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RESOLVING QCD JETS BEYOND LEADING ORDER :
QUARK DECAY PROBABILITIES

J. Kalinowski *)

Institute of Theoretical Physics
Warsaw University

K. Konishi +)

CERN - Geneva

P. N. Scharbach

Niels Bohr Institute - Copenhagen

and

T.R. Taylor *)

Institute of Theoretical Physics
Warsaw University

A B S T R A C T

Corrections beyond the leading order of perturbative QCD are computed for the evolution of jets and scaling violation in the parton fragmentation functions. We present the prescription for calculating the jet calculus vertices to higher orders, and give complete results for the quark decay probabilities in the next-to-leading approximation. These contain, as special cases, the generalized Altarelli-Parisi decay functions. For two-particle correlations among fast particles, the next-to-leading corrections to the p_T spectrum inside a jet are small.

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+) Address after 1 December 1980 : Istituto di Fisica, Università di Pisa, Italy.

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1. - INTRODUCTION

The pattern of hadronic final states in hard processes, as expected from perturbative QCD, is quite complex due to the importance of higher order radiative corrections. Although lowest-order calculations à la Sterman-Weinberg¹⁾ are useful for a limited class of jet measures or cross-sections, a summation of all terms $\sim (\alpha_s^m(Q^2) \log^n Q^2)^L$ is in general required for a quantitative understanding of final states. The inclusion of terms of higher order in α_s results in an intuitive picture of QCD jets, which is best grasped by the concept of fractals²⁾.

In the leading log approximation (LLA), the resummation of these higher order terms for jet evolution is summarized in a simple parton-probabilistic algorithm (the jet calculus)³⁾. Numerous tests of QCD have been found as its applications, which are particularly appropriate for the e^+e^- annihilation process. Furthermore, some generalizations have recently led to a consistent over-all picture of jet evolution in hard processes, in real space-time as well as in colour, momentum and rapidity^{*)}.

In Ref. 4), which will be referred to as I in this paper, it was shown that the simple parton-probabilistic picture is a general feature of contributions associated with collinear mass singularities, not necessarily restricted to LLA. As is now widely acknowledged⁵⁾, scaling violation effects are governed by collinear mass singularities. In I, it was found possible, by using the Kinoshita-Lee-Nauenberg (KLN) theorem⁶⁾ and the factorization property of mass singularities⁵⁾, to generalize the simple algorithm of jet calculus beyond LLA.

In this paper we continue the investigation of I. Our main result is the complete set of quark decay probabilities, computed up to the next-to-leading order. As particular cases, they contain the generalized Altarelli-Parisi parton decay functions, $P_{Qj,Q_1}^{NL}(x)$, $P_{\bar{Q}j,Q_1}^{NL}(x)$ and $P_{G,Q}^{NL}(x)$. We have not yet computed the decay probabilities of gluons. However, we give precise prescriptions for computing them, so that one can check our calculations and/or extend the results to include gluon decays.

As a first phenomenological implication of the results of I and this paper we discuss how the leading log results for the p_T^2 structure of jets are modified by the next-to-leading corrections. For the inclusive two-particle cor-

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*) For a review, see, for example, the last work of Ref. 2).

relation among fast particles, the corrections due to the next-to-leading terms are found to be quite small, less than 15% with respect to the LLA predictions in a wide range of transverse momentum.

In I, a simple reduction of the unitarity equations was introduced, in a form which manifests the full mass-singularity structure of the final states. One starts with the off-shell unitarity equation ⁷⁾ (we take the e^+e^- annihilation process for the purpose of illustration),

$$\sigma_t \propto \text{Disc}_{Q^2} \left\{ \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \dots \right\} \quad (1)$$

written in axial-gauge QCD. In Eq. (1), the crossed lines represent cut exact quark or gluon propagators ; uncut propagators are also exact ones, and vertices are one-particle irreducible (1pI).

Three operations (A)-(C) are then performed on Eq. (1) :

Operation (A)

For each crossed line, write

$$\begin{aligned} q \text{---} \times \text{---} &= \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \dots, \text{ if } \sqrt{q^2} \geq M_0 \\ &= \text{---} \times \text{---}, \text{ if } \sqrt{q^2} < M_0 \end{aligned} \quad (2)$$

where M_0 is an arbitrary mass scale. This is to be repeated until all crossed lines have off-shell mass less than M_0 .

Operation (B)

Introduce effective vertices which are sums of two-particle irreducible (2pI) subgraphs in all zero-momentum channels, e.g.,

$$\begin{aligned} \text{---} \bigcirc \text{---} &= \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} &= \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \dots, \text{ etc.} \end{aligned} \quad (3)$$

The unitarity equations can be expressed entirely in terms of these vertices which, in an axial gauge of QCD, are free of collinear singularities at fixed q^2 : the integration over q^2 then brings in at most one collinear log factor.

Operation (C)

The operation of a projection operator $\mathcal{P}$ applied to each internal line defines new effective vertices,

$\mathcal{P}$ projects out only those terms giving rise to collinear singularities on integration over q^2 .

The reduction of the unitarity equations through (A) to (C) thus leads to effective diagrams of the form of the square of a tree amplitude. In particular, effective parton decay vertices for $i \rightarrow a(x_1) + \dots + a(x_m) + X$,

$$V_{i \rightarrow \{a\}}^{(m)} [\alpha_s(q^2); \{x_j\}]$$

where q^2 is the squared mass of the parton i , can be defined. These are closely related to the vertices in Eq. (4) as will be seen in Section 2.3.

$V_{i \rightarrow \{a\}}^{(m)}$ can be interpreted as a short-distance "hard process" probability, with q^2 playing the rôle of the large momentum scale. The whole process of jet evolution is iterative, hence the multiparton distributions can be computed from a simple parton probabilistic algorithm: to get the ℓ parton spectra,

- (i) write all tree graphs for $i \rightarrow b_1 + \dots + b_\ell$; in the next-to-leading order, only vertices $V^{(m)}$ with $m \leq 3$ are relevant;
- (ii) the "propagator" in the effective tree graphs is the normalized fragmentation function $D_{b,a}(q_1^2, q_2^2; z)$;
- (iii) integrate over internal parton masses with weight $1/q_1^2$; sum over internal parton species and over inequivalent trees.

