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ANTICOMMUTING VARIABLES, INTERNAL DEGREES OF FREEDOM, AND THE WILSON LOOP

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A B S T R A C T

In this paper we show that it is possible to give a real physical meaning to theories in which internal degrees of freedom are described by Grassmann variables. The physical theory is defined by means of an averaging procedure in terms of a distribution function in the Grassmann restricted space satisfying all the physical requirements.

If we use this result for a scalar particle with inner degrees of freedom (electric charge, colour,...) interacting with Yang-Mills gauge fields, it turns out that we can define two different classical theories.

Taking the average of the coupled particle-field equations of motion, we recover the usual classical theory. Taking instead the average of the solution of such equations we get a theory which is free from all the classical infinities (and so of the causal defects, like runaway solution or pre-acceleration) but also of all the effects of the same order in the charges (like radiation). The main point is that the processes of averaging and integrating the equations of motion do not commute.

Then for the case of colour degrees of freedom we study the quantization of the theory by the path-integral method and we show that the functional integration can be done for an arbitrary gluon field simply by using the classical solution. As a result we obtain an expression for the Wilson loop as a functional integral for the internal fermionic degrees of freedom.

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1. - INTRODUCTION

During the last few years we have been interested in dynamical theories described by Lagrangians involving ordinary c numbers and anticommuting numbers (Grassmann variables). This general approach has been defined as "pseudomechanics" ^{1),2)} due to the special nature of the variables occurring in the problem. We have described elsewhere our general philosophy about these theories, but let us review here some of the general aspects ³⁾. The main (and quite obvious) observation is that quantum observables having a finite number of eigenvalues, that we will call microscopical observables, are such that their classical limit is equal to zero. One can then take two attitudes about this situation :

- i) ignore completely such variables in the classical theory ;
- ii) try to keep track of them by using a description which, in some sense, is in between the classical and the quantum theory.

In order to realize the second possibility it is useful to consider the microscopical observables as infinitesimal quantities ; then, in order to be algebraically consistent we have to assume that they are elements of a Grassmann algebra. Although this description appears to be a rather formal one, there is an immediate relevant result. In fact, one can quantize back the pseudoclassical theory and the outcome is that the microscopical observables must belong to a Clifford algebra.

This is a quite strong prediction because Clifford algebras have essentially only one irreducible representation and this means that the spectrum of the microscopical observables is completely fixed. It has been shown that spin, colour and electric charge of fundamental fermions (quarks and leptons) fit in a marvelous way within this scheme ⁴⁾⁻⁶⁾. Despite this enormous success of the scheme, we want to make quite clear that the pseudoclassical mechanics, as it stands, cannot be interpreted from a physical point of view, although it is an extremely powerful method at the intermediate stage between classical and quantum mechanics as we said before.

However, it is a natural question to ask if there is some possibility of attaching a more direct physical meaning to the scheme. Such a possibility was proposed in a beautiful paper by Berezin and Marinov ²⁾. They formulate the pseudomechanics as a Cauchy problem for a distribution function in the phase space, associated to a Liouville equation (see Section 3). In this way, to each

dynamical variable, both a c number and a Grassmann number, one can associate a real number, that is its average evaluated by means of the density function.

However, in the cases analyzed by Berezin and Marinov (non-relativistic and relativistic spinning particles) the distribution function turns out to be non-positive definite except in the trivial case in which all the relevant averaged quantities vanish. This prevents again to attach a direct physical meaning to the corresponding pseudomechanics.

In this paper we apply the Berezin-Marinov method to the case of internal degrees of freedom of elementary particles, by analyzing as a particular case the colour degrees of freedom. Surprisingly enough, in this case the distribution function is positive definite. We suspect strongly that this is related to the fact that in this case the restricted phase space is even-dimensional, whereas for the spinning particle it is odd-dimensional. We have not made any attempt to prove this conjecture in general, but we know that also for the electric charge the distribution function is positive definite (again an even-dimensional restricted phase space).

In this way, one gets a classical description of the internal degrees of freedom. However, this does not mean that at this point one can take the equation of motion of the averaged quantities and assume them as fundamental equations. In fact, the process of solving the equations of motion involving Grassmann variables and the process of averaging do not commute. This is easily understood by observing that if θ is a complex Grassmann variable $\langle(\theta^*\theta)^2\rangle = 0$ whereas $\langle\theta^*\theta\rangle^2 \neq 0$, because by definition $\langle\theta^*\theta\rangle$ is a real number. Therefore, the first step must be always to solve the equations of motion for the variables and after take the average. Attempts to invert the procedure lead necessarily to contradictions.

We see that one is not allowed to forget about the pseudomechanical treatment, but rather one has to solve the pseudomechanical equations of motion and only after this can one attribute a physical meaning to the various quantities by taking their expectation values. This point is of the utmost importance, because - as we will show in a separate paper - by describing the electric charge by means of Grassmann variables and following the procedure formerly described one is able to avoid the problem of the infinite self-mass of the electron.

The paper is organized as follows. In Section 2 we recall our formulation of the interactions of relativistic particles with Yang-Mills fields (for sake of simplicity, we will limit ourselves to the discussion of scalar particles

although the treatment of the internal degrees of freedom can be extended to spinning particles too). In Section 3 we recall in detail the Berezin-Marinov formulation. We will define our distribution function in the case of a gauge group $SU(3)_C$ and we will show its positivity. We will write the Liouville equation and its characteristic equations (equations of motion for the dynamical variables).

