$. However, one easily finds that the following two interesting limits exist:

$$i) \lim_{q^0 \rightarrow 0^+} \lim_{q \rightarrow 0} R^{1P}(p, q) = 0$$

and

$$ii) \lim_{q^0 \rightarrow 0^+} \lim_{q^2 \rightarrow 0} \lim_{p \rightarrow 0} R^{1P}(p, q) = -\frac{1}{2M^2} |\langle p=0 | \mathcal{H}_I(0) | p=0 \rangle|^2$$

Since

$$\lim_{q^\mu \rightarrow 0} (R(p, q) - R^{1P}(p, q)) = -2 \langle p | \mathcal{H}_I(0) (p^0 - H)^{-1} H_I | p \rangle_{PI}$$

we obtain the following two possibilities for the evaluation of $\frac{\partial^2 M^2}{\partial \lambda^2}$:

$$\frac{\partial^2 M^2}{\partial \lambda^2} = -\lim_{q^0 \rightarrow 0^+} \lim_{q \rightarrow 0} R(p, q) + \frac{1}{2p^0^2} |\langle p | \mathcal{H}_I(0) | p \rangle|^2 \quad (3.5)$$

or

$$\frac{\partial^2 M^2}{\partial \lambda^2} = -\lim_{q^0 \rightarrow 0^+} \lim_{q^2 \rightarrow 0} \lim_{p \rightarrow 0} R(p, q) \quad (3.6)$$

The formulae in eqs. (3.2), (3.5) and (3.6) are discussed in Appendix C in the case of free fields.

We end this section with the decomposition of $R(p, q)$ into manifestly connected pieces.

C. Cluster decomposition of $R(p, q)$

The cluster expansion leads to a very convenient representation of $R(p, q)$. We do not repeat its derivation here, but refer to the literature⁽¹⁴⁾.

We shall see in section 5C that we need a slightly generalized version of $R(p, q)$: One has to consider $R(p, q)$ as the limit of the nonforward matrix element

$$R_{\Delta}(P, q) = i \int d^4x \exp[iqx] \langle \underline{p}_2 | \theta(x^0) [\mathcal{H}_I(\frac{x}{2}), \mathcal{H}_I(-\frac{x}{2})] | \underline{p}_1 \rangle_{\text{connected}}$$

where

$$\Delta^{\mu} = (p_2 - p_1)^{\mu} \quad ; \quad P^{\mu} = \frac{1}{2} (p_1 + p_2)^{\mu} \quad ; \quad p_1^{\mu} \equiv p^{\mu}$$

and we must calculate $R(p, q)$ via

$$R(p, q) = \lim_{\Delta^{\mu} \rightarrow 0} R_{\Delta}(P, q) \quad (3.7)$$

The dispersion relation and cluster decomposition for the quantity $R_{\Delta}(P, q)$ read

$$R_{\Delta}(P, q) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dq'_0}{q'_0 - q_0 - i\epsilon} V_{\Delta}(q', P) \quad (3.8a)$$

where

$$q'^{\mu} = (q'_0, \underline{q}) \quad (3.8b)$$

and

$$V_{\Delta}(q, P) = \frac{1}{2} \int d^4x \exp[iqx] \langle \underline{p}_2 | [\mathcal{H}_I(\frac{x}{2}), \mathcal{H}_I(-\frac{x}{2})] | \underline{p}_1 \rangle_{\text{connected}}$$

For $q^0 > 0$, one has

$$\begin{aligned}
 V_{\Delta}(q, P) &= V_D + V_Z + V_E \\
 V_D &= \frac{1}{2}(2\pi)^4 \sum_{\alpha} \langle p_2 | \mathcal{H}_I(0) | \alpha \rangle_c \langle \alpha | \mathcal{H}_I(0) | p_1 \rangle_c \delta^4(P + q - p_{\alpha}) \\
 V_Z &= \pm \frac{1}{2}(2\pi)^4 \sum_{\alpha} \langle 0 | \mathcal{H}_I(0) | p_1, \alpha \rangle_c \langle \alpha, p_2 | \mathcal{H}_I(0) | 0 \rangle_c \delta^4(q - P - p_{\alpha}) \\
 V_E &= \frac{1}{2}(2\pi)^4 \sum_{\alpha} \langle p_2 | \mathcal{H}_I(0) | p_1, \alpha \rangle_c \langle \alpha | \mathcal{H}_I(0) | 0 \rangle_c \delta^4(q + \frac{\Delta}{2} - p_{\alpha}) \\
 &\quad + \frac{1}{2}(2\pi)^4 \sum_{\alpha} \langle 0 | \mathcal{H}_I(0) | \alpha \rangle_c \langle \alpha, p_2 | \mathcal{H}_I(0) | p_1 \rangle_c \delta^4(q - \frac{\Delta}{2} - p_{\alpha})
 \end{aligned} \tag{3.9}$$

The minus sign in the expression for V_Z refers to the case where the particle under consideration is a fermion. The index c means "connected". The so-called direct graphs V_D , Z-graphs V_Z and the "second class" graphs V_E are visualized in Fig. 3.

Eqs. (3.5) - (3.9) constitute our final tool for the evaluation of $\frac{\partial^2 M^2}{\partial \lambda^2}$ in QCD. It remains to discuss the meaning of the parameter λ and the convergence of the dispersion relation in eq. (3.8a). This is done in the next section.

4. CALCULATING NONANALYTIC TERMS IN QCD

In order to apply the mass formulae of section 3 in QCD, we must

- i) give the connection between the parameter λ and the quark mass matrix
- ii) discuss the convergence of the dispersion relation in eq. (3.8a).

We discuss these two points in turn. At the end of this section we summarize the procedure to find LNAC to hadron masses in QCD.

A. Connection between λ and the mass matrix

Let us start the discussion in QED, where the bare Lagrangian is given by

$$\begin{aligned}
 \mathcal{L}_{QED} &= \bar{\Psi}_0 i \gamma^{\mu} \partial_{\mu} \Psi_0 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
 &\quad - e_0 \bar{\Psi}_0 \gamma_{\mu} \Psi_0 A_0^{\mu} - m_0 \bar{\Psi}_0 \Psi_0
 \end{aligned} \tag{4.1}$$

$\psi_0, A_0^\mu, c_0, m_0$ denote bare quantities. Suppose we evaluate the physical mass of the electron as a function of m_0, e_0 and a cut-off Λ : $M = M(e_0, m_0, \Lambda)$. It is then easy to calculate the variation of M with the parameter m_0 : In order to construct the Hamiltonian from (4.1), we need the fields and their derivatives at a fixed time, and these values are independent of any mass⁽¹⁵⁾. Therefore the bare mass m_0 plays the role of the coupling constant λ , see eq. (3.1); all other quantities in eq. (4.1) are independent of m_0 . We conclude therefore from eq. (3.2) that

$$m_0 \frac{\partial M}{\partial m_0} = \frac{1}{2M} \frac{1}{2} \sum_s \langle p, s | m_0 \bar{\psi}_0 \psi_0 | p, s \rangle \quad (4.2)$$

