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A STUDY OF EXACTLY SOLVABLE LATTICE GAUGE THEORIES
IN TWO SPACE-TIME DIMENSIONS

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A B S T R A C T

A study of exactly solvable two-dimensional $U(N)$ and $SU(N)$ lattice gauge theories is presented where use is made of Wilson's action or an alternative action suggested by Manton. We study possible traces of the Gross-Witten third-order phase transition for finite N . When making use of Manton's action we observe that there is no sign of phase transition at all, even for the $U(\infty)$ theory. Some possible implications for higher dimensions are briefly touched upon.

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1. INTRODUCTION

The lattice regularization procedure of gauge theories, as introduced by Wilson^{1,2)}, provides a non-perturbative method of studying the large scale behaviour of QCD. Using such a regularization technique, a demonstration that quark confinement in QCD is compatible with asymptotic freedom³⁾ is in sight.

It is of crucial importance that the lattice gauge theory does not exhibit a phase transition which invalidates an extrapolation from the strong to the weak coupling regime of the theory. It is believed that QCD (without quarks) does not have such a phase transition. Recent Monte Carlo calculations⁴⁾ support the conjecture that there is no phase transition in the bulk phase, but a roughening transition⁵⁾ in the string tension.

Recently, it was shown by Gross and Witten^{6,7)} that, in two dimensions, the $U(N)$ lattice gauge theory, with Wilson's action, exhibits a third order phase transition when N approaches infinity. This phase transition is due to the presence of an infinite number of degrees of freedom in group space. In four dimensions and in the independent link approximation⁸⁾, it has been shown⁹⁾ that the same type of phase transition occurs in the large N limit. In Ref. 6) it was, in fact, argued that the third order phase transition observed in the $N=\infty$ limit in two dimensions also could occur in four dimensions and that a sign of this phase transition should be seen in the $SU(N)$ case for finite N . The possible existence of such a phase transition in four dimensions may be related to the sharp crossover between the strong and weak coupling regimes observed in Monte Carlo calculations⁴⁾. It is therefore of general interest to study the nature of this phase transition and its effect on $U(N)$ or $SU(N)$ lattice gauge theories. In two dimensions it is wellknown^{2,10)} that these models can be exactly solved in the sense that the solution of the problem can be reduced to the evaluation of a one-link integral. In the present paper we will present the results of such a study. We find that there is a sign of the Gross-Witten phase transition for finite N , but it is more pronounced for the unitary groups than for the special unitary groups.

The Wilson action on the lattice has the property that it reduces to the standard QCD (without quarks) action in the naive continuum limit. This property is a necessary requirement for any lattice regularization of the continuum action but it is, however, not sufficient to uniquely determine the form of the action. In fact, other choices have been suggested in the literature¹¹⁾. Recently, Manton suggested¹²⁾ a lattice gauge theory action which shares many of the properties of the Wilson action. In Wilson's approach¹⁾, one considers the action

$$S_W(U) = - \frac{1}{g^2} \sum_P (\text{Tr}(U(P) + U(P)^\dagger) - 2 \cdot \mathbf{1}) \quad (1)$$

where $U(P)$ stands for an oriented product of the group elements of the P 'th plaquette and the sum is over all plaquettes. In Manton's approach the action is constructed in terms of the shortest distance in group space, denoted by $d(U(P), \mathbf{1})$, from $U(P)$ to the unit matrix, i.e.,

$$S_M(U) = \frac{1}{g^2} \sum_P (d(U(P), \mathbf{1}))^2 \quad (2)$$

In Ref. 12), a method is given for calculating $d(U, \mathbf{1})$ for a given group element U . Below we will present some results on two-dimensional lattice $SU(N)$ and $U(N)$ gauge theory when using the action proposed by Manton. For finite N there is no sign of a phase transition in the specific heat or in the β function. In fact, by making use of the Manton's action there will be no phase transition in two dimensions, even in the limit $N \rightarrow \infty$. As a result of this analysis we find a very rapid convergence of the $1/N$ -expansion.