Furthermore, we will study the equations of motion for the relevant averaged quantities. In Section 4 we will study the unaveraged equations of motion and we will show that there are no infinities arising if one tries to integrate the averaged equations of motion.

Finally in Section 5 we will study the quantization by the path integral methods. This has been already done ⁷⁾ but we think that it is worth presenting our calculation, because we will show that the functional integration can be made for an arbitrary gluon field simply by using the classical solution. The relevant result here is that one gets an expression for the Wilson loop as a functional integral for the internal fermionic degrees of freedom. This formula has been recently applied to the problem of connecting the string formulation with QCD, because it offers the possibility of a simple regularization of the Wilson loop in the context of quantum field theory ⁸⁾.

2. - INTERACTIONS OF RELATIVISTIC PARTICLES WITH YANG-MILLS FIELDS

We want to describe the coupling of a scalar particle with a non-Abelian gauge field. We will assume that the gauge group is $SU(N)$. First of all, we describe the internal degrees of freedom of our particle in terms of a set of Grassmann variables $\{\theta_\alpha\}$ belonging to the fundamental representation of $SU(N)$. The free Lagrangian describing the system is ⁵⁾ :

$$L_{free} = -m\sqrt{\dot{x}^2} + \frac{i}{2} \sum_{\alpha=1}^N (\theta_\alpha^* \dot{\theta}_\alpha - \dot{\theta}_\alpha^* \theta_\alpha) \quad (2.1)$$

This Lagrangian is invariant under the global transformation

$$\delta \theta = -i (\tau_a \alpha^a) \theta \quad (2.2)$$

where τ_a are the infinitesimal generators of $SU(N)$ in the fundamental representation ; therefore, they satisfy the algebra :

$$[\tau_a, \tau_b] = i f_{ab}^c \tau_c \quad (2.3)$$

The Lagrangian (2.1) gives rise to second-class constraints, correspondingly we have to use Dirac brackets. The fundamental brackets are :

$$\begin{aligned} \{\theta_\alpha, \theta_\beta\}^* &= \{\theta_\alpha^*, \theta_\beta^*\}^* = 0 \\ \{\theta_\alpha, \theta_\beta^*\}^* &= -i \delta_{\alpha\beta} \end{aligned} \quad (2.4)$$

The generators of the transformation (2.2) are given by :

$$I_a = \theta^* \tau_a \theta \quad (2.5)$$

and they satisfy the algebra

$$\{I_a, I_b\}^* = f_{ab}^c I_c \quad (2.6)$$

If we want the invariance (2.2) to become a local one, we need to introduce a gauge field and replace the time derivative acting on the θ 's with the covariant derivative along the path :

$$\begin{aligned} \frac{d}{dz} \theta &= \dot{x}_\mu \frac{\partial}{\partial x_\mu} \Big|_{x(z)} \theta \Rightarrow \\ \Rightarrow \dot{x}_\mu D^\mu \Big|_{x(z)} \theta &= \left(\frac{d}{dz} + i g \tau_a A_\mu^a \dot{x}^\mu \right) \theta \end{aligned} \quad (2.7)$$

In this way we get the Lagrangian

$$L = -m \sqrt{\dot{x}^2} + \frac{i}{2} \sum_{\alpha=1}^N (\theta_{\alpha}^* \dot{\theta}_{\alpha} - \dot{\theta}_{\alpha}^* \theta_{\alpha}) - g I_a A_{\mu}^a \dot{x}^{\mu} \quad (2.8)$$

The Euler-Lagrangian equations of motion are

$$m \frac{d}{dz} \frac{\dot{x}^{\mu}}{\sqrt{\dot{x}^2}} = g F_{\mu\nu}^a \dot{x}^{\nu} I_a \quad (2.9)$$

$$\dot{\theta} + i g z_a A_{\mu}^a \dot{x}^{\mu} \theta = 0 \quad (2.10)$$

From (2.5) and (2.10) we get the equations of motion for the SU(N) generators :

$$\dot{I}_a - g f_{ab}^c \dot{x}^{\mu} A_{\mu}^b I_c = 0 \quad (2.11)$$

Equations (2.10) and (2.11) show that both θ and I_a have zero covariant derivatives and therefore they undergo a parallel transport during the motion.

To complete this section, we recall also that by quantization we get the following canonical anticommutation relation

$$[b_{\alpha}, b_{\beta}]_+ = [b_{\alpha}^{\dagger}, b_{\beta}^{\dagger}]_+ = 0$$

$$[b_{\alpha}, b_{\beta}^{\dagger}]_+ = \delta_{\alpha\beta} \quad (2.12)$$

where $b_{\alpha}, b_{\alpha}^{\dagger}$ are the quantum operators corresponding to $\theta_{\alpha}, \theta_{\alpha}^*$. The states of the system are of the form

$$b_{\alpha_1}^{\dagger} \dots b_{\alpha_n}^{\dagger} |p\rangle \quad (2.13)$$

where $|p\rangle$ is the "vacuum" state representing a scalar particle of momentum p . The states (2.13) transform like a completely antisymmetric representation of dimension $\binom{N}{K}$ of $SU(N)$. Therefore, the space of the states of our particle transforms like the following reducible representation of $SU(N)$ of dimension $\sum_{K=0}^N \binom{N}{K} = 2^N$:

$$1 \oplus N \oplus \frac{N(N-1)}{2} \oplus \dots \oplus \left[\frac{N(N-1)}{2} \right]^* \oplus N^* \oplus 1 \quad (2.14)$$