This result coincides with eq. (2.8) of ref. (16). (The authors of this reference made use of the Callan-Symanzik equation to get this result.) Clearly we are not interested in the variation of M with the bare mass m_0 . But we may replace m_0 by $m_R(\mu)$,

$$m_0 = m_R(\mu) Z_m$$

where $m_R(\mu)$ is an arbitrary mass which plays the role of a coupling constant. μ denotes a mass scale introduced in the regularization scheme. The renormalization factor may be chosen to be mass independent⁽¹⁷⁾. Renormalized operators $(\bar{\psi}\psi)_R(\mu)$ are introduced in the usual way,

$$m_0 \bar{\psi}_0 \psi_0 = m_R(\mu) (\bar{\psi}\psi)_R(\mu)$$

This allows us to rewrite eq. (4.2),

$$\frac{\partial M}{\partial m_R} = \frac{1}{2M} \frac{1}{2} \sum_s \langle p, s | (\bar{\psi}\psi)_R | p, s \rangle$$

Now we turn to QCD. For simplicity we restrict ourselves to 2 quark flavors u, d from here on. The Lagrangian in question is

$$\begin{aligned} \mathcal{L}_{\text{QCD}} = & i \bar{u} \not{\partial} u + i \bar{d} \not{\partial} d - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - m_0^u \bar{u} u - m_0^d \bar{d} d \\ & + \text{gauge fixing terms} + \text{ghost terms} \end{aligned} \quad (4.3)$$

in usual notation⁽¹⁸⁾.

The relation between the bare masses m_u^0, m_d^0 and $m_u(\mu), m_d(\mu)$ reads as above,

$$m_i^0 = m_i(\mu) Z_m, \quad i = u, d$$

with Z_m flavor- and mass independent⁽¹⁹⁾. (This guarantees that quark mass ratios will be μ -independent.) Furthermore we introduce the renormalized operators $(\bar{u}u)_R$

$$m_u^0 \bar{u} u = m_u(\mu) (\bar{u} u)_R(\mu)$$

and similarly for $(\bar{d}d)_R$. It is most convenient to have only one coupling constant in which we expand the hadron masses. Therefore we set

$$m_u(\mu) = m(\mu) \xi_u, \quad m_d(\mu) = m(\mu) \xi_d \quad (4.4)$$

where $m(\mu)$ is an arbitrarily chosen mass > 0 , and ξ_u, ξ_d are positive numbers. We omit from here on the label μ .

This discussion shows that we may write the symmetry breaking part of the QCD Hamiltonian as

$$H_I = \int (m_u^0 \bar{u} u + m_d^0 \bar{d} d) d^3x = m \int \mathcal{H}_\xi(x) d^3x$$

with

$$\mathcal{H}_\xi = \xi_u (\bar{u} u)_R + \xi_d (\bar{d} d)_R \quad (4.5)$$

Hence the parameter m can be used as a coupling constant, on the same footing as the parameter λ in eq. (3.1). The extension to more flavors is obvious.

Now we turn to the discussion of point ii).

B. Convergence of mass formula

The convergence of the dispersion relation in eq. (3.8a) is governed by the short-distance behaviour of the commutator $[\mathcal{H}_\xi(x), \mathcal{H}_\xi(0)]$, which may be analyzed using its operator product expansion⁽²⁰⁾. This is the nice feature of our choice for the representation of the retarded commutator $R(p, q)$ via the dispersion integral (3.8a): since QCD is asymptotically free⁽²¹⁾, we may unambiguously discuss the ultraviolet behaviour of the dispersion integral in eq. (3.8a).

In fact this discussion is easy. Since there are no gauge-invariant operators of dimension less than or equal to 2 which might contribute to the connected part of the matrix element in question, we can conclude that the integral in (3.8a) is in fact convergent. Furthermore the ambiguity⁽²²⁾ in the definition of the retarded commutator in eq. (3.4) does not invalidate eq. (3.2): There is one retarded decomposition of the commutator which agrees with the formula for $\frac{\partial^2 M^2}{\partial \lambda^2}$ in eq. (3.2), namely the representation $R(p,q)$ given in eq. (3.8a). Now let $\tilde{R}(p,q)$ be any other retarded decomposition. Then $\tilde{R}(p,q) = R(p,q) + (\text{Polynomial in } v \text{ and } q^2)$. This polynomial does not contribute to $\frac{\partial^2 M^2}{\partial \lambda^2}$, since we have to evaluate $\tilde{R}(p,q)$ at $q^\mu = 0$, see eq. (3.2). Hence eq. (3.2) is valid for any decomposition $\tilde{R}(p,q)$. In the following we choose the one defined in eq. (3.8a).

C. Final recipe to find LNAC in QCD

The evaluation of LNAC in QCD runs as follows:

We look for singularities of $\frac{\partial^2 M^2}{\partial m^2}$ as $m \rightarrow 0$. This can be done via eqs. (3.5)-(3.9): Replace λ by m (eq. 4.4) and $\mathcal{H}_I(0)$ by $\mathcal{H}_\xi(0)$ (eq. 4.5). Then go to eq. (3.9) and take into account those intermediate states which give divergent contributions in the chiral limit. Prove that all other intermediate states may be neglected. The final result is then found by a quadrature.

We summarize at the beginning of the next section all assumptions which allow the evaluation of LNAC in QCD, like spectrum of H_{QCD} in the chiral limit, current algebra theorems and so on.

5. LNAC TO HADRON MASSES. THE CASE OF BOSONS

In this and the following section we evaluate the effect of the quark mass term $\bar{q} M q$ on the hadron spectrum in the leading nonanalytic approximation. Before we plunge into calculations, we wish to list here all the assumptions from which our final results follow.

A. Basic assumptions

1) The Lagrangian of the world is the QCD-Lagrangian, eq. (4.3). The corresponding Hamiltonian is written as

$$H_{QCD} = H_{QCD}^0 + m \int \mathcal{H}_\xi(\underline{x}) d^3x \quad (5.1)$$

where

$$\mathcal{H}_\xi = \xi_u(\bar{u}u)_R + \xi_d(\bar{d}d)_R$$

see section 4.A. Here and later on the mass scale μ in the mass parameter m and in $\mathcal{H}_\xi$ is suppressed.

2) Chiral symmetry is spontaneously broken. In the symmetry limit, we have an isotriplet of massless pseudoscalar Goldstone bosons (3 pions). No other massless particles are present.