The problem is therefore reduced to the computation of a finite number of effective vertices $V^{(m)}$ and, as a particular case, of the generalized Altarelli-Parisi⁸⁾ parton decay functions.

The task of the next two sections is to give the precise prescription for computing these quantities, and to present a complete set of quark decay probabilities to the next-to-leading order.

2. - PRESCRIPTION FOR $V_{j \rightarrow \{a\}}^{(\ell)}$ AND $\Pi_{b,a} \equiv V_{a \rightarrow b}^{(1)}$

2.1. - Convention, regularization scheme, gauge choice and renormalization prescription

We first summarize our conventions which are particularly useful in describing the detailed structure of jets and in extending the jet calculus beyond LLA ⁴⁾.

The calculations are performed taking advantage of the factorization property of mass singularities. The method consists of directly computing cross-sections from Feynman graphs in an axial gauge.

Consider, for the purpose of illustration, the single-parton inclusive cross-section in a hard process. Its moment with respect to the momentum fraction x factorizes in the form ⁵⁾

$$\int_0^1 dx \, x^n \left(\frac{d\sigma}{dx} \right)_a = \sum_{i,b} G_i(\alpha_s(Q^2); Q^2/M^2; n) \cdot D_{b,i}(M^2; M_0^2; n) \Gamma_{a,b}(M_0^2; n; 1/\epsilon) \quad (5)$$

where we have introduced two arbitrary mass scales M_0 and M , and the function Γ contains all the collinear mass singularities appearing as $1/\epsilon$ poles in dimensional regularization. The function $D_{b,i}$ describing the evolution between the two mass scales M and M_0 satisfies a generalized Altarelli-Parisi equation ⁴⁾

$$M^2 \frac{d}{dM^2} D_{a,i}(M^2; M_0^2; n) = \sum_b \Pi_{a,b}(\alpha_s(M^2); n) D_{b,i}(M^2; M_0^2; n) \quad (6)$$

where $\Pi_{a,b}$ is the moment of $\Pi_{a,b}[\alpha_s(M^2); \bar{x}]$, and the latter has an expansion in powers of $\alpha_s(M^2)$:

$$\Pi_{a,b} = \frac{\alpha_s(M^2)}{2\pi} P_{a,b}^{AP}(x) + \left(\frac{\alpha_s(M^2)}{2\pi} \right)^2 P_{a,b}^{NL}(x) + \dots \quad (7)$$

In Eq. (7), $P_{a,b}^{AP}$ and $P_{a,b}^{NL}$ are, respectively, the usual Altarelli-Parisi probabilities and the next-to-leading corrections to them.

The first moment of $\Gamma_{a,b}$ summed over the parton species a is the total decay rate of the parton b , and hence, according to the KLN theorem, is free of mass singularities. The factorized structure of Eq. (5) allows a redefinition of Γ such that

$$\sum_a \Gamma_{a,b} (M_o^2; n=1; 1/\epsilon) = 1 \quad (8)$$

to all orders of α_s , which will be our convention.

In discussing inclusive hadron spectra, the singular function Γ can be absorbed into a redefined non-perturbative fragmentation function.

The same argument as given above for Γ can be repeated for the combination $D \cdot \Gamma$ in Eq. (5) : we fix our convention such that

$$\sum_b D_{b,i} (M^2; M_o^2; 1) = 1 \quad (9)$$

exactly. In other words, D can be interpreted as a normalized inclusive probability. Accordingly, $P_{b,a}^{NL}(x)$ satisfies the sum rules

$$\sum_b \int_0^1 dx \, x \, P_{b,a}^{NL}(x) = 0 \quad (10)$$

$$\sum_b e_b \int_0^1 dx \, P_{b,a}^{NL}(x) = 0 \quad (11)$$

where e_b is the charge of the parton b . [In fact, these sum rules hold for $\Pi_{a,b}(\alpha_s(M^2); x)$.]

Analogous arguments can be repeated for multiparton spectra. Only pairs of partons having an invariant mass larger than M_o will be regarded as distinct. The mass-singular contributions from the region below M_o can be factorized into non-perturbative fragmentation functions. The inclusive vertices $V_{j \rightarrow \{a\}}^{(l)}$ satisfy, by convention, the usual energy and charge sum rules. These conventions allow the probabilistic interpretation of jet evolution and the jet calculus rules : the generating functional of n -parton spectra satisfies a branching process equation of the Kolmogorov type.

Throughout, we use the dimensional regularization method for regularizing the ultra-violet, collinear and soft (infra-red) divergent integrals. The ultra-violet poles are subtracted upon renormalization : we use the $\overline{MS}$ scheme ⁹⁾ as the renormalization prescription.

We work in a light-like axial gauge, in an infinite momentum frame

$$n_\mu A^\mu = 0, \quad n^2 = 0$$

$$n \cdot q \neq 0, \quad n \cdot p_{T_i} = 0 \quad (12)$$

where q and p_{T_i} are the momenta related to the partons in a jet, as defined in Appendix A. Individual Feynman diagrams in light-like gauges suffer from peculiar divergences which are of neither the usual ultra-violet nor infra-red type. However, we find ⁴⁾ that in the next-to-leading order these singularities completely cancel. All remaining divergent integrals can now be dimensionally required in $4-2\epsilon$ dimensions. As a convenient choice, we shall set μ^2 , the renormalization scale, to be equal to q^2 , the mass squared of the decaying parton. Then all factors $\log q^2/\mu^2$ are absent.

2.2. - Factorization of evolution below the mass scale M_0

An effect of the factorization of evolution below a given mass scale M_0 is that only pairs of partons having an invariant mass $\sqrt{q^2}$ such that $\sqrt{q^2} \geq M_0$ are regarded as distinct. Any group of partons forming an invariant mass less than M_0 cannot be resolved and hence summation over them should be performed. As a consequence, M_0 enters as the lower limit of intermediate parton masses.