2. LATTICE GAUGE THEORY WITH WILSON'S ACTION

As mentioned above, the two-dimensional lattice gauge theory with a gauge group G , $U(N)$ or $SU(N)$ in our case can be solved in the sense that all physical quantities can be obtained from the one-link partition function integral

$$z(g^2, N) = \int_G dU \exp\left(\frac{1}{g^2} \text{Tr}(U + U^\dagger)\right) \quad (3)$$

where dU is the invariant measure on the group G . The internal energy per unit volume, $\langle E \rangle$, is obtained from

$$\langle E \rangle = - \frac{\partial}{\partial \beta} \ln z(g^2, N) \quad (4)$$

where $\beta = N/\lambda$, λ is the "temperature given by $\lambda = ng^2$ and g is the bare coupling constant. The specific heat, C , is then obtained from

*) For a detailed analysis of this question we refer to Ref. 13).

$$C = \frac{\partial}{\partial \lambda} \langle E \rangle \quad (5)$$

Another quantity of physical interest is the β function of the gauge theory. One then computes the expectation value of the Wilson loop operator. It is easy to show that, in two dimensions, this expectation value can be obtained from the one-link expectation value w ,

$$\begin{aligned} w(g^2, N) &= \langle \frac{1}{N} \text{Tr} U \rangle = \frac{1}{z} \int_G dU \frac{1}{N} \text{Tr} U \exp\left(\frac{1}{g^2} \text{Tr}(U + U^\dagger)\right) = \\ &= \frac{1}{2N} \frac{\partial}{\partial \beta} \ln z(g^2, N) \end{aligned} \quad (6)$$

If the lattice spacing is a , the string tension, σ , is defined by $\sigma = -1/a^2 \cdot \ln w(g^2, N)$. If we let the bare coupling constant g be dependent on a in such a way that σ is independent of the lattice spacing, we obtain the β function by differentiation

$$\beta(\lambda) = -a \frac{d \lambda(a)}{da} = -2 \frac{\ln w(\lambda)}{w'(\lambda)/w(\lambda)} \quad (7)$$

where $w(\sigma) = w(g^2, N)$.

If $G = U(N)$ the integral (3) can be evaluated¹⁴⁾ by means of Weyl's parametrization of the reduced group measure¹⁵⁾, i.e.,

$$\begin{aligned} z(\lambda) = z(g^2, N) &= \frac{1}{N!} \int_G \left(\prod_{\mu} \frac{d\alpha_{\mu}}{2\pi} \right) \left(\prod_{\mu \neq \nu} \left| 2 \sin \frac{\alpha_{\mu} - \alpha_{\nu}}{2} \right| \right) \cdot \\ &\exp\left(\frac{2}{g^2} \sum_{\nu=1}^N \cos \alpha_{\nu} \right) = \det M \end{aligned} \quad (8)$$

where

$$(M)_{i,j} = I_{i-j}(2N/\lambda); \quad i, j = 1, 2, \dots, N \quad (9)$$

The specific heat per degree of freedom, C/N^2 , can then be expressed in terms of modified Bessel functions. A graphical representation of the result is given in Fig. 1a with a comparison with the Gross-Witten result. At low temperatures we obtain

$$C/N^2 = \begin{cases} \frac{1}{2} + \frac{\lambda}{8} & \text{for } U(1) \\ \frac{1}{2} + \frac{\lambda}{32} & \text{for } U(2) \\ \frac{1}{2} + \frac{\lambda}{72} & \text{for } U(3) \end{cases} \quad (10)$$

A rather pronounced approach to the Gross-Witten result appears when N increases. The result of an evaluation of the β function (7) is presented in Fig. 1b.