3) Beside these Goldstone bosons, the spectrum of H_{QCD}^0 contains

- isomultiplets of bosons with definite G-parity
- isomultiplets of fermions

In the following we denote the common mass within a boson (fermion) multiplet with $M_B^0(M_F^0)$.

4) We consider all particles to be stable.

5) The current algebra low-energy theorems for on-shell π -hadron scattering are valid.

6) The symmetry breaking part of the Hamiltonian has the following property:
Let

$$V(q) = \int d^4x \exp[iq \cdot x] \langle \alpha | [\mathcal{H}_\xi(x), \mathcal{H}_\xi(0)] | \alpha \rangle$$

where $|\alpha\rangle$ is any one-particle eigenstate of the total Hamiltonian H_{QCD} . We assume that $V(q)$ does not develop mass singularities as $m \rightarrow 0$, i.e.,

$$\lim_{m \rightarrow 0} V(q) = V^0(q) \quad (5.2)$$

exists.

These assumptions allow the evaluation of LNAC to the mass of pions (eq. (5.8)), massive bosons (eq. (5.11)) and fermions (eqs. (6.5) and (6.6)) via eqs. (3.5) - (3.9) and the procedure described at the end of section 4.

Comments:

Evidently assumption 1) must be replaced in later work in order to include more quark flavors and electromagnetic interactions. The origin of the quark masses is left open in this approach. Assumptions 2) and 5) will have to be modified accordingly. Clearly assumption 4) will be dropped in later work, and our scheme for the evaluation of LNAC will need modification in the presence of unstable particles. Finally we come to point 6). The softness property of the commutator $V(q)$ as stated in eq. (5.2) will allow us to single out in a simple way those intermediate states in eq. (3.9) which potentially might contribute to LNAC. In order to understand better the meaning of this assumption, we may relate $V(q)$ to a better known quantity as follows. Since $\mathcal{H}_\xi(0) \sim \bar{q}q$, we expect that the commutator of the electromagnetic currents $\bar{q}Q\gamma\mu q$ has the same behaviour as $V(q)$ in the chiral limit. Then eq. (5.2) presumes that the inclusive cross section

$$e + d \rightarrow e + X$$

does not develop mass singularities as $m \rightarrow 0$. The proof of this property requires the solution of the confinement problem in QCD. For a partial answer to these questions in the framework of perturbation theory see ref. (23).

We now consider the above mentioned isomultiplets in turn. The procedure will be as follows. First we eliminate many-pion intermediate states in eq. (3.9) with the help of assumptions 5) and 6) above. Then we evaluate the contributions from the remaining graphs to LNAC. The rest of this section deals with pions and massive bosons, whereas the calculation for fermions is done in section 6.

B. Pions. Elimination of many-pion intermediate states

We see from eq. (3.9) that we can dispose of massive intermediate states: They cannot contribute to LNAC, since the threshold in the dispersion relation (eq. (3.8a)) stays above zero also in the chiral limit. Therefore pion-intermediate states are the only candidates which may lead to nonanalytic terms. They are shown in Fig. 4.

Now we prove that in fact only the graphs D, Z, E_1 and E_2 contribute. They lead to absorptive parts which are $O(1)$ as $q^\mu \rightarrow 0$, see below. So we have to show that the rest of the diagrams in Fig. 4 have absorptive parts which are softer at $q^\mu \rightarrow 0$.

Since we only want to prove finiteness of the corresponding contributions to $R(p,q)$, we may work in the chiral limit from the very beginning. Furthermore

we set $p^\mu = 0$ and go to $q^\mu \rightarrow 0$ according to eq. (3.5), e.g., we also set $q = 0$ from the beginning. Consider now the graphs $D(n)$. Then

$$V^0(q^0) \sim \int d\mu_1 \dots d\mu_{n+1} \delta^4(q - \sum_{i=1}^{n+1} p_i) |\langle \pi^a(0) | \mathcal{H}_\xi(0) | \pi^{b_1}(p_1) \dots \pi^{b_{n+1}}(p_{n+1}) \rangle_0|^2$$

where $|\pi^{a_1}(p_1) \dots \pi^{a_n}(p_n) \rangle_0$ is an n -pion eigenstate of the symmetric Hamiltonian H_{QCD}^0 .

Set $q^0 = \eta \tilde{q}^0$, $\tilde{q}^0$ fixed and let $\eta \rightarrow 0$. Then we find

$$V^0(\eta \tilde{q}^0) \sim \eta^{2n-2} V^0(\tilde{q}^0), \quad \eta \rightarrow 0 \quad (5.3)$$

Since, due to the softness of the $\pi\pi$ -scattering amplitude (assumption 5), the matrix elements ${}_0\langle \pi^a(0) | \mathcal{H}_\xi(0) | \pi^{a_1} \dots \pi^{a_n} \rangle_0$ are $O(1)$ as all the momenta vanish. Since we assumed that $V(q)$ does not have mass singularities, we conclude that

$$V^0(q^0) \sim (q^0)^{2n-2} \text{ as } q^0 \rightarrow 0$$

We see that only the term with $n = 1$ may contribute. Its coefficient is proportional to ${}_0\langle \pi^a(0) | \mathcal{H}_\xi(0) | \pi^{b_1}(0) \pi^{c_1}(0) \rangle_0^2$ which vanishes (parity). Hence the graphs $D(n)$ can be neglected in the calculation of LNAC. The proof that $Z(n)$ and $E_i(n)$ do not contribute runs exactly the same way.

The benefit from our assumption that $V(q)$ does not develop mass singularities is the following. It might happen that the power counting in eq. (5.3) is wrong, since some inverse powers of p_ν might be converted to inverse powers of M_π as $m \rightarrow 0$. Then

$$V(q^0) \rightarrow \frac{1}{M_\pi^\alpha} (q^0)^\beta, \quad m \rightarrow 0$$

but power counting would say $V(q^0) \rightarrow (q^0)^{(\beta-\alpha)}$. Although $\beta - \alpha \geq 0$, we had to conclude that the graphs in question are not only more singular than D , Z and E_i , but, what is worse, one had to know the absorptive parts $V(q^0)$ all the way up to $q^0 \rightarrow \infty$ in order to find LNAC: We see therefore that assumption 6) is essential if we want to say anything about LNAC.