However, this is not all. In accordance with the convention Eq. (8), all finite corrections arising from the evolution below M_0 should be shifted into the parton distributions describing the evolution above M_0 . This means that, in computing $V_{i \rightarrow \{a\}}^{(l)}$, a multiplicative factor

$$\begin{aligned} \Delta \tilde{\Gamma} &= \sum_a \int_0^1 dz \, z \, T_{a,b}^{\Gamma}(M_0^2; z; 1/\epsilon) \\ &= \int_0^{M_0^2} dq^2 \left\{ \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \dots \right\} \quad (13) \\ &= 1 + \sum_{m=1}^{\infty} \alpha_s^m C_m \end{aligned}$$

should be included for each observed parton. A subtle point should be mentioned here. Consider the $\alpha_s^n(q^2)$ term in $V^{(\ell)}$. Then only the terms $C_1, \dots, C_{n-\ell+1}$ are relevant. By our choice, $\mu^2 = q^2$, they contain in general powers of $\log(q^2/M_0^2)$ which should be regarded as part of the evolution above M_0 . These terms correspond to contributions which appear on iteration of the $\delta(1-x)$ piece of the generalized Altarelli-Parisi kernel $\Pi[\alpha_s(q^2); x]$ of lower orders. In our approach these terms are computed at each order by the use of the sum rules (See I and Section 3.3 below). Therefore, to avoid double counting, only

$$\Delta \Gamma \equiv \Delta \tilde{\Gamma} - [\text{terms containing } \log q^2/M_0^2] \quad (14)$$

should be kept for computing the vertices $V^{(\ell)}$.

2.3. - Projection operator $\mathcal{P}$

We now define the projection operator $\mathcal{P}$. A diagram for $i(q) \rightarrow a_1(p_1) + \dots + a_m(p_m)$ looks like (i = quark)

$$+ \dots \quad (15)$$

where hatched vertices are two-particle irreducible (2PI) in all zero-momentum channels [See Eq. (4)]. Let us, for the moment, concentrate on the first term in Eq. (15). It is of the form ($d\Omega_m$ = invariant phase volume at fixed q^2)

$$(q^2)^{-2} \not{q} \mathcal{A} \not{q} dq^2 d\Omega_m \quad (16)$$

where $\mathcal{A}$ is the square of a sum of amplitudes as in Eq. (15) and is a matrix in spinor indices. After averaging over initial and summing over final spin and colour states, Eq. (16) becomes of the form

$$dq^2 d\Omega_m \frac{1}{q^2} [\not{q} F + \sum_i \not{\epsilon}_i G_i + \not{q} H] \quad (17)$$

where the kinematics in an infinite momentum frame (IMF) have been used. We choose to work in an IMF in the direction of the momentum of the decaying parton, q^μ , which allows a symmetric treatment of the final partons. See Appendix A for the kinematics in this system.

With the choice of the gauge vector n^μ , Eq. (12), only the first term of Eq. (17) can develop a collinear singularity. A projection operator introduced in Ref. 10),

$$P_{ph}^{(in)} = \frac{1}{4(q \cdot n)} \text{Tr} [\not{n} \not{q} \not{q} \not{n}] \quad (18)$$

extracts the desired physical spin sum.

Furthermore, only the evolution above $q^2 > M_0^2$ should be taken into account according to our definition of parton evolution. This corresponds to the introduction of another projection operator

$$P_{M_0}^{(in)} = \delta^4(q - q') \Theta(q'^2 - M_0^2) \quad (19)$$

There are also non-logarithmic contributions arising from the kinematical bounds on the integration over internal masses and from the phase volume in $4-2\epsilon$ dimensions. Define

$q_p^2 \equiv$ the off-shell mass squared of the parent parton,

then a third projection operator

$$P_{log}^{(out)} \equiv \log \frac{q_p^2}{M_0^2} \left(q_p^2 \frac{\partial}{\partial q_p^2} \right) \int d\epsilon \delta(\epsilon) \quad (20)$$

sets $\epsilon = 0$ and picks up the term linear in $\log q_p^2/M_0^2$.

The last thing to be taken into account is the multiplicative factor Δl for each observed parton leg in $V_{j \rightarrow \{a\}}^{(l)}$, discussed in the previous subsection.

To summarize, we define the projection operator

$$P = P_{log}^{(out)} \cdot P_{M_0}^{(in)} \cdot P_{ph}^{(in)} \quad (21)$$

where the superscripts (in) and (out) mean operation inside and outside the q^2 integration. For 2pI diagrams, the operation $\mathcal{J}$ on the initial parton line and multiplication of factors $\Delta\Gamma$ give directly a contribution to $V_{i \rightarrow a_1 + \dots + a_m}^{(\ell)}$.

For 2pR graphs, the collinear singularities corresponding to subsystems should be subtracted by the use of the projection operator $1 - \mathcal{J}$. This subtraction should be done iteratively, starting from the smallest 2pR subgraph, as in the usual renormalization programme. This means that the operations $\Delta\Gamma$ and $1 - \mathcal{J}$ should be applied in an ordered way starting from the final external lines.

We finally get *)

$$V_{i \rightarrow \{a(x_j)\} + X}^{(\ell)} [\alpha_s(q^2), \{x_j\}] = \sum_{\substack{\text{(unobserved)} \\ \text{partons}}} \sum_{\text{(trees)}} \mathcal{P} \rightarrow q$$

where a summation over unobserved partons is understood. In the next subsection, we illustrate the prescription by considering the next-to-leading contribution of a simple rainbow diagram shown in Fig. 1.

2.4. - Illustration : calculation of a rainbow diagram

For the purpose of illustrating the prescription for $V^{(m)}$, we consider the next-to-leading contribution from the rainbow diagram shown in Fig. 1. Let us first operate $\mathcal{J} \equiv \mathcal{J}_{\log M_0}^{(\text{out})} \mathcal{J}_{\text{ph}}^{(\text{in})} \mathcal{J}_{\text{ph}}^{(\text{in})}$ on the intermediate line with momentum r , in order to find out exactly what should be subtracted. The operation $\mathcal{J}_{\text{ph}}$ after the final spin summation extracts the term proportional to r' from the inner part of the diagram. All other terms are included in the next-to-leading contribution, upon operating $\mathcal{J}$ on the line q .