3. - THE DISTRIBUTION FUNCTION

The general idea in the Berezin and Marinov paper²⁾ is that one should establish a relation between abstract mechanics and observable quantities by using a distribution function ρ in the phase space. The distribution ρ is supposed to satisfy a Liouville equation

$$\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0 \quad (3.1)$$

where $\{, \}$ is the Poisson bracket. Clearly, the equations of motion for the phase space variables are nothing but the characteristic equations of the partial differential equation (3.1). The next step is to associate to each function f in the phase space a number $\langle f \rangle$; this number is assumed to be the observed value of f and it is given by

$$\langle f \rangle = \int_{\text{phase space}} \rho f \, d\mu \quad (3.2)$$

where $d\mu$ is the measure in the phase space.

Let us review briefly this formalism in the simple case of a charged non-relativistic spinning particle interacting with a constant magnetic field $\vec{B}$ 1), 2). The Lagrangian describing the system is

$$L = \frac{i}{2} \vec{\xi} \cdot \dot{\vec{\xi}} + \frac{m}{2} \vec{v}^2 + e \vec{v} \cdot \vec{A} + g \vec{S} \cdot \vec{B} \quad (3.3)$$

where ξ_i ($i = 1, 2, 3$) are anticommuting variables, $\vec{v}$ is the three-velocity of the particle, g is the magnetic moment,

$$\vec{S} = -\frac{i}{2} \vec{\xi} \wedge \vec{\xi} \quad (3.4)$$

and $\vec{A}$ is the vector potential given by

$$\vec{A} = -\frac{1}{2} \vec{x} \wedge \vec{B} \quad (3.5)$$

From such a Lagrangian we get the Hamiltonian

$$H = \frac{1}{2m} (\vec{p} - e\vec{A})^2 - g \vec{S} \cdot \vec{B} \quad (3.6)$$

By requiring the normalization condition

$$\int \rho d\mu = 1 ; d\mu = i d\xi_3 d\xi_2 d\xi_1 \quad (3.7)$$

Berezin and Marinov ²⁾ did the choice

$$\begin{aligned} \rho &= -i \xi_1 \xi_2 \xi_3 + \vec{c}(t) \cdot \vec{\xi} = \\ &= -\frac{i}{6} \epsilon_{ijk} \xi_i \xi_j \xi_k + \vec{c}(t) \cdot \vec{\xi} \end{aligned} \quad (3.8)$$

The meaning of the vector $\vec{c}(t)$ is immediately obtained by taking the average of $\vec{S}$:

$$\langle \vec{S} \rangle = \int \rho \vec{S} d\mu = \vec{c}(t)$$

Furthermore, by using the Liouville equation we get that the average of $\vec{S}$ satisfies the same equation of motion of $\vec{S}$ itself:

$$\dot{\vec{c}}(t) = -g \vec{B} \wedge \vec{c}(t) \quad (3.9)$$

which says that $\vec{c}$ performs a precession with angular velocity $g|\vec{B}|$. However, the problem is that ρ should satisfy some positivity condition. For instance, we could require

$$\int \rho f^* f d\mu \geq 0 \quad (3.10)$$

for any function f . It turns out that in the actual case it is possible to satisfy (3.10) only in the trivial case $\vec{c} = 0$. This fact prevents us from giving a real physical meaning to this procedure. An analogous fact is seen to happen also in the case of a relativistic electron. However, as it will be shown, this difficulty disappears in the case of internal degrees of freedom described by anticommuting variables. The reason for that is not completely clear to us, but we guess that this is due to the fact that the phase space is even-dimensional for internal degrees of freedom, whereas it is odd-dimensional for a spinning particle both in the relativistic and in the non-relativistic case. Let us illustrate this fact, in the simple case of the electric charge.

According to our general presentation in Section 2 we take a gauge group $U(1)$ and a complex Grassmann variable θ undergoing the local gauge transformation :

$$\delta \theta(z) = -i \alpha(x(z)) \theta(z) \quad (3.11)$$

The gauge invariant Lagrangian will be

$$L = -m \sqrt{\dot{x}^2} + \frac{i}{2} (\theta^* \dot{\theta} - \dot{\theta}^* \theta) - g \theta^* \theta \dot{x}_\mu A^\mu \quad (3.12)$$

with corresponding equations of motion

$$m \frac{d}{dz} \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} = g \theta^* \theta F_{\mu\nu} \dot{x}^\nu \quad (3.13)$$

$$\dot{\theta} + i g \dot{x}_\mu A^\mu \theta = 0 \quad ; \quad \frac{d}{dz} (\theta^* \theta) = 0$$

The last equation is nothing but the conservation of the electric charge. If we limit ourselves to treat statistically only the Grassmann part, the more general density function will be :

$$\varphi(\theta, \theta^*) = a + b \theta + b^* \theta^* + \theta^* \theta \quad (3.14)$$

where we have required

$$\begin{aligned} \varphi^* &= \varphi \\ \int \varphi d\theta d\theta^* &= 1 \end{aligned} \quad (3.15)$$