C. Pions. Leading nonanalytic pieces

We start with the diagrams D and Z. From eq. (3.6) we easily find

$$\left(\frac{\partial^2 M_\pi^2}{\partial m^2}\right)_{D+Z} = -2 \left[H(4M_\pi^2, M_\pi^2) + H(0, M_\pi^2) \right] \frac{1}{4M_\pi^2} \left[H(4M_\pi^2, M_\pi^2) - H(0, M_\pi^2) \right] \quad (5.4a)$$

where

$$\delta_b^a H(s, M_\pi^2) = \langle \pi^a(p) \pi^b(p') \text{ out} | \mathcal{H}_F(0) | 0 \rangle ; \quad s = (p+p')^2$$

$|\pi^a(p) \pi^b(p') \text{ out}\rangle$ is an eigenstate of the total Hamiltonian H_{QCD} , and $|0\rangle$ its vacuum. Next we write the difference $H(s, M_\pi^2) - H(0, M_\pi^2)$ as

$$H(s, M_\pi^2) - H(0, M_\pi^2) = \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'(s'-s)} \text{Im} H(s', M_\pi^2) \quad (5.4b)$$

$$\delta^{ab} \text{Im} H(s, M_\pi^2) = -\frac{i}{2} \sum_d \langle \pi^a(p) \pi^b(p') \text{ in} | (S-1) | d \text{ in} \rangle \langle 0 | \mathcal{H}_F(0) | d \text{ in} \rangle^*$$

S denotes the S -operator. The ultraviolet convergence of the dispersion integral is not related to the short-distance behaviour of QCD. We have to make the assumption that the singularity of the integral in eq. (5.4b) at $M_\pi \rightarrow 0$ is not affected by its ultraviolet behaviour.

The LNAC from 2 pion intermediate states in eq. (5.4b) gives

$$\left(\frac{\partial^2 M_\pi^2}{\partial m^2}\right)_{D+Z} = \frac{1}{m^2} \frac{1}{8\pi^2} \frac{\sigma_\pi^2}{F_\pi^0{}^2} (\delta_c^c - 1) \ln \frac{M_\pi^2}{\Lambda^2} \quad (5.5)$$

where we have defined

$$\delta^{ab} \sigma_\pi = \langle \pi^a(p) | [m_u(\bar{u}u)_R + m_d(\bar{d}d)_R] | \pi^b(p) \rangle_0 \quad (5.6a)$$

and F_π^0 is the pion decay constant in the chiral symmetric world,

$$\langle 0 | A_a^\mu(x) | \pi^b(p) \rangle_0 = i p^\mu \delta_a^b F_\pi^0 \exp[-ipx] \quad (5.6b)$$

The value of the cutoff Λ which is present in eq. (5.5) needs a careful discussion once we want to extract quark mass ratios from our analysis. We leave this point open in the present article.

Many-pion intermediate states in (5.4b) do not contribute to LNAC. This is shown with the same arguments as we used above in case of the graphs $D(n)$.

Now we come to diagrams E_1 and E_2 (Fig. 4). First we consider E_1 . The lower part is given by the matrix element $\langle \pi^a(p_2) | \mathcal{H}_\xi(0) | \pi^a(p_1) \pi^b(k) \pi^c(k') \rangle$ in. We show in Fig. 5 those contributions to this matrix element which make it ill-defined in the limit $\Delta^\mu = (p^2 - p'^2)^\mu \rightarrow 0$. This singularity is fictitious and has nothing to do with massless pions ($M_\pi \neq 0$ at this stage of the calculation!). It forces us to consider $R(p, q)$ as the limit

$$R(p, q)_{E_1+E_2} = \lim_{\Delta^\mu \rightarrow 0} R_\Delta(p, q)_{E_1+E_2}$$

In fact it is not hard to see that the sum of the absorptive parts $V_\Delta(p, q)_{E_1+E_2}$ stays finite as $\Delta^\mu \rightarrow 0$.

We encounter a second difficulty when we do the dispersion integral in eq. (3.8a): The absorptive part $V(p, q)_{E_1+E_2}$ has an (integrable) singularity at $q^2 = 4\mu^2$ for $\underline{p} = \underline{q} = 0$. We only note that a careful analysis shows that one gets the correct answer for the nonanalytic piece in $\frac{\partial^2 M_\pi^2}{\partial m^2}$ if we first set $M_\pi^2 = 0$ in $V(p, q)_{E_1+E_2}$. Then the application of standard current algebra techniques to the matrix element $\langle \pi^a(p_1) | \mathcal{H}_\xi(0) | \pi^a(p_2) \pi^b(k) \pi^c(k') \rangle$ in leads to

$$\left(\frac{\partial^2 M_\pi^2}{\partial m^2} \right)_{E_1+E_2} = - \frac{1}{m^2} \frac{3}{16\pi^2} \frac{G_\pi^2}{F_\pi^2} \ln \frac{M_\pi^2}{\Lambda^2} \quad (5.7)$$

We have checked in the nonlinear σ -model (section 2) that the procedure described here to deal with the difficulties inherent in the graphs E_1 and E_2 gives the correct result (at the one loop level) for the nonanalytic piece of $\frac{\partial^2 M_\pi^2}{\partial m^2}$.

Adding eq. (5.5) and (5.7) we conclude that

$$M_\pi^2 = G_\pi \left(1 + (32\pi^2 F_\pi^2)^{-1} G_\pi \ln \frac{G_\pi}{\Lambda^2} \right) \quad (5.8)$$

This result agrees with eq. (2.8b), see eqs. (2.7a), (5.6a).

D. Leading nonanalytic corrections (heavy bosons)

The relevant graphs are again shown in Fig. 4. Since we assume all particles to be stable, graphs like those in Fig. 6 are excluded from our considerations.

Firstly we eliminate nonleading contributions. From eq. (5.4a) we see that D and Z do not lead to LNAC, since M_π^2 in the denominator of eq. (5.4a) is replaced by $(M_B^0)^2$. The same argument as in the case of the pions shows that $D(n)$ and $E_i(n)$ may be neglected. What remains is the evaluation of E_1 and E_2 . A calculation identical to the pion case leads to

$$\left(\frac{\partial^2 M_B^2}{\partial m^2} \right)_{E_1+E_2} = - \frac{3}{16\pi^2} \frac{\sigma_\pi \sigma_B}{m^2 F_\pi^2} \ln \frac{M_\pi^2}{\Lambda^2} \quad (5.9)$$

where we used the notation

$$\frac{1}{2s+1} \sum_{s_z} \langle B^a(k); s, s_z | m \mathcal{H}_\xi(0) | B^b(k); s, s_z \rangle_0 = \delta^{ab} \sigma_B \quad (5.10)$$

$|B^a(k); s, s_z \rangle_0$ is a one-boson eigenstate of H_{QCD}^0 with spin $s(s_z)$, isospin $I^3 = a$. Since this state has definite G-parity, only the isoscalar part of $m \mathcal{H}_\xi(0)$ contributes to this matrix element. Notice that the spin of the boson does not lead to any complication in the calculation. Its only occurrence is in a spin average in the definition of σ_B , see above. Sometimes life is easy.