We now turn to the case^{*)} $G = SU(N)$. For large N one expects no essential difference when comparing with the $U(N)$ results above. For $N = 2$ and 3 the $SU(N)$ integral (3) can be computed to be [see, e.g., Ref. 7). One could also make use of (8) by imposing the constraint $\alpha_1 + \dots + \alpha_N = 0$]

$$z(\lambda) = \begin{cases} I_0(8/\lambda) & \text{for } SU(2) \\ 2 \sum_{m=0}^{\infty} \frac{\beta^{2m}}{(m+1)!(m+2)!} \sum_{n=0}^{\infty} \frac{(2\beta)^n}{n!} \binom{3m+3}{2m+n+3} & \text{for } SU(3) \end{cases} \quad (11)$$

The specific heat per degree of freedom, $C/(N^2 - 1)$, is given in Fig. 2a with a comparison with the Gross-Witten result. At low temperatures we have

$$C/(N^2 - 1) = \begin{cases} \frac{1}{2} - \frac{\lambda}{32} & \text{for } SU(2) \\ \frac{1}{2} - \frac{\lambda}{72} & \text{for } SU(3) \end{cases} \quad (12)$$

Although one sees a slow approach to the $U(\infty)$ result, the trace of the third order

^{*)} The $SU(2)$ case has also been studied by P. Hasenfratz¹⁶⁾.

phase transition is much more pronounced for the unitary groups^{*)}. The β function (7) with a comparison with the Gross-Witten result is shown in Fig. 2b. In two dimensions we therefore conclude that the $N=\infty$ limit is approached rather slowly, indicating a slow convergence of the $1/N$ -expansion. In the next section we see that the situation changes when Manton's action is used.

3. LATTICE GAUGE THEORY WITH MANTON'S ACTION

In this section we consider Manton's action (2) for the gauge groups $G = SU(N)$ and $U(N)$. As in the case of Wilson's action, one can reduce the solution of the model to the evaluation of a one-link integral^{**)}. By making use of Weyl's parametrization of the group measure¹⁵⁾ we obtain for $G=U(N)$ partition function

$$z(\lambda) = \frac{1}{N!} \int_B \left(\prod_{\mu} \frac{d\alpha_{\mu}}{2\pi} \right) \left(\prod_{\mu \neq \nu} \left| 2 \sin \frac{\alpha_{\mu} - \alpha_{\nu}}{2} \right| \right) \exp\left(-\frac{1}{g^2} \sum_{\nu} \alpha_{\nu}^2\right) \quad (13)$$

and the one-plaquette expectation value of the Wilson loop operator

$$w(\lambda) = \frac{1}{z} \int_B \left(\prod_{\mu} \frac{d\alpha_{\mu}}{2\pi} \right) \left(\prod_{\mu \neq \nu} \left| 2 \sin \frac{\alpha_{\mu} - \alpha_{\nu}}{2} \right| \right) \frac{1}{N} \left(\sum_{\lambda} \exp(i\alpha_{\lambda}) \right) \exp\left(-\frac{1}{g^2} \sum_{\nu} \alpha_{\nu}^2\right) \quad (14)$$

where $B = \{ -\pi \leq \alpha_{\nu} \leq \pi ; \nu = 1, \dots, N \}$ corresponds to the first Brillouin zone of the magnetic weight lattice¹⁸⁾ of $U(N)$. Here we also notice^{**)} that when going from Wilson's action to Manton's action, $2(1 - \cos \alpha)$ is replaced by α^2 . It is straightforward to obtain the expressions corresponding to (13) and (14) for $G = SU(N)$. In (13) and in (14) one then imposes the constraint $\alpha_1 + \dots + \alpha_N = 0$ and replaces the first $U(N)$ Brillouin zone with the first $SU(N)$ Brillouin zone. The normalization of the $U(N)$ group integral has to be modified accordingly. In Fig. 3a we display the specific heat per degree of freedom for the groups $U(1)$, $U(2)$, $U(3)$ and $U(\infty)$. There is no sign of any phase transition when Manton's action is used. The curves are well approximated by

*) It is, furthermore, amusing to notice that within two decimals the specific heat per degree of freedom for $SU(N)$ and $U(N)$ are equal at the transition point.

**) For details we refer to Ref.13).

the first terms in their weak coupling expansion, i.e.,

$$C/N^2 = \frac{1}{2} - \left(1 - \frac{1}{N^2}\right) \frac{\lambda}{12} + \left(1 - \frac{1}{N^2}\right) \frac{\lambda^2}{192} - \left(2 - \frac{5}{N^2} + \frac{3}{N^4}\right) \cdot \frac{\lambda^4}{138240} + O(\lambda^5) \quad (15)$$

In Fig. 3b the corresponding curves are given for the β function. Here we also notice the absence of any sign of any phase transition.