It is easily seen that for an arbitrary function $f(\theta, \theta^*)$ the positivity condition cannot be satisfied unless we take $a = b = 0$. However, if we take into consideration the space of the analytic functions of θ , i.e.

$$f(\theta) = \alpha + \beta \theta \quad (3.16)$$

then it is possible to satisfy in a non-trivial way the positivity requirement. In fact,

$$\begin{aligned} &\int \varphi(\theta, \theta^*) f^*(\theta) f(\theta) d\theta d\theta^* = \\ &= |\alpha|^2 + a |\beta|^2 + (b^* \alpha^* \beta - b \alpha \beta^*) \end{aligned} \quad (3.17)$$

which is definite positive for any analytic function f , if

$$\begin{aligned} a &> 0 \\ b &= 0 \end{aligned} \quad (3.18)$$

We notice also that the restriction to analytic functions is completely natural because the quantum eigenstates of the creation (or annihilation) Fermi oscillators⁹⁾ are nothing but analytic functions of the kind (3.16).

The Lagrangian (3.12) is homogeneous of first degree in the time derivative ; therefore, the canonical Hamiltonian will be zero. The corresponding first class constraint is

$$(p_\mu - g \theta^* \theta A_\mu)^2 - m^2 = 0 \quad (3.19)$$

where

$$p_\mu = - \frac{\partial L}{\partial \dot{x}^\mu} = m \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} + g \theta^* \theta A_\mu \quad (3.20)$$

In the proper time gauge $\dot{x}^2 = 1$ we can work with the Hamiltonian

$$H = - \frac{1}{2m} \left[(p_\mu - g \theta^* \theta A_\mu)^2 - m^2 \right] \quad (3.21)$$

from which one can derive the equations of motion (3.13) by means of the Dirac brackets (2.4). The Liouville equation gives us

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial a}{\partial \tau} = - \{ \rho, H \}^* = 0 \quad (3.22)$$

Furthermore, we have :

$$\langle \theta^* \theta \rangle = \int \rho \theta^* \theta d\theta d\theta^* = a \quad (3.23)$$

This equation, together with (3.22), tells us that the expectation value of the electric charge is a constant of motion. The average of the equations (3.13) gives rise to the usual Maxwell-Lorentz equations

$$m \frac{d}{d\tau} \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} = g a F_{\mu\nu} \dot{x}^\nu$$

$$\frac{d}{d\tau} a = 0 \quad (3.24)$$

As we will show in more detail in a separate paper, the use of Eqs. (3.13) removes the classical difficulties, like the infinite self-mass, which are present in Eqs. (3.24).

Therefore, we have shown that it is perfectly conceivable to describe the electric charge by means of Grassmann variables in such a way that it is still possible to give a classical interpretation to the theory.

Now, let us study in detail the case of a colour gauge group $SU(3)_C$. Here we take the complex Grassmann variables θ_α ($\alpha = 1, 2, 3$) undergoing the local gauge transformation

$$\delta \theta(z) = -i \left(\frac{\lambda_a}{2} \alpha^a(x(z)) \right) \theta(z) \quad (3.25)$$

where λ_a are the Gell-Mann's matrices. The Lagrangian and the equations of motion are obtained from the general case, Eqs. (2.8) and (2.9), (2.10), (2.11).

Again one can write down the more general density function satisfying (3.15). Furthermore, one requires the positivity condition (3.10) on the space of the analytic functions :

$$\begin{aligned} f(\theta) = & \alpha + \beta_\alpha \theta_\alpha + \gamma_\alpha \varepsilon_{\alpha\beta\gamma} \theta_\beta \theta_\gamma + \\ & + \delta \varepsilon_{\alpha\beta\gamma} \theta_\alpha \theta_\beta \theta_\gamma \end{aligned} \quad (3.26)$$

The result is that the more general density function satisfying all these conditions is

$$\begin{aligned} \rho(\theta, \theta^*) = & c + \theta^* W \theta + 2(\tau_\alpha V) (\theta^* \cdot \theta)^2 - \\ & - 4(\theta^* V \theta)(\theta^* \cdot \theta) + \frac{1}{6} (\theta^* \cdot \theta)^3 \end{aligned} \quad (3.27)$$

where

$$c = c^* > 0$$

and the matrices V and W are both Hermitian and definite positive. In fact, one gets :

$$\begin{aligned} \int \rho \rho^* \rho d\mu(\theta, \theta^*) &= \int \rho \rho^* \rho \prod_{\alpha=1}^3 d\theta_\alpha d\theta_\alpha^* = \\ &= |\alpha|^2 + 36c|\delta|^2 + 4\beta V \beta^* + 4\gamma^* W \gamma \end{aligned} \quad (3.28)$$

which is indeed a definite positive expression under our hypothesis on the quantities c , V and W .