We conclude from eq. (5.9) that

$$M_B^2 = (M_B^0)^2 + \sigma_B \left(1 - 3(32\pi^2 F_\pi^2)^{-1} \sigma_\pi \ln \frac{\sigma_\pi}{\Lambda^2} \right) \quad (5.11)$$

The reason that $\frac{\partial^2 M_B^2}{\partial m^2}$ is only logarithmically divergent hinges on the fact that the graph shown in Fig. 7a) vanishes in the chiral limit (G-parity). Otherwise we would have to conclude that

$$M_B^2 = (M_B^0)^2 + \sigma_B + \text{const} \cdot \sigma_\pi^{3/2}$$

see next section.

6. LNAC FOR FERMIONS

We turn now to fermion masses. Let us first single out the most singular graphs in Fig. 4. Clearly D and Z do not contribute for the same reason as before. We find

$$V^0(q^0) \sim (q^0)^{2n-3}, \quad q^0 \rightarrow 0 \quad \text{for } D(n), n \geq 1$$

and

$$V^0(q^0) \sim (q^0)^{2n-5}, \quad q^0 \rightarrow 0 \quad \text{for } E_i(n), n \geq 2 \quad (6.1)$$

Use has been made of low-energy theorems for π -hadron $\rightarrow \pi$ -hadron scattering. Simple power counting suggests therefore that D(1) and $E_i(2)$ lead to singular absorptive parts $V^0(q^0) \sim (q^0)^{-1}$ as $q^0 \rightarrow 0$. However, it can be shown that the graphs $E_i(2)$ are in fact O(1) as $q^0 \rightarrow 0$. This is most easily seen by writing the π -fermion scattering amplitude (which occurs in the leading contribution to $E_i(2)$, see Fig. 5a), b)) as

$$\begin{aligned} i(2\pi)^4 \delta^4(P_f - P_i) T_{\pi f}^{aa} &= \langle F, \pi^a(k') | (S-1) | F, \pi^a(k) \rangle_0 \\ &= -\frac{1}{F_\pi^2} k'^\mu k^\nu \langle F' | T[\tilde{A}_\mu^a(k') \tilde{A}_\nu^a(k)] | F \rangle_0 \end{aligned} \quad (6.2)$$

where $\tilde{A}_\mu^a(k) = \int d^4x e^{ikx} \hat{A}_\mu^a(x)$, $\langle 0 | \hat{A}_\mu^a(x) | \pi^b(k) \rangle_0 = 0$ and no summation over the index a is understood. $|F, \pi^b(k) \rangle_0$ is a one fermion-one meson eigenstate of H_{QCD}^0 with all quantum numbers (but isospin of pion) suppressed. Saturating with intermediate states, one finds that $\sum_{\text{spin}} T_{\pi f}^{aa} = 0(k, k')$ as $k, k' \rightarrow 0$, and therefore $E_i(2)$ do not contribute, since the result (6.1) assumes $\sum_{\text{spin}} T_{\pi f}^{aa} = 0(k, k')$ as $k, k' \rightarrow 0$. We are left therefore with the evaluation of D(1). The relevant contributions are shown in Fig. 7. Only the graph Fig. 7a) adds to LNAC. For its evaluation, we use the generalized Goldberger-Treiman relation

$$\langle F'(p') | F(p), \pi^a(p'-p) \rangle_0 = \frac{1}{F_\pi^0} (p'-p)^\mu \langle F'(p') | \hat{A}_\mu^a(0) | F(p) \rangle_0$$

with obvious notation. In the frame where $p^\mu = (M_F^0, \underline{0})$, we set

$$\begin{aligned} \frac{1}{2S+1} \sum_{S_2, S_2'} |(\underline{p}' - \underline{p})^\mu \langle F(\underline{p}') | \hat{A}_\mu^a(0) | F(\underline{p}) \rangle_0|^2 = \\ = 4I(I+1)(2I+1) \chi_{I_3' I_3}^a (M_F^0)^2 (G_A^F)^2 (\underline{p}')^2 + \dots \end{aligned} \quad (6.3)$$

$I(I_3)$ denotes the isospin(third component) of the state $|F(\underline{p})\rangle_0$. The matrix $\chi_{I_3' I_3}^a$ is subject to the normalization

$$\sum_{a, I_3', I_3} \chi_{I_3' I_3}^a = 1 \quad (6.4)$$

and the dots in (6.3) stand for terms which vanish faster than $(\underline{p}')^2$ as $\underline{p}' \rightarrow 0$. For nucleons, $G_A^F = g_A \approx 1.25$.

Having settled the notation, it is easy to find for the mass of a fermion with I-spin component I_3 ,

$$\frac{\partial^2 M_F^2}{\partial m^2} = -4I(I+1)(2I+1) \sum_{a, I_3'} \chi_{I_3' I_3}^a (G_A^F)^2 \frac{M_F^0}{F_\pi^0{}^2} \frac{3}{64\pi} \frac{\sigma_\pi^{3/2}}{m^2}$$

Therefore,

$$M_F^2 = (M_F^0)^2 + \sigma_F - 4I(I+1)(2I+1) \sum_{a, I_3'} \chi_{I_3' I_3}^a (G_A^F)^2 \frac{1}{16\pi} \frac{M_F^0}{F_\pi^0{}^2} \sigma_\pi^{3/2} \quad (6.5)$$

with

$$\sigma_F = \frac{1}{2S+1} \sum_{S_2} \langle F(\underline{p}=0) | m \mathcal{H}_\xi(0) | F(\underline{p}=0) \rangle_0$$

For simplicity we suppress all quantum numbers in the states $|F\rangle_0$. Let us specialize to the nucleons. One finds

$$M_N^2 = (M_N^0)^2 + \sigma_N - 3g_A^2 M_N^0 (16\pi F_\pi^0{}^2)^{-1} \sigma_\pi^{3/2} \quad (6.6)$$

M_N^0 is the chiral symmetric nucleon mass, and $\sigma_N = \frac{1}{2} \sum_{S_2} \langle N(\underline{p}=0) | m \mathcal{H}_\xi(0) | N(\underline{p}=0) \rangle_0$. Remarkably enough there is not cutoff Λ present in eq. (6.6), compare the

corresponding equations for the bosons, eqs (5.8) and (5.11). From (6.6) we see that the LNAC gives the following contribution to $M_N - M_N^0$:

$$(M_N - M_N^0)_{\text{LNAC}} = -3g_A^2 (32\pi F_\pi^2)^{-1} M_\pi^3 \sim -15 \text{ MeV}$$

This concludes our discussion of LNAC to fermion masses.

7. CONCLUSIONS AND OUTLOOK

1) We are concerned in this article with the evaluation of leading non-analytic corrections (LNAC) to hadron masses which are considered as a function of the chiral symmetry breaking parameters m_u , m_d and m_s in the QCD-Lagrangian.