The term proportional to r' leads to

$$\left(\frac{\alpha_s}{2\pi}\right)^2 c_F^2 \int [z_1(1-z_1)]^{-\epsilon} \{ \hat{P}(z_1) - \epsilon z_1 \} (q^2)^{-1-\epsilon} R dq^2 dz_1 \frac{(4\pi)^{2\epsilon}}{\{\Gamma(1-\epsilon)\}^2} \quad (23)$$

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*) Half of the self-energy correction on the initial parton line q should be included. See I.

where

$$R = \int dz_2 [z_2(1-z_2)]^{-\varepsilon} \left\{ \hat{P}(z_2) - \varepsilon z_2 \right\} \int_0^{q^2(1-z_1)} dr^2 (r^2)^{-1-\varepsilon} \cdot \left[1 - \frac{r^2}{q^2(1-z_1)} \right]^{-\varepsilon} \delta(x_1 - z_1) \delta(x_2 - (1-z_1)z_2); \quad (24)$$

$$\hat{P}(z) = \left\{ 1 + (1-z)^2 \right\} / z, \quad (25)$$

and x_1, x_2 are the momentum fractions of the final gluons. Let us first note that

$$\int_0^{q^2(1-z_1)} dr^2 (r^2)^{-1-\varepsilon} \left[1 - \frac{r^2}{q^2(1-z_1)} \right]^{-\varepsilon} = \int_0^{q^2(1-z_1)} dr^2 (r^2)^{-1-\varepsilon} + \varepsilon \frac{\pi^2}{6} (q^2)^{-\varepsilon} + O(\varepsilon^2) \quad (26)$$

The second term in Eq. (26) is annihilated by the operation $\mathcal{P}_{\log}^{(out)}$ on the r line, and therefore should be included in the next-to-leading contribution. This term survives only when combined with $\varepsilon^{-1}\delta(z_1)$ or $\varepsilon^{-1}\delta(z_2)$ coming from the expansion ^{*)}

$$z^{-1-\varepsilon} = -\frac{1}{\varepsilon} \delta(z) + \left(\frac{1}{z} \right)_+ - \varepsilon \left(\frac{\log z}{z} \right)_+ + O(\varepsilon^2) \quad (27)$$

in Eq. (24), and hence is to be included as a contribution to the inclusive vertex $V_{Q \rightarrow QG}^{(2)}$. In the first term of Eq. (26), we now operate with $\mathcal{P}_{M_0^2}^{(in)} \equiv \delta^4(r-r')$ $\theta(r'^2 - M_0^2) dr'^2$. It is convenient to rewrite the corresponding part of R in the form

$$R = \int dz_2 \left[\left\{ \hat{P}(z_2) - \varepsilon [z_2 + \hat{P}(z_2) \log z_2(1-z_2)] \right\}_+ - \delta(z_2) \left\{ \frac{z}{\varepsilon} + \frac{z}{2} + \varepsilon \left(\frac{z}{2} - \frac{\pi^2}{3} \right) \right\} \right] \cdot \int_{M_0^2}^{q^2(1-z_1)} dr^2 (r^2)^{-1-\varepsilon} \delta(x_1 - z_1) \delta(x_2 - (1-z_1)z_2) \quad (28a)$$

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^{*)} $[f(z)]_+$ is the usual distribution defined as $[f(z)]_+ = f(z) - \delta(z) \int_0^1 dy f(y)$.

$$\begin{aligned}
 &= \int dz_2 \left\{ \hat{P}(z_2) - \varepsilon [z_2 + \hat{P}(z_2) \log z_2 (1-z_2)] \right\} \int_{M_0^2}^{q^2(1-z_1)} d^2(r^2)^{-1-\varepsilon} \delta(x_1-z_1) \delta(x_2-(1-z_1)z_2) \\
 &\quad + \delta(x_2) \delta(x_1-z_1) \frac{\{q^2(1-z_1)\}^{-\varepsilon} - (M_0^2)^{-\varepsilon}}{\varepsilon} \cdot \left\{ \frac{2}{\varepsilon} + \frac{3}{2} + \varepsilon \left(\frac{7}{2} - \frac{\pi^2}{3} \right) \right\}.
 \end{aligned}
 \tag{28b}$$

We now turn to $\Delta\Gamma$ discussed in Section 2.2. In the region $0 \leq r^2 \leq M_0^2$ the quark of momentum p and gluon of momentum k are regarded as a single quark state with momentum $r = p + k$. We therefore get for $\Delta\Gamma$ the same integral as in Eq. (28a) between the limits $0 \leq r^2 \leq M_0^2$ and with the replacement of the delta functions by $\delta(x_1-z_1)\delta((1-x_1-x_2)-(1-z_1))$. But then the first term in the z_2 integrand gives a vanishing contribution because of the $\{ \}_+$ regularization. As a result, we get for the order $O(\alpha_s)$ term,

$$\Delta\Gamma(\text{Fig. 1}) = \left(\frac{\alpha_s}{2\pi} \right) C_F \frac{(M_0^2)^{-\varepsilon}}{\varepsilon} \left\{ \frac{2}{\varepsilon} + \frac{3}{2} + \varepsilon \left(\frac{7}{2} - \frac{\pi^2}{3} \right) \right\}
 \tag{29}$$

Because of $\delta(x_2)$, the second term of Eq. (28b) and $\Delta\Gamma$ contribute to the same physical process, $Q \rightarrow Q(1-x_1) + G(x_1)$, as the diagrams with virtual corrections, Figs. 2e,f,g,h,j and l. For the C_F^2 part of $V_{Q \rightarrow QG}^{(2)}(1-x, x)$ the addition of the above terms gives

$$- C_F^2 \left\{ \hat{P}(x) \left[2 \text{Li}_2(x) + \frac{3}{2} \log(1-x) \right] + 1 \right\}
 \tag{30}$$

where $\text{Li}_2(x)$ is the usual dilogarithmic function defined as

$$\text{Li}_2(x) = - \int_0^x \frac{\log(1-z)}{z} dz
 \tag{31}$$

The remaining part of the rainbow graph contribution from Eq. (23) is given by

$$\begin{aligned}
 & \left(\frac{\alpha_s}{2\pi} \right)^2 c_F^2 \int dz_1 [z_1(1-z_1)]^{-\varepsilon} \{ \hat{P}(z_1) - \varepsilon z_1 \} (q^2)^{-1-\varepsilon} dq^2 \\
 & \cdot \int dz_2 \left[\{ \hat{P}(z_2) - \varepsilon(z_2 + \hat{P}(z_2) \log z_2(1-z_2)) \}_+ \right] \int_{M_0^2}^{q^2(1-z_1)} dr^2 (r^2)^{-1-\varepsilon} \delta(x_1 - z_1) \delta(x_2 - (1-z_1)z_2)
 \end{aligned}
 \tag{32}$$