In the proper time gauge we have the Hamiltonian

$$H = -\frac{1}{2m} \left[\left(p_\mu - g \Theta^* \frac{\lambda_a}{2} \Theta A_\mu^a \right)^2 - m^2 \right] \quad (3.29)$$

Then, from the Liouville equation we get

$$\begin{aligned} \frac{dc}{dz} &= 0 \quad ; \quad \frac{d}{dz} \text{Tr } V = 0 \\ \frac{dW}{dz} + ig \dot{x}^\mu A_\mu^a \left[\frac{\lambda_a}{2}, W \right] &= 0 \\ \frac{dV}{dz} + ig \dot{x}^\mu A_\mu^a \left[\frac{\lambda_a}{2}, V \right] &= 0 \end{aligned} \quad (3.30)$$

From which it follows also that

$$\frac{d}{dz} \text{Tr } W = 0 \quad (3.31)$$

If we span W and V on a complete set of 3×3 matrices (the λ 's are the usual Gell-Mann matrices)

$$W = w_0 \mathbb{1} + w^a \frac{\lambda_a}{2} ; \quad w_0 = \frac{1}{3} \text{Tr } W ; \quad w^a = \text{Tr} (\lambda_a W)$$

$$V = v_0 \mathbb{1} + v^a \frac{\lambda_a}{2} ; \quad v_0 = \frac{1}{3} \text{Tr } V ; \quad v^a = \text{Tr} (\lambda_a V) \quad (3.32)$$

we get the equations

$$\begin{aligned} \frac{dc}{dz} &= \frac{dw_0}{dz} = \frac{dv_0}{dz} = 0 \\ \frac{dw^a}{dz} - g f_{bc}^a \dot{x}^\mu A_\mu^b w^c &= 0 \\ \frac{dv^a}{dz} - g f_{bc}^a \dot{x}^\mu A_\mu^b v^c &= 0 \end{aligned} \quad (3.33)$$

showing that the quantities v^a, w^a have a zero covariant derivative.

By using the formulae

$$\begin{aligned}\int \rho \, \partial_\alpha^* \Theta_\beta \, d\mu &= 4 \, V_{\beta\alpha} \\ \int \rho \, \partial_\alpha^* \Theta_\beta \, \partial_\gamma^* \Theta_\delta \, d\mu &= W_{\lambda\tau} \, \varepsilon_{\alpha\gamma\lambda} \, \varepsilon_{\beta\delta\tau} \\ \int \rho \, \partial_\alpha^* \Theta_\beta \, \partial_\gamma^* \Theta_\delta \, \partial_\lambda^* \Theta_\epsilon \, d\mu &= c \, \varepsilon_{\alpha\gamma\lambda} \, \varepsilon_{\beta\delta\epsilon}\end{aligned}\quad (3.34)$$

we can immediately find the meaning of the quantities introduced in (3.32) :

$$\begin{aligned}\langle \Theta^* \cdot \Theta \rangle &= 12 \, v_0 \\ \langle \Theta^* \frac{\lambda_a}{2} \Theta \rangle &= 2 \, v^a \\ \langle (\Theta^* \cdot \Theta)^2 \rangle &= 6 \, W_0 \\ \langle (\Theta^* \frac{\lambda_a}{2} \Theta) (\Theta^* \Theta) \rangle &= -\frac{1}{2} \, W^a \quad ; \quad \langle (\Theta^* \cdot \Theta)^3 \rangle = 6c\end{aligned}\quad (3.35)$$

Other useful relations are

$$\begin{aligned}\langle (\Theta^* \frac{\lambda_a}{2} \Theta) (\Theta^* \frac{\lambda_b}{2} \Theta) \rangle &= \frac{1}{2} (d_{abc} \, W^c - W_0 \, \delta_{ab}) \\ \langle (\Theta^* \frac{\lambda_a}{2} \Theta) (\Theta^* \frac{\lambda_b}{2} \Theta) (\Theta^* \frac{\lambda_c}{2} \Theta) \rangle &= \frac{1}{2} c \, d_{abc} \\ \langle (\Theta^* \frac{\lambda_a}{2} \Theta) (\Theta^* \frac{\lambda_b}{2} \Theta) (\Theta^* \cdot \Theta) \rangle &= -\frac{1}{2} c \, \delta_{ab}\end{aligned}\quad (3.36)$$

Here

$$\frac{1}{2} d_{abc} = \text{Tr} \left[\left\{ \frac{\lambda_a}{2}, \frac{\lambda_b}{2} \right\} \frac{\lambda_c}{2} \right]$$

By using (3.35) and (3.33) we are able to get immediately the averaged form of the equations of motion (2.9) and (2.11) ($\langle \dot{f} \rangle = d/d\tau \langle f \rangle$) :

$$\begin{aligned}m \frac{d}{d\tau} \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} &= g \, F_{\mu\nu}^a \, \dot{x}^\nu (2 v^a) \\ \dot{v}^a - g \, f_{bc}^a \, \dot{x}^\mu A_\mu^b \, v^c &= 0\end{aligned}\quad (3.37)$$

These are completely classical equations of motion and the system has a real classical interpretation as we announced. However, as we shall see in the next section, it is not correct to assume (3.37) as the initial point of the theory, because in doing so one introduces unavoidable difficulties such as infinities, runaway solutions, and so on, exactly, as it happens in the usual treatment of the electromagnetic interaction.