2) Let $H = H_0 + \lambda H_I$ by any Hamiltonian, and $M(\lambda)$ the mass of a one-particle state: $H|\underline{p} = 0\rangle = M(\lambda)|\underline{p} = 0\rangle$. Nonanalyticity shows up in singular derivatives $\frac{\partial^n M}{\partial \lambda^n}$ as $\lambda \rightarrow 0$.

3) We show in section 3 that the Fourier transform of the retarded commutator,

$$R(p, q) = i \int d^4x \exp[iq \cdot x] \langle \underline{p} | \theta(x^0) [\mathcal{H}_I(\frac{x}{2}), \mathcal{H}_I(-\frac{x}{2})] | \underline{p} \rangle$$

is closely related to $\frac{\partial^2 M^2}{\partial \lambda^2}$,

$$\frac{\partial^2 M^2}{\partial \lambda^2} = -\lim_{q^0 \rightarrow 0^+} \lim_{q^2 \rightarrow 0} \lim_{\underline{p} \rightarrow 0} R(p, q)$$

Eqs. (3.5) - (3.9) then allow the evaluation of the singular part in $\frac{\partial^2 M^2}{\partial \lambda^2}$.

4) In section 4 we show how to apply these equations to QCD, where it is known that $\frac{\partial^2 M^2}{\partial m^2}$ is singular as $m \rightarrow 0$. (For the exact relation between m and the quark masses m_u , m_d see text). For simplicity we restrict our discussion to 2 flavors (up and down).

5) The LNAC for Goldstone bosons (pions) are evaluated in section 5. We find

$$M_\pi^2 = G_\pi \left(1 + (32\pi^2 F_\pi^2)^{-1} G_\pi \ln \frac{G_\pi}{\Lambda^2} \right) + o(m^2 \ln m)$$

For massive particles of any spin, one has

$$M_\alpha^2 = (M_\alpha^0)^2 + \sigma_\alpha - C_\alpha (G_A^\alpha)^2 \sigma_\pi^{3/2} + o(m^{3/2})$$

The index α denotes the particle under consideration. The axial vector coupling constant G_A^α is defined in eq. (6.3), and the constant C_α can be read off from eq. (6.5). For the nucleons this formula becomes

$$M_N^2 = (M_N^0)^2 + \sigma_N - 3g_A^2 (16\pi F_\pi^2)^{-1} M_N^0 \sigma_\pi^{3/2} + o(m^{3/2})$$

with $g_A \sim 1.25$.

If $G_A^\alpha = 0$, the LNAC become

$$M_\alpha^2 = (M_\alpha^0)^2 + \sigma_\alpha \left(1 - 3 \sigma_\pi (32\pi^2 F_\pi^2)^{-1} \ln \frac{\sigma_\pi}{\Lambda^2} \right) + o(m^2 \ln m)$$

The assumptions which allow a proof of these results are summarized in section 5.A. (The result for the pions presumes in addition that one finds the correct chiral limit of the form factor $\langle p' | \mathcal{H}_\xi(0) | p \rangle$ from a once subtracted dispersion relation. See eq. (5.4b) and comments thereafter.)

6) The calculation of LNAC is not at all complicated by the spin of the particle under consideration. In particular we did not have to introduce interpolating fields.

7) A very similar procedure would apply to the evaluation of nonanalytic corrections to other matrix elements, e.g. F_π . Here the quantity in question is

$$\int d^4x \exp[iqx] \langle 0 | T[A_\mu^a(0) \mathcal{H}_I(x)] | \pi^b(p) \rangle = \Gamma_\mu^{ab}(p, q)$$

The vertex function $\Gamma_\mu^{ab}(p, q)$ is closely related to $\frac{\partial F_\pi}{\partial m}$ and known to be singular as $m \rightarrow 0$, see eq. (2.8a) and ref. (8).

8) In order to extract quark mass ratios, we have to extend the formalism described there to $SU(3) \otimes SU(3)$, including electromagnetism and unstable particles. These calculations are presently underway ⁽¹¹⁾.

Acknowledgements

The approach to chiral perturbation theory as presented here was suggested to one of us (J.G.) by H. Leutwyler. We thank all the members of the Institute of Theoretical Physics of the University of Berne for discussions. We thank especially H. Leutwyler for continuous help and for reading the manuscript. P. Minkowski's remarks for clarifying the concept of quark mass parameters in QCD are gratefully acknowledged. One of us (J.G.) thanks R. Crewther for discussions and for bringing the methods of ref. (14) to our attention. The warm hospitality at the CERN TH Division group is gratefully acknowledged by J.G.

APPENDIX A

We show that our result for M_π^2 (see eq. 2.8b),

$$M_\pi^2 = m_o^2 \left(1 + (32\pi^2 F_\pi^2)^{-1} m_o^2 \ln \frac{m_o^2}{\Lambda^2} \right) \quad (A.1)$$

agrees with the calculation done by Langacker and Pagels in ref. (8). They define M_π^2 in eq. (2.4) as

$$M_\pi^2 = -\epsilon_o Z_\pi^{1/2}(M_\pi^2) F_\pi(M_\pi^2)^{-1}$$

Now we rewrite

$$\frac{Z_\pi^{1/2}(M_\pi^2)}{F_\pi(M_\pi^2)} = \frac{Z_\pi^{1/2}(0)}{F_\pi(0)} \frac{F_\pi(0)}{F_\pi(M_\pi^2)} \frac{Z_\pi^{1/2}(M_\pi^2)}{Z_\pi^{1/2}(0)}$$

and use their result (eqs. (2.19), (2.18) for the ratios $F_\pi(0)/F_\pi(M_\pi^2)$, $Z_\pi^{1/2}(M_\pi^2)/Z_\pi^{1/2}(0)$). One finds

$$M_\pi^2 = -\epsilon_o \frac{Z_\pi^{1/2}(0)}{F_\pi^o} \left(1 + (32\pi^2 F_\pi^2)^{-1} M_\pi^2 \ln \frac{M_\pi^2}{\Lambda^2} \right) \quad (A.2)$$

According to our eqs. (2.6) and (2.10) we have

$$\frac{Z_\pi^{1/2}(0)}{F_\pi^o} = -\frac{1}{c} \left(1 + (2c^2 i)^{-1} \Delta_c(0,0) \right)$$

Since $\epsilon_o = \mu^2 c$ in our notation,

$$-\epsilon_o \frac{Z_\pi^{1/2}(0)}{F_\pi^o} = \mu^2 \left(1 + (2c^2 i)^{-1} \Delta_c(0,0) \right) = m_o^2$$

see eq. (2.7b). This proves the equality of the expressions given in eq. (A.1) and (A.2) for the LNAC, since

$$M_\pi^2 = m_o^2 + O(m_o^4 \ln m_o^2)$$

APPENDIX B

The proof of eq. (3.2) is done here in close analogy to the proof of Feynman's formula (and its generalization) in nonrelativistic quantum mechanics, see e.g. ref. (24). Our one-particle state vectors are normalized as follows:

$$\langle \underline{p}', r' | \underline{p}, r \rangle = 2(2\pi)^3 (M^2(r) + \underline{p}^2)^{1/2} \delta_{r'r} \delta^3(\underline{p}' - \underline{p})$$

where the index γ summarizes all quantum numbers necessary to specify the state vector $|\underline{p}, \gamma\rangle$. The Hamiltonian depends upon a parameter λ ,

$$H(\lambda) |\underline{p}, r\rangle = (M^2(r) + \underline{p}^2)^{1/2} |\underline{p}, r\rangle \equiv p^0(r) |\underline{p}, r\rangle$$

(the index γ contains λ).