We now set $\varepsilon = 0$ and operate with $\mathcal{J}_{\log}^{(out)}(r)$, obtaining $[\hat{P}(z_2)]_+ \log(q^2/M_0^2)$, which is the leading log piece to be subtracted. After the subtraction we operate with $\mathcal{J}$ on the line q and obtain

$$\left(\frac{\alpha_s}{2\pi} \right)^2 c_F^2 \int_0^1 \hat{P}(z_1) [\hat{P}(z_2)]_+ (\log(1-z_1) \delta(x_1 - z_1) \delta(x_2 - (1-z_1)z_2)) dz_1 dz_2
 \tag{33}$$

Taking into account the indistinguishability of the final gluons we arrive at the following next-to-leading contribution to $V_{Q \rightarrow QGG}^{(3)}(1-x_1-x_2, x_1, x_2)$:

$$c_F^2 \left\{ \left[\hat{P}\left(\frac{x_1}{1-x_2}\right) \right]_+ \hat{P}(x_2) \frac{\log(1-x_2)}{1-x_2} + \left[\hat{P}\left(\frac{x_2}{1-x_1}\right) \right]_+ \hat{P}(x_1) \frac{\log(1-x_1)}{1-x_1} \right\}
 \tag{34}$$

The full next-to-leading result from Fig. 1 for $V_{Q \rightarrow QGG}^{(3)}$ is the sum of (34) and the contribution from the $(1-\mathcal{J}_{ph})$ part of the intermediate state. A calculation of the latter term gives

$$\begin{aligned}
 & c_F^2 \left\{ -\frac{1}{x_1 x_2} \left[\left\{ 1 + (1-x_2)^2 \right\} \left\{ 1 + \left(\frac{1-x_1-x_2}{1-x_2} \right)^2 \right\} \right. \right. \\
 & \left. \left. + \left\{ 1 + (1-x_1)^2 \right\} \left\{ 1 + \left(\frac{1-x_1-x_2}{1-x_1} \right)^2 \right\} \right] + (1-x_1-x_2) \left[\frac{1}{(1-x_1)^2} + \frac{1}{(1-x_2)^2} \right] \right\}
 \end{aligned}
 \tag{35}$$

The above example illustrates our prescription for extracting the jet calculus vertices from Feynman diagrams and should be sufficient to show how analogous computations can be done for other 2PR graphs.

3. - RESULTS FOR QUARK DECAY PROBABILITIES TO THE NEXT-TO-LEADING ORDER

In this section, we present a complete set of quark decay probabilities to the next-to-leading order. They are

$$\begin{aligned} V_{Q \rightarrow a}^{(1)} [\alpha_s(q^2); x] &\equiv \Pi_{a,Q} [\alpha_s(q^2); x] \\ &= \frac{\alpha_s(q^2)}{2\pi} P_{a,Q}^{AP}(x) + \left(\frac{\alpha_s(q^2)}{2\pi} \right)^2 P_{a,Q}^{NL}(x), \end{aligned} \quad (36)$$

$$\begin{aligned} V_{Q \rightarrow a, a_2}^{(2)} [\alpha_s(q^2); x_1, x_2] &= \frac{\alpha_s(q^2)}{2\pi} \hat{P}_{a, Q}^{AP}(x_1) \delta(x_1 + x_2 - 1) \\ &+ \left(\frac{\alpha_s(q^2)}{2\pi} \right)^2 V_{Q \rightarrow a, a_2}^{(2)}(x_1, x_2), \end{aligned} \quad (37)$$

and

$$\begin{aligned} V_{Q \rightarrow a, a_2 a_3}^{(3)} [\alpha_s(q^2); x_1, x_2, x_3] &= \left(\frac{\alpha_s(q^2)}{2\pi} \right)^2 V_{Q \rightarrow a, a_2 a_3}^{(3)}(x_1, x_2, 1-x_1-x_2) \\ &\cdot \delta(x_1 + x_2 + x_3 - 1), \end{aligned} \quad (38)$$

where $P_{a,Q}^{AP}(x)$ is the usual Altarelli-Parisi parton decay function, $\hat{P}_{a,Q}^{AP}(x)$ is the same but without $\delta(1-x)$ terms. The new quantities $V_{Q \rightarrow a_1 a_2 a_3}$, $V_{Q \rightarrow a_1 a_2}$, and $P_{a,Q}^{NL}$ are presented below; the Feynman diagrams contributing to them are shown in Fig. 2.

3.1. - Three-body decay probabilities

There are two possible three-body decays: $Q \rightarrow QGG$ and $Q \rightarrow QQ\bar{Q}$, corresponding to the diagrams Fig. 2a,b,c,d and Fig. 2i,k, respectively. To the next-to-leading order, they are exclusive processes, and can be written as

$$\begin{aligned} V_{Q_i \rightarrow Q_i(x_3) + G(x_1) + G(x_2)}^{(3)} &\equiv V_{Q_i \rightarrow Q_i G G}(1-x_1-x_2, x_1, x_2) \delta(1-x_1-x_2-x_3) \\ &= C_F^2 [A(x_1, x_2) + B(x_1, x_2)] + \frac{1}{2} C_A C_F [C(x_1, x_2) + D(x_1, x_2) - B(x_1, x_2)], \end{aligned} \quad (39)$$

$$\begin{aligned} V_{Q_i \rightarrow Q_i(x_1) + Q_j(x_2) + \bar{Q}_j(x_3)}^{(3)} &\equiv V_{Q_i \rightarrow Q_i Q_j \bar{Q}_j}(x_1, x_2, 1-x_1-x_2) \delta(1-x_1-x_2-x_3) \\ &= C_F(C_F - C_A/2) \delta_{ij} E(x_1, x_2) + C_F T_R [F(x_1, x_2) + \delta_{ij} F(x_2, x_1)] \end{aligned} \quad (40)$$

where $T_R = \frac{1}{2}$ and $C_F = \frac{4}{3}$, $C_A = 3$ for $SU(3)_{\text{colour}}$. Quark-flavour indices are denoted by i, j . The functions A to F are given in Appendix B.