4. - EQUATIONS OF MOTION

In this section we will study the coupled equations for particles and fields in their non-averaged version and we will show that it is possible to make a choice of the density function such that all the classical infinities disappear. We have the system of equations

$$m \frac{d}{dz} \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} = g F_{\mu\nu}^a \dot{x}^\nu I_a \quad (4.1)$$

$$\dot{I}_a - g f_{ab}^c \dot{x}^\mu A_\mu^b I_c = 0 \quad (4.2)$$

$$\begin{aligned} \partial^\mu F_{\mu\nu}^a &= g f_{bc}^a A^b{}^\mu F_{\mu\nu}^c = \\ &= -g \int dz \dot{x}_\nu(z) I_a(z) \delta^4(x(z) - z) \end{aligned} \quad (4.3)$$

$$\dot{\Theta} + ig \frac{\lambda_a}{2} A_\mu^a \dot{x}^\mu \Theta = 0 \quad (4.4)$$

$$I_a(z) = \Theta^*(z) \frac{\lambda_a}{2} \Theta(z) \quad (4.5)$$

Now we can expand any of our quantities in a polynomial of the Grassmann variables taken at a given time, for instance $\tau = 0$. In fact, by using (4.4) we can always evaluate $\theta(\tau)$ in terms of $\theta(0)$. In the case of $A_\mu^a(x)$ we will get

$$A_\mu^a(x) = A_{\mu, in}^a(x) + B_\mu^a(x) \quad (4.6)$$

where $B_\mu^a(x)$ is at least bilinear in the Grassmann variables and $A_{\mu, in}^a(x)$ satisfies pure Yang-Mills equations

$$\partial^\mu F_{\mu\nu, in}^a - g f_{bc}^a A_{in}^b F_{\mu\nu, in}^c = 0 \quad (4.7)$$

$$F_{\mu\nu, in}^a = \partial_\mu A_{\nu, in}^a - \partial_\nu A_{\mu, in}^a - g f_{bc}^a A_{\mu, in}^b A_{\nu, in}^c \quad (4.8)$$

Correspondingly

$$F_{\mu\nu}^a = F_{\mu\nu, in}^a + G_{\mu\nu}^a \quad (4.9)$$

where $G_{\mu\nu}^a$ is at least bilinear in Grassmann variables. By substituting (4.9) into (4.1), one finally gets :

$$m \frac{d}{dz} \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} = g F_{\mu\nu, in}^a \dot{x}^\nu I_a + g G_{\mu\nu}^a \dot{x}^\nu I_a \quad (4.10)$$

The second term on the right-hand side is at least quadrilinear in Grassmann variables, and it is the term which is responsible for the classical infinities, like self-mass and so on. For instance, if we had to do with an Abelian theory ($f_{ab}^c = 0$) we would get

$$B_\mu^a(y) = g(\theta^* \tau_a \theta) \int dz \dot{x}_\mu(z) \mathcal{D}(y - x(z)) \quad (4.11)$$

where

$$\square \mathcal{D}(x-y) = - \delta^4(x-y) \quad (4.12)$$

Putting (4.11) into (4.10) we get

$$\begin{aligned}
 m \frac{d}{dz} \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} &= g F_{\mu\nu, a}^a(x(z)) \dot{x}^\nu(z) I_a + \\
 &+ g^2 I_a I_b \dot{x}^\nu(z) \int dz' \left[\dot{x}^\nu(z') \frac{\partial}{\partial x^\nu(z)} - \right. \\
 &\quad \left. - \dot{x}_\mu(z') \frac{\partial}{\partial x^\nu(z)} \right] D(x(z) - x(z'))
 \end{aligned} \tag{4.13}$$

and we see that a divergence arises when $x(\tau') \rightarrow x(\tau)$. In the case of a non-Abelian theory the structure of this term is much more complicated. However, as we noticed already, it is at least quadrilinear in the Grassmann variables. Therefore, if we choose

$$C = \psi_0 = \psi^a = 0 \tag{4.14}$$

in the density function, the average will be zero. Therefore, by taking

$$\begin{aligned}
 \rho(\theta, \theta^*) &= 2 (T_\nu V) (\theta^* \cdot \theta)^2 - \\
 &\quad - 4 (\theta^* V \theta) (\theta^* \cdot \theta) + \frac{1}{6} (\theta^* \cdot \theta)^3 = \\
 &= \frac{1}{6} \langle \theta^* \cdot \theta \rangle (\theta^* \cdot \theta)^2 - \\
 &\quad - 2 \langle \theta^* \frac{\lambda_a}{2} \theta \rangle (\theta^* \frac{\lambda_a}{2} \theta) (\theta^* \cdot \theta) + \frac{1}{6} (\theta^* \cdot \theta)^3
 \end{aligned} \tag{4.15}$$

we get the following averaged form for Eq. (4.1) :

$$m \frac{d}{dz} \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} = g F_{\mu\nu, a}^a \dot{x}^\nu (2v^a) \tag{4.16}$$

which is free of infinities, because it does not involve self-reaction terms. On the contrary, had we started with Eq. (3.37), we would have found the usual difficulties. We see that the processes of averaging and integrating the equations of motion do not commute.

We observe also that the choice (4.13) for the density function is not unnatural, because after all the only observables of the theory are the generators of $U(3)$, and one expects that the density function depends only on such quantities.

5. - PATH INTEGRAL QUANTIZATION

In this section we will develop the path integral quantization of the theories analyzed in this paper. First of all, we will develop the case of a gauge group $U(1)$. According to the usual path integral quantization, we get from Eq. (3.12) :

$$\begin{aligned} \langle x_f, \theta_f^* ; z_f | x_i, \theta_i ; z_i \rangle &= \int_{x_i, \theta_i}^{x_f, \theta_f^*} \mathcal{D}(x_\mu) \mathcal{D}(\theta, \theta^*) \cdot \\ &\cdot e^{i \int_{z_i}^{z_f} dz \left[-m \sqrt{\dot{x}^2} + \frac{i}{2} (\theta^* \dot{\theta} - \dot{\theta}^* \theta) - g \theta^* \theta \dot{x}_\mu A^\mu \right]} \\ &\cdot e^{\frac{1}{2} \left[\theta_f^* \theta(z_f) + \theta^*(z_i) \theta_i \right]} \end{aligned} \quad (5.1)$$

where the last exponential takes into account the fact that the theory is of first order in the time derivative of Grassmann variables ⁹⁾.