First we consider the nondegenerate case.

B.1 Nondegenerate case

We drop in the following the index γ in M^2 , p^0 and $|\underline{p}\rangle$, and write $H(\lambda) = H$. The proof starts from the identity

$$\int f(\underline{p}') p^0 \langle \underline{p}' | \underline{p} \rangle d^3 p' = \int f(\underline{p}') \langle \underline{p}' | H | \underline{p} \rangle d^3 p' \quad (\text{B.1})$$

where $f(\underline{p})$ is an arbitrary function, independent of λ . We differentiate both sides with respect to λ ,

$$2(2\pi)^3 f(\underline{p}) \frac{\partial M^2}{\partial \lambda} = \int f(\underline{p}') [\langle \underline{p}' | \frac{\partial H}{\partial \lambda} | \underline{p} \rangle + \{ p^0 \langle \underline{p}' | \underline{p} \rangle + p^0 \langle \underline{p}' | \underline{p} \rangle' \}] d^3 p' \quad (\text{B.2})$$

where

$$|\underline{p}\rangle' \equiv \frac{\partial}{\partial \lambda} |\underline{p}\rangle$$

The expression in curled brackets may be rewritten as follows. Differentiate the identity

$$\langle \underline{p}' | \underline{p} \rangle = 2(2\pi)^3 p^0 \delta^3(\underline{p}' - \underline{p})$$

and insert the result in eq. (B.2). One finds

$$\frac{\partial M^2}{\partial \lambda} = \langle \underline{p} | \frac{\partial \mathcal{H}(0)}{\partial \lambda} | \underline{p} \rangle \quad (\text{B.3})$$

where

$$H = \int \mathcal{H}(\underline{x}) d^3x$$

The derivative of the state vector $|\underline{p}\rangle$ is found similarly. Differentiating the identities

$$\langle \alpha | H | \underline{p} \rangle = 0$$

$$\langle \alpha | \underline{p} \rangle = 0$$

($|\alpha\rangle$ denotes a many-particle state) one finds

$$\langle \alpha | \underline{p}' \rangle = \langle \alpha | \frac{\partial H}{\partial \lambda} | \underline{p} \rangle (p_0 - E_\alpha)^{-1}$$

with

$$H |\alpha\rangle = E_\alpha |\alpha\rangle$$

Therefore

$$\begin{aligned} |\underline{p}'\rangle &= \sum_{\alpha} (2\pi)^3 \delta^3(\underline{p} - \underline{p}_\alpha) |\alpha\rangle \langle \alpha | \frac{\partial \mathcal{H}(0)}{\partial \lambda} | \underline{p} \rangle (p_0 - E_\alpha)^{-1} \\ &+ \int d\mu(q) |\underline{q}\rangle \langle \underline{q} | \underline{p}' \rangle \end{aligned} \quad (\text{B.4})$$

Eqs. (B.3) and (B.4) allow us to derive the result in eq. (3.2) easily: Differentiate eq. (B.3) once more and insert the above expression for $|\underline{p}'\rangle$. Then

$$\frac{\partial^2 M^2}{\partial \lambda^2} = \frac{1}{2p_0^2} |\langle \underline{p} | \frac{\partial \mathcal{H}}{\partial \lambda} | \underline{p} \rangle|^2 + 2 \langle \underline{p} | \frac{\partial H}{\partial \lambda} \frac{1-P}{p_0 - H} \frac{\partial \mathcal{H}}{\partial \lambda} | \underline{p} \rangle$$

where $\mathcal{P} = \int d\mu(q) |\underline{q}\rangle \langle \underline{q}|$ is the projection operator on the one-particle subspace.

B.2 Degenerate case

The one-particle states carry the additional label γ , $|\underline{p}, \gamma\rangle$. Let I denote the set of indices for which the mass values are degenerate,

$$M_\gamma = M_{\gamma'} \quad \text{for } \gamma, \gamma' \in I$$

The procedure to find the derivatives $\frac{\partial M^2(\gamma)}{\partial \lambda}$, $\frac{\partial^2 M^2}{\partial \lambda^2}$ goes exactly as before: Differentiate the identity

$$\int f(\underline{p}') p_0(\gamma) \langle \underline{p}', \gamma | H | \underline{p}, \gamma \rangle d^3 p' = \int f(\underline{p}') \langle \underline{p}', \gamma | H | \underline{p}, \gamma \rangle d^3 p'$$

Then

$$\frac{\partial M^2(\gamma)}{\partial \lambda} = \langle \underline{p}, \gamma | \frac{\partial H}{\partial \lambda} | \underline{p}, \gamma \rangle \quad (\text{B.5})$$

and

$$\begin{aligned} |\underline{p}, \gamma'\rangle &= \sum_{\underline{q}} (2\pi)^3 \delta^3(\underline{p} - \underline{q}) |\underline{q}, \gamma'\rangle \langle \underline{q}, \gamma' | \frac{\partial H}{\partial \lambda} | \underline{p}, \gamma \rangle (p_0(\gamma) - E_{\underline{q}})^{-1} \\ &\quad + \sum_{\gamma'} d\mu(q) |\underline{q}, \gamma'\rangle \langle \underline{q}, \gamma' | \underline{p}, \gamma \rangle \end{aligned}$$

The sum in the last term runs over all one-particle states. Differentiating eq. (B.5) leads to eq. (3.2) in the text.