3.2. - Two-body inclusive decay probabilities

The diagrams contributing to inclusive two-body decays to this order are those listed in Section 3.1 as well as the remaining ones from Fig. 2, which represent exclusive two-body decays. Adding all contributions, we find

$$V_{q_i \rightarrow q_i G}^{(2)}(x_1, x_2) = V_{q_i \rightarrow q_i G G}^{(3)}(x_1, x_2, 1-x_1-x_2) + G(x_1) \delta(1-x_1-x_2) \quad (41)$$

$$V_{q_i \rightarrow q_i q_j}^{(2)}(x_1, x_2) = V_{q_i \rightarrow q_i q_j \bar{q}_j}^{(3)}(x_1, x_2, 1-x_1-x_2) \quad (42)$$

$$V_{q_i \rightarrow q_i \bar{q}_j}^{(2)}(x_1, x_2) = V_{q_i \rightarrow q_i q_j \bar{q}_j}^{(3)}(x_1, 1-x_1-x_2, x_2) \quad (43)$$

$$V_{q_i \rightarrow q_j \bar{q}_j}^{(2)}(x_1, x_2) = V_{q_i \rightarrow q_i q_j \bar{q}_j}^{(3)}(1-x_1-x_2, x_1, x_2) \quad (44)$$

$$V_{q_i \rightarrow G G}^{(2)}(x_1, x_2) = V_{q_i \rightarrow q_i G G}^{(3)}(1-x_1-x_2, x_1, x_2) \quad (45)$$

$$V_{q_i \rightarrow G \bar{q}}^{(2)} = 0, \quad (46)$$

where $V^{(3)}$ are as given in Eqs. (39), (40) and

$$\begin{aligned} G(x) = & -C_F^2 \left\{ \frac{1+x^2}{1-x} \left[2 \text{Li}_2(1-x) + \frac{3}{2} \log x \right] + 1 \right\} \\ & + C_F C_A \left\{ \frac{1+x^2}{1-x} \left[\frac{67}{18} + 2 \text{Li}_2(1-x) \right] + 1 \right\} \\ & - \frac{10}{9} C_F N_f T_R \frac{1+x^2}{1-x}. \end{aligned} \quad (47)$$

3.3. - Single-parton inclusive vertices (generalized Altarelli-Parisi probabilities)

It is convenient to calculate the single-parton inclusive vertices with the aid of the sum rules which, as demonstrated in I, are satisfied by convention to all orders by our parton decay probabilities. The momentum sum rules read :

$$\sum_d \int_0^{1-x_1-x_2} dx_3 x_3 V_{a \rightarrow bcd}^{(3)}(x_1, x_2, x_3) = (1-x_1-x_2) V_{a \rightarrow bc}^{(2)}(x_1, x_2) \quad (48)$$

$$\sum_c \int_0^{1-x_1} dx_2 x_2 V_{a \rightarrow bc}^{(2)}(x_1, x_2) = (1-x_1) V_{a \rightarrow b}^{(1)}(x_1) \equiv (1-x_1) P_{b,a}^{NL}(x_1) \quad (49)$$

$$\sum_b \int_0^1 dx_1 x_1 P_{b,a}^{NL}(x_1) = 0 \quad (50)$$

while the charge sum rule is

$$\int_0^1 dx \left\{ P_{q_j, q_i}^{NL}(x) - P_{\bar{q}_j, q_i}^{NL}(x) \right\} = 0. \quad (51)$$

To order $O(\alpha^2)$, the first sum rule, Eq. (48), is satisfied automatically. The second one, Eq. (49), can be used to determine the probabilities $P_{\bar{Q}, Q}^{NL}$, $P_{G, Q}^{NL}$ and $P_{Q, Q}^{NL}$ up to the term proportional to $\delta(x-1)$. We get :

$$P_{q_j, q_i}^{NL}(x_1) = \sum_b \frac{1}{1-x_1} \int_0^{1-x_1} dx_2 x_2 V_{q_i \rightarrow q_j b}^{(2)}(x_1, x_2) + S \delta(x_1-1) \delta_{ij} \quad (52)$$

$$P_{\bar{q}_j, q_i}^{NL}(x_1) = \sum_b \frac{1}{1-x_1} \int_0^{1-x_1} dx_2 x_2 V_{q_i \rightarrow \bar{q}_j b}^{(2)}(x_1, x_2) \quad (53)$$

$$P_{G, q_i}^{NL}(x_1) = \sum_b \frac{1}{1-x_1} \int_0^{1-x_1} dx_2 x_2 V_{q_i \rightarrow G b}^{(2)}(x_1, x_2) \quad (54)$$

The charge sum rule, Eq. (51), can now be used to determine S . We obtain the following full expression for P_{Q_i, Q_i}^{NL} :

$$P_{Q_i, Q_i}^{NL}(x_1) = \left\{ \sum_b \frac{1}{1-x_1} \int_0^{1-x_1} dx_2 x_2 V_{Q_i \rightarrow Q_i b}^{(2)}(x_1, x_2) \right\}_+ + \delta(x_1-1) \sum_{j=1}^{N_f} \int_0^1 \frac{dy}{1-y} \int_0^{1-y} dx_2 x_2 V_{Q_i \rightarrow \bar{Q}_j Q_j}^{(2)}(y, x_2) \quad (55)$$

where N_f is the number of active flavours. By using the charge sum rule we have avoided the direct evaluation of the two-loop quark self-energy corrections. Moreover, the last of the momentum sum rules, Eq. (50), provides us with a non-trivial self-consistency check of the foregoing construction. A straightforward

manipulation demonstrates that Eq. (50) is indeed satisfied by our probabilities. After substituting Eqs. (41)-(46) for the two-parton inclusive vertices into Eqs. (52)-(55), we derive the $O(\alpha^2)$ probabilities

$$P_{q_j, q_i}^{NL}(x) = \delta_{ij} \left\{ [C_F^2 P_1 + \frac{1}{2} C_F C_A P_2 + C_F N_f T_R P_3] + C_F (C_F - C_A/2) T \delta(1-x) \right\} + C_F T_R P_4, \quad (56)$$

$$P_{G, q_i}^{NL}(x) = C_F^2 P_5 + \frac{1}{2} C_F C_A P_6 + C_F N_f T_R P_7, \quad (57)$$

$$P_{\bar{q}_j, q_i}^{NL}(x) = \delta_{ij} C_F (C_F - C_A/2) P_8 + C_F T_R P_4, \quad (58)$$

where

$$P_1 = 2 \frac{1+x^2}{1-x} \log x \log(1-x) + \left[\frac{3}{1-x} - 5 - 3x \right] \log x + \left[\frac{1}{2}(1+x) - 2 \frac{1+x^2}{1-x} \right] (\log^2 x - 5(1-x)), \quad (59)$$