The integration with respect to the anticommuting variables can be easily performed by translating the integration variable with respect to the solution of the classical equation of motion (3.13) given by

$$\theta_{\text{class.}}(z) = e^{-ig \int_{C'} A_\mu dx^\mu} \theta_i \quad (5.2)$$

where C' is the classical path between x_i and $x_\mu(\tau)$, and θ_i is the boundary value at $\tau = \tau_i$ given in (5.1). Analogously,

$$\Theta_{class.}^*(z) = \Theta_f^* e^{-ig \int_{C''} A_\mu dx^\mu}$$

where C'' is the classical path between $x_\mu(\tau)$ and x_f , and Θ_f^* is the boundary value at $\tau = \tau_f$ given in (5.1) :

and therefore $C = C' + C''$ is the classical path between x_i and x_f .

By making a change of variables

$$\Theta(z) = \Theta_{class.}(z) + \gamma(z)$$

$$\Theta^*(z) = \Theta_{class.}^*(z) + \gamma^*(z) \quad (5.3)$$

with

$$\gamma(z_i) = 0, \quad \gamma^*(z_f) = 0 \quad (5.4)$$

we get

$$\begin{aligned} & \langle x_f, \Theta_f^*; z_f | x_i, \Theta_i; z_i \rangle = \\ & = \int_{x_i}^{x_f} \mathcal{D}(x_\mu) N e^{i \int_{z_i}^{z_f} dz [-m \sqrt{\dot{x}^2}]} e^{\Theta_f^* (e^{-ig \int_C A_\mu dx^\mu}) \Theta_i} \end{aligned} \quad (5.5)$$

where N is a normalization factor given by :

$$\begin{aligned} N = & \int_0^1 \mathcal{D}(\gamma, \gamma^*) \cdot \\ & \cdot e^{i \int_{z_i}^{z_f} dz \left[\frac{i}{2} (\gamma^* \dot{\gamma} - \dot{\gamma}^* \gamma) - g \gamma^* \gamma \dot{x}_\mu A^\mu \right]} \end{aligned} \quad (5.6)$$

This can easily be evaluated and we get ¹⁰⁾

$$N = 1 \quad (5.7)$$

To get contact with physics we must evaluate (5.5) among charged states which are nothing but the "occupation number" eigenstates. The transition wave functions are

$$\begin{aligned} \langle 0 | \theta \rangle &= 1 \\ \langle 1 | \theta \rangle &= \theta \end{aligned} \quad (5.8)$$

By multiplying (5.5) for these functions and integrating over θ_f^* and θ_i with a measure $9) e^{(\theta_f^* \theta_f + \theta_i^* \theta_i)}$ we get

$$\begin{aligned} \langle x_f, 0; z_f | x_i, 0; z_i \rangle &= \\ &= \int_{x_i}^{x_f} \mathcal{D}(x_\mu) e^{i \int_{z_i}^{z_f} d\tau [-m \sqrt{\dot{x}^2}]} \end{aligned} \quad (5.9)$$

$$\begin{aligned} \langle x_f, 1; z_f | x_i, 1; z_i \rangle &= \\ &= \int_{x_i}^{x_f} \mathcal{D}(x_\mu) e^{i \int_{z_i}^{z_f} d\tau [-m \sqrt{\dot{x}^2} - g \dot{x}_\mu A^\mu]} \end{aligned} \quad (5.10)$$

which are the usual expressions for the propagators of particles with charges 0 and g¹¹⁾.

Exactly the same considerations can be made in the non-Abelian case. By considering again the $SU(3)_c$ case we define

$$\begin{aligned} \langle x_f, \theta_f^*; z_f | x_i, \theta_i; z_i \rangle &= \int_{x_i, \theta_i}^{x_f, \theta_f^*} \mathcal{D}(x_\mu) \mathcal{D}(\theta, \theta^*) \cdot \\ &e^{i \int_{z_i}^{z_f} d\tau \left[-m \sqrt{\dot{x}^2} + \frac{i}{2} \sum_{a=1}^3 (\theta_a^* \dot{\theta}_a - \dot{\theta}_a^* \theta_a) - g \theta^* \frac{\lambda_a}{2} \dot{x}^\mu A_\mu^a \right]} \\ &\cdot e^{\frac{1}{2} [\theta_f^* \cdot \theta(z_f) + \theta^*(z_i) \cdot \theta_i]} \end{aligned} \quad (5.11)$$