For completeness we discuss the approach of $|\underline{p}, \gamma\rangle$ to the "symmetry limit" $\lambda = 0$. Consider the identity

$$\int f(\underline{p}') \langle \underline{p}', \gamma' | H | \underline{p}, \gamma \rangle d^3 p' = 0, \quad \gamma \neq \gamma'$$

Differentiation leads to

$$\begin{aligned} (2\pi)^3 f(\underline{p}) \langle \underline{p}, \gamma' | \frac{\partial H}{\partial \lambda} | \underline{p}, \gamma \rangle &= - \int f(\underline{p}') p_0(\gamma) \langle \underline{p}', \gamma' | \underline{p}, \gamma \rangle d^3 p' \\ &\quad - \int f(\underline{p}') p_0(\gamma') \langle \underline{p}', \gamma' | \underline{p}, \gamma \rangle d^3 p' \end{aligned}$$

Now set $\lambda = 0$. From

$$\langle p', \gamma' | p, \gamma \rangle + \langle p', \gamma' | p, \gamma \rangle = 0, \quad \gamma \neq \gamma'$$

we conclude that

$$\langle p', \gamma' | \frac{\partial \mathcal{H}}{\partial \lambda} | p, \gamma \rangle \Big|_{\lambda=0} = 0 \quad \text{if } \gamma \neq \gamma'; M_\gamma = M_{\gamma'}$$

in agreement with a well-known result from quantum mechanics: The one-particle eigenstates of H reach the limit $\lambda \rightarrow 0$ such that the perturbation is diagonal in the subspace of degenerate mass values.

APPENDIX C

We illustrate the use of eqs. (3.2) - (3.9) in the free field example

$$\mathcal{L}(x) = \frac{1}{2} \partial_\mu \varphi(x) \partial^\mu \varphi(x) - \frac{\lambda}{2} \varphi^2(x)$$

The mass is $M^2 = \lambda$, and therefore

$$\frac{\partial M^2}{\partial \lambda} = 1, \quad \frac{\partial^2 M^2}{\partial \lambda^2} = 0 \quad (\text{C.1})$$

Consider first eq. (3.2). The check for $\frac{\partial M^2}{\partial \lambda}$ is trivial. The second derivative $\frac{\partial^2 M^2}{\partial \lambda^2}$ is a sum of 2 terms. The first one gives

$$\frac{1}{2p_0^2} |\langle p | \frac{1}{2} \varphi^2(0) | p \rangle|^2 = \frac{1}{2(p_0^2 + M^2)} \quad (\text{C.2})$$

The second one may be evaluated by inserting a complete set of intermediate states. The only states which contribute to the connected piece of $\langle p | \mathcal{H}(0) \frac{1 - P_0}{p_0 - H} H | p \rangle$ are 3-particle states. Each matrix element $\langle p | \frac{1}{2} \varphi^2(0) | p_1, p_2, p_3 \rangle$ contributes with the disconnected piece

$$\langle p | \frac{1}{2} \varphi^2(0) | p_1, p_2, p_3 \rangle = 2(2\pi)^3 p^0 \sum_{i=1}^3 \delta^3(p - p_i)$$

The connected part is then found to be

$$2 \int \langle p | \frac{1}{2} \varphi^2(0) \frac{1-P}{p^0-H} \frac{1}{2} \varphi^2(\underline{x}) | p \rangle_{\text{conn.}} d^3x = -\frac{1}{2p_0^2} \quad (\text{C.3})$$

Adding (C.2) and (C.3) gives $\frac{\partial^2 M^2}{\partial \lambda^2} = 0$, in agreement with eq. (C.1).

Now we discuss formulae (3.5) and (3.6) in the text.

The retarded amplitude may be obtained from the dispersion relation eq. (3.8a). We prefer instead to calculate it via the time ordered product ($q^0 > 0$),

$$R(p, q) = T(p, q) = i \int d^4x \exp[iq \cdot x] \langle p | T \left[\frac{1}{2} \varphi^2(\frac{x}{2}) \frac{1}{2} \varphi^2(-\frac{x}{2}) \right] | p \rangle$$

From

$$T(p, q) = - \left\{ (q^2 + 2pq + i\varepsilon)^{-1} + (q^2 - 2pq + i\varepsilon)^{-1} \right\}$$

we may work out the limits $q^\mu \rightarrow 0$ as indicated in eqs. (3.5) and (3.6),

$$\lim_{q^0 \rightarrow 0^+} \lim_{\underline{q} \rightarrow 0} R(p, q) = \frac{1}{2p_0^2}$$

$$\lim_{q^0 \rightarrow 0^+} \lim_{q^2 \rightarrow 0} \lim_{\underline{p} \rightarrow 0} R(p, q) = 0$$

Taking into account eq. (C.2), we see that $\frac{\partial^2 M^2}{\partial \lambda^2} = 0$, q.e.d.

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Figure Captions

- Fig. 1: The two tadpole graphs which contribute to the pion propagator in lowest order. a) denotes the term due to $(2c^2)^{-1} \partial_\mu \Sigma \partial^\mu \Sigma$, and b) represents the contribution from $\mu^2 (2c^2)^{-1} \Sigma^2$.
- Fig. 2: The graphs which contribute to F_π up to $O(c^{-2})$. The double line denotes the axial current, the single line the pion. The notation for c) and d) is the same as in Fig. 1. The last graph represents the contribution from mass renormalization.
- Fig. 3: The cluster decomposition (eq. (3.9)). We have set $q_1 = q + \Delta/2$, $q_2 = q - \Delta/2$, $q_1 + p_1 = q_2 + p_2$. a), b) and (c), d)) correspond to V_D , V_Z and V_E respectively.
- Fig. 4: Intermediate states in the expansion of $V_\Delta(P, q)$ (eq. (3.9)). Solid line: Particle under consideration. Dashed line: Pion. Wavy line: $\mathcal{H}_\xi(0)$. Only the graphs which might contribute to LNAC are shown. All particles are considered to be stable, so graphs like those in Fig. 6 are absent.
- Fig. 5: Lower part of the graph E_1 in Fig. 4. The contributions a) and b) are ambiguous in the limit $\Delta^\mu = (p^2 - p^1)^\mu \rightarrow 0$.
- Fig. 6: Example of a graph which is absent for massive bosons, since all particles are considered to be stable.
- Fig. 7: Contribution to $D(1)$. a), b) and c) are absent in the chiral limit if the solid line represents a boson of definite G-parity.

FIG . 1

FIG . 2

FIG . 3

FIG. 4

$$\begin{array}{c} \diagup \\ \text{---} \end{array} \begin{array}{c} \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \diagup \\ \text{---} \end{array} \begin{array}{c} \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \diagup \\ \text{---} \end{array} \begin{array}{c} \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \diagup \\ \text{---} \end{array} \begin{array}{c} \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \diagup \\ \text{---} \end{array} \begin{array}{c} \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \left\{ \begin{array}{c} \diagup \\ \text{---} \end{array} \begin{array}{c} \diagdown \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\}_{1PI}$$

a) b) c) d) e)

FIG . 5

FIG . 6

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \left\{ \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\}_{1PI}$$

a) b) c) d)

FIG.7