The specific heats per degree of freedom for the $SU(N)$ groups are also shown in fig.3a. They are surprisingly close to the $N=\infty$ limit, indicating a rapid convergence of the $1/N$ -expansion. In the weak coupling expansion

$$C/(N^2 - 1) = \frac{1}{2} - \frac{\lambda}{12} + \frac{\lambda^2}{192} - \left(2 - \frac{3}{N^2}\right) \frac{\lambda^4}{138240} + O(\lambda^5) \quad (16)$$

this phenomenon shows up as a (miraculous?) N independence up to order three in the temperature. The β function, shown in Fig.3b for $SU(2)$, $SU(3)$ and $SU(\infty)$, also varies little with N .

In the limit $N=\infty$ we have solved¹³⁾ the lattice gauge theory analytically in two dimensions with the results

$$C/N^2 = \frac{1}{\lambda^2} \int_0^\lambda \frac{dx}{2} \frac{x^2}{\exp(x/2) - 1} \quad (17)$$

for the specific heat per degree of freedom, and

$$w(\lambda) = \frac{2}{\lambda} (1 - \exp(-\lambda/2)) \quad (18)$$

for the one-plaquette expectation value of the Wilson loop operator, from which the β function can be easily evaluated. The corresponding curves agree with those for $SU(2)$, $SU(3)$ in Fig. 3. There is no phase transition at all in the limit $N=\infty$ for the $U(N)$ lattice gauge theory and the $SU(2)$ and $SU(3)$ theories are surprisingly good approximations of the $U(\infty)$ theory.

4. CONCLUSIONS AND FINAL REMARKS

We have presented a study of two-dimensional lattice gauge theories with Wilson's action and Manton's action. In the case of Wilson's action we have found that there is a sign of the third order phase transition of Gross and Witten for finite N . There is, however, a more pronounced trace of this phase transition for $U(N)$ groups as compared to $SU(N)$ groups. We conclude that there is both a qualitative as well as a quantitative difference between $SU(N)$ and $U(N)$ lattice gauge theories for finite N , at least in two dimensions.

When making use of the action suggested by Manton the situation changes drastically. The $SU(N)$ and $U(N)$ theories are, in fact, very similar in this case. In the limit $N = \infty$ there is no phase transition at all when using Manton's action¹³⁾, and we have noticed that the $1/N$ -expansion converges rapidly in an unexpected way.

Further studies of Manton's action now suggest themselves¹³⁾. It was conjectured in Ref. 6) that the third order phase transition observed in two dimensions may be related to the sharp crossover in the string tension observed in four-dimensional Monte Carlo calculations. It would therefore be interesting to study Monte Carlo simulation of four-dimensional lattice gauge theory using Manton's action since, in this case, there is no phase transition in two dimensions. Such a study is at present in progress¹⁹⁾. Related topics for example, the glueball mass and strong coupling expansion for Manton's action will be discussed elsewhere.

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FIGURE CAPTIONS

- Figs. 1a and 1b : $U(N)$, $N = 1, 2, 3$ lattice gauge theories with Wilson's action are compared with the Gross-Witten $U(\infty)$ solution. In Fig. 1a the specific heat per degree of freedom given and in Fig. 1b the corresponding β functions.
- Figs. 2a and 2b : $SU(2)$ and $SU(3)$ lattice gauge theories with Wilson's action are compared with the Gross-Witten $U(\infty)$ solution. In Fig. 2a the specific heats per degree of freedom are given and in Fig. 2b the β functions.
- Figs. 3a and 3b : The specific heat per degree of freedom (Fig. 3a) and the β function (Fig. 3b) for the $U(1)$, $U(2)$, $SU(2)$, $SU(3)$ and $U(\infty)$ lattice gauge theories with Manton's action.

Fig. 1

Fig. 2

Fig. 3