$$P_2 = \frac{1+x^2}{1-x} \left[\log^2 x - \frac{11}{3} \log(1-x) + \frac{67}{9} - \frac{\pi^2}{3} \right] + 2(1+x) \log x + \frac{29}{3} (1-x), \quad (60)$$

$$P_3 = \frac{2}{3} \frac{1+x^2}{1-x} \left[\log(1-x) - \frac{5}{3} \right] - \frac{2}{3} (1-x), \quad (61)$$

$$P_4 = (1+x) \log^2 x + \frac{\log x}{x} \left[\frac{4}{3} - x^2 \right] + \frac{1}{9x} [13 - 45x + 45x^2 - 13x^3], \quad (62)$$

$$P_5 = - \frac{1+(1-x)^2}{x} \left[2 \text{Li}_2(x) + \frac{\pi^2}{3} + \frac{3}{2} \log(1-x) \right] + (1 - \frac{1}{2}x) \log^2 x + x \log x + \frac{1}{2x} [5 - 9x + 2x^2], \quad (63)$$

$$P_6 = \frac{1+(1-x)^2}{x} [\pi^2 - 4 \text{Li}_2(1-x)] - 2 \frac{\log^2 x}{x} [4 + 2x + 3x^2] - \frac{\log x}{x} \left[\frac{62}{3} - 2x^2 \right] + \frac{1}{9x} [-45 + 118x - 14x^2 + 26x^3] + \frac{2(2+2x+x^2)}{x} \left[2 \text{Li}_2(-x) + 2 \log x \log(1+x) + \frac{\pi^2}{6} \right], \quad (64)$$

$$P_7 = - \frac{10}{9} \frac{1+(1-x)^2}{x}, \quad (65)$$

$$P_g = 2(1+x)\log x + 4(1-x) + \frac{1+x^2}{1+x} \left\{ \log^2 x - 4\log x \log(1+x) - 4\text{Li}_2(-x) - \frac{\pi^2}{3} \right\}, \quad (66)$$

$$T \equiv \int_0^1 dx P_g(x) = \frac{13}{4} - \frac{\pi^2}{2} + 2\zeta(3), \quad \zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3} = 1.20205... \quad (67)$$

Equations (39)-(67) of this section form the main body of our results. It is important to note that the expressions for the jet calculus vertices given above depend on the scheme used. In particular, the generalized Altarelli-Parisi probabilities of Eqs. (56)-(58) differ, for example, from those calculated in Refs. 10) and 11). This scheme-dependence must disappear in physical quantities which should be computed consistently within the same scheme. For $e^+e^- \rightarrow q(\text{non-singlet}) + X$, this is in fact the case as was explicitly demonstrated in I.

4. - DISCUSSION

Complete results to the next-to-leading order would require the knowledge of gluon decay probabilities, which we have not yet computed. It is, however, possible to gain some insight into the importance of the corrections beyond LLA using the quantities already computed.

Consider the two-particle correlation inside a jet. We observe the inclusive production of two particles with momentum fractions x_1 and x_2 , and with a fixed relative transverse momentum p_T . p_T should be in the range

$$\Lambda^2 \ll p_T^2 \ll Q^2 \quad (68)$$

where Q is the large momentum scale of the hard process under consideration ; we shall also restrict ourselves to those processes where the hard vertex is electroweak, such as e^+e^- annihilation or leptonproduction.

We take the n^{th} moment in x_1 and the first moment in x_2 and sum over the particle species in leg 2. Choosing n sufficiently large, we shall assume that the parton with momentum fraction x_1 is the quark emerging from

the hard vertex. Large n means a large momentum fraction x_1 , and since gluons tend to be soft, this assumption seems to be reasonable, even beyond LLA.

The corresponding cross-section can be computed by use of the jet calculus rules to be (See Fig. 3) :

$$\left(\frac{d\sigma}{dp_T^2} \right)_{n,1} \propto \frac{1}{p_T^2} \sum_{a,b} D_{a,q}(p_T^2; M_o^2; n) D_{a,b}(p_T^2; M_o^2; 1) \cdot V_{q \rightarrow qb}^{(2)}(\alpha_s(p_T^2); n, 1) D_{a,q}(q^2; p_T^2; n+1) \quad (69)$$

where $V_{q \rightarrow qb}^{(2)}(\alpha_s(p_T^2); n, 1)$ is a double moment,

$$V_{q \rightarrow qb}^{(2)}(\alpha_s(p_T^2); n, 1) = \int dx_1 dx_2 x_1^n x_2 V_{q \rightarrow qb}^{(2)}(\alpha_s(p_T^2); x_1, x_2) \quad (70)$$

In deriving Eq. (69), we have set

$$q^2 \approx p_T^2 \quad (71)$$

where q^2 is the squared mass of the intermediate parton line (Fig. 3). For the leading log terms, Eq. (71) is a consequence of the strong ordering in transverse momenta of emitted partons, and of the kinematics³⁾. For the next-to-leading terms, the simple kinematical relation is lost because the vertex $V^{(2)}$ is inclusive. However, since $V^{(2)}$ is free of collinear singularities, the proportionality $q^2 \propto p_T^2$ holds, which is sufficient to the order considered. Although a more rigorous justification of Eq. (69) is needed, we shall use it as a reasonable assumption.

Also, the process-dependent "hard subprocess cross-section" has been omitted in Eq. (69) : with a single species of parton (quark) dominating in the intermediate state, it gives just a multiplicative factor independent of p_T .

By using the momentum sum rule, Eq. (49), and the sum rule for $V^{(2)}$,

$$\sum_b V_{a \rightarrow qb}^{(2)}(\alpha_s(p_T^2); n, 1) = \Pi_{a,q}(\alpha_s(p_T^2); n) - \Pi_{a,q}(\alpha_s(p_T^2); n+1) \quad (72)$$

one can rewrite eq. (69) in a simpler form

$$\left(\frac{d\sigma}{dp_T^2}\right)_{n,1} \propto \frac{\partial}{\partial p_T^2} \left\{ D_{q,q}(p_T^2; M_0^2; n) D_{q,q}(Q^2; p_T^2; n+1) \right\} \quad (73)$$

where Eq. (6) has been used.