The classical solution is now given by

$$\Theta_{\text{class.}}(\tau) = T \left(e^{-ig \int_c A_\mu^a \frac{\lambda_a}{2} dx^\mu} \right) \Theta_i \quad (5.12)$$

and analogously for $\Theta_{\text{class.}}^*$. By performing again the change of variables Eq. (5.3) in Eq. (5.11), together with the boundary conditions (5.4), we obtain :

$$\begin{aligned} \langle x_f, \Theta_f^*; \tau_f | x_i, \Theta_i; \tau_i \rangle &= \\ &= \int_{x_i}^{x_f} \mathcal{D}(x_\mu) e^{i \int_{\tau_i}^{\tau_f} d\tau [-m \sqrt{\dot{x}^2}]} \\ &\cdot e^{\Theta_f^* T \left(e^{-ig \int_c A_\mu^a \frac{\lambda_a}{2} dx^\mu} \right) \Theta_i} \end{aligned} \quad (5.13)$$

The relevant wave functions are

$$\begin{aligned} \langle 0 | \Theta \rangle &= 1 \equiv \langle \underline{1} | \Theta \rangle \\ \langle \underline{1}; \alpha | \Theta \rangle &= \Theta_\alpha \equiv \langle \underline{3}; \alpha | \Theta \rangle \\ \langle \underline{2}; \alpha, \beta | \Theta \rangle &= \Theta_\alpha \Theta_\beta \equiv \langle \underline{3}^*; \alpha, \beta | \Theta \rangle \\ \langle \underline{3}; \alpha, \beta, \gamma | \Theta \rangle &= \Theta_\alpha \Theta_\beta \Theta_\gamma \equiv \langle \underline{1}'; \alpha, \beta, \gamma | \Theta \rangle \end{aligned} \quad (5.14)$$

where we have introduced also symbols corresponding to the various $SU(3)_C$ representations. For instance, in the case of the transition $\underline{3} \rightarrow \underline{3}$ we get

$$\begin{aligned} \langle x_f, \underline{3}; \alpha; \tau_f | x_i, \underline{3}; \beta; \tau_i \rangle &= \\ &= \int_{x_i}^{x_f} \mathcal{D}(x_\mu) e^{i \int_{\tau_i}^{\tau_f} d\tau (-m \sqrt{\dot{x}^2})} \\ &\cdot \left[T \left(e^{-ig \int_{\tau_i}^{\tau_f} A_\mu^a \frac{\lambda_a}{2} \dot{x}^\mu d\tau} \right) \right]_{\alpha\beta} \end{aligned} \quad (5.15)$$

It is interesting to notice that contrarily to the Abelian case, this expression cannot be interpreted as a sum over some paths weighted by a "classical" action. In fact, here, we have to deal with an exponential which is a matrix itself. However, this interpretation is retained in the original expression in terms of Grassmann variables. The interesting case is when we consider a path corresponding to a closed loop ; we get

$$\begin{aligned}
 \langle X, \underline{z}; z | X, \underline{z}; z \rangle &= \\
 &= \int_X^X \mathcal{D}(X_\mu) e^{i \int_{z_i}^{z_f} dz (-m \sqrt{\dot{X}^2})} \cdot \\
 &\cdot \text{Tr} \left[T \left(e^{-ig \oint A_\mu^a \frac{\lambda_a}{2} dx^\mu} \right) \right]
 \end{aligned} \tag{5.16}$$

which gives us a nice interpretation of the Wilson loop. Analogously, one finds :

$$\begin{aligned}
 \langle X, \underline{z}^*; z | X, \underline{z}^*; z \rangle &= \\
 &= \int_X^X \mathcal{D}(X_\mu) e^{i \int_{z_i}^{z_f} dz (-m \sqrt{\dot{X}^2})} \cdot \\
 &\cdot \text{Tr} \left[T \left(e^{ig \oint A_\mu^a \frac{\lambda_a}{2} dx^\mu} \right) \right]
 \end{aligned} \tag{5.17}$$

The fact that the Wilson loop can be rewritten as a functional integral over one-dimensional Fermi fields (our Grassmann variables θ_α) has received recently a considerable attention.

In fact, in doing so one is able to find a convenient renormalization procedure to deal with the divergence associated to the field A_μ being defined along a line ⁸⁾.

6. - CONCLUSIONS

In this paper we have shown that it is possible to give a real physical meaning to theories in which internal degrees of freedom are described by anti-commuting variables. This has been done defining the physical theory by means of an averaging procedure in terms of a density function satisfying all the necessary physical requirements.

As a result, we have seen that it is possible to get a classical theory free of the usual diseases plaguing the usual approaches. We refer, in particular, to the difficulties one usually encounters in the Dirac-Maxwell theory. We have also examined the path integral quantization of this theory. This problem was already studied by Samuel ⁷⁾, Hosoya ^{7),12)} and Ishida ⁷⁾. However, we think

that our approach is more convenient because we show how to evaluate the path integral directly in terms of the classical solution. Furthermore, we have shown that in the Abelian case one recovers, after integration over the Grassmann variables, the usual expression for the path integral. An analogous treatment in the non-Abelian case leads directly to the expression for the Wilson loop.

We see that our "pseudoclassical" treatment of the internal degrees of freedom represents the background necessary in order to deal with the problem of interpreting the relativistic strings as condensed state of gluons ¹³⁾.

In our picture, we have internal degrees of freedom $\theta_\alpha(\tau)$ which are defined on the classical trajectories of real relativistic particles. The gauge fields are coupled to the oscillations of these internal variables, and they give rise to effective strings tying together the particles. This approach can easily be extended to the more realistic case of spinning relativistic particles interacting via gauge fields. In fact it has been recently shown how to describe the propagation of spinning particles in terms of path integrals ¹⁴⁾.

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