We have studied this expression numerically to next-to-leading log accuracy. By solving Eq. (6), D can be written as an exponential of the integral $\int_{M_0^2}^{Q^2} (dq^2/q^2) \Pi[\alpha_s(q^2); n]$, which can then be expanded to order $O(\alpha_s(p_T^2))$. The two-loop corrections ¹²⁾ to the running coupling constant have also been taken into account. The result for the case $n = 3$ is plotted in Fig. 4, as a ratio to the corresponding result in LLA. Two renormalization subtraction schemes, $\overline{MS}$ and $\overline{MS}^{(9)}$, are employed. It can be seen from Fig. 4 that the corrections due to the next-to-leading terms are quite small: if both are normalized to the LLA value at $p_T = 5$ GeV [the plotted ratios are then independent of Q^2/Λ^2 within the range (68)], the correction is less than $\pm 15\%$ for $5 \text{ (GeV)} \leq p_T \leq 60 \text{ (GeV)}$. The smallness is mainly due to the fact that in Eq. (69) only the differences of the n^{th} and $(n+1)^{\text{th}}$ moments of Π enter. Another reason is that the $\alpha_s^2(q^2)$ term in Π (analogue of the second order anomalous dimension) and the two-loop correction to $\alpha_s(q^2)$ in the first term in Π , have opposite signs and tend to cancel each other.

We thus find a first indication that for inclusive p_T correlations among fast particles inside a jet ($p_T \ll Q$), the next-to-leading corrections are quite small and the jet-calculus results in LLA already provide a good approximation. This is to be contrasted with some other processes ¹³⁾⁻¹⁵⁾ where the next-to-leading corrections have been found to be large or very sensitive to the choice of $\alpha_s(Q^2)$. The difference may be attributed to the highly inclusive nature of our measure, and also to the fact that we are interested here only in the shape of the p_T distribution, and not in the absolute normalization.

Another possible use of our results would be in the improvement of Monte-Carlo models for jet evolution, constructed in LLA ¹⁶⁾. Although some next-to-leading effects are included in these models by the exact treatment of decay kinematics, it is an interesting and quite relevant question whether the inclusion of full next-to-leading corrections - which contain intrinsically new contributions such as additional diagrams - would substantially modify the results of Refs. 16). For certain aspects involving soft gluons, there exists

some hint that they will, as has been discussed in Section 4.1 of I in connection with certain double soft divergences. Unfortunately, a more thorough analysis must await the completion of the calculation of gluon decay probabilities.

Some other possible applications, related to a calorimeter measure of jet-energy spread, and to the question of average multiplicity/moments in QCD, have also been discussed in I.

To summarize, we have presented a simple scheme for computing higher order contributions to jet evolution and scaling violation effects. Our scheme extends to higher orders the parton-probabilistic picture of jet evolution^{8),3)}. As an illustration and as first examples, we computed all the quark decay probabilities in the next-to-leading order, and among them, the generalized Altarelli-Parisi decay functions, $P_{Qj,Qi}^{NL}(x)$, $P_{\bar{Q}j,Qi}^{NL}(x)$ and $P_{G,Q}^{NL}(x)$. A numerical analysis of inclusive two-particle correlations indicates that the corrections beyond the leading order to the p_T spread of jets are reasonably small. We expect that further effort along the lines developed in this paper will make possible a comprehensive phenomenology of QCD jets and scaling violation beyond the leading order.

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APPENDIX A - KINEMATICS IN THE INFINITE MOMENTUM FRAME

We performed all the calculations in the infinite momentum frame of the parent parton, so as to allow a symmetric treatment of final particles. For illustration, consider the process $\gamma^* \rightarrow p+q$, $q \rightarrow k_1+k_2+k_3$ shown in Fig. 4. Assume that the particles k_1 , k_2 and k_3 belong to an effective vertex so that an eventual collinear log factor arises from the integration over q^2 .

The invariant phase volume is given by ($p^2 = k_1^2 = 0$)

$$\begin{aligned} d\Omega_4(Q \rightarrow p + \sum_i k_i) &= \frac{d^3p}{(2\pi)^3 2p_0} \prod_i \frac{d^3k_i}{(2\pi)^3 2k_{i0}} (2\pi)^4 \delta^4(Q - p - \sum_i k_i) \\ &= d\Omega_2(Q \rightarrow p+q) (2\pi)^3 dq^2 d\Omega_3(q \rightarrow k_1+k_2+k_3) \end{aligned} \quad (A.1)$$

where $d\Omega_2$ and $d\Omega_3$ are the usual two-body and three-body invariant phase volumes, respectively.

For the purpose of computing logarithmic contributions coming from the q^2 integration, we can set $q^2 = 0$ in $d\Omega_2(Q \rightarrow p+q)$. Then the integration over $dq^2 d\Omega_3(q \rightarrow k_1+k_2+k_3)$ can be done independently of the vertex, $Q \rightarrow p+q$.

We go to the following IMF system defined by

$$k_i = (x_i q^* + \frac{p_{i\perp}^2}{2x_i q^*}, p_{i\perp}, x_i q^*) \quad , \quad i = 1, 2, 3$$

$$q = k_1 + k_2 + k_3$$

$$q^2 = \sum_{i=1}^3 p_{i\perp}^2 / x_i$$

with

$$\sum_{i=1}^3 x_i = 1 \quad ; \quad \sum_{i=1}^3 p_{i\perp} = 0 \quad (A.2)$$

and $q^* \rightarrow \infty$.

The invariant phase volume $dq^2 d\Omega_3$ becomes, in $N = 4-2\epsilon$ space-time dimensions,

$$\begin{aligned}
 & dq^2 d\Omega_3 (q \rightarrow k_1 + k_2 + k_3) \\
 &= \frac{dq^2 dx_1 d^{N-2} p_{1\perp}}{2x_1 (2\pi)^{N-1}} \frac{dx_2 d^{N-2} p_{2\perp}}{2x_2 (2\pi)^{N-1}} \frac{1}{x_3} \delta(q^2 - \sum_{i=1}^3 \frac{p_{i\perp}^2}{x_i}) \\
 &= \frac{dx_1 dx_2 d^{N-2} p_{1\perp} d^{N-2} p_{2\perp}}{4(2\pi)^{2N-2} x_1 x_2 (1-x_1-x_2)} . \tag{A.3}
 \end{aligned}$$

Invariants such as $(k_1+k_2)^2$, $(q+k_3)^2$ and $k_1 \cdot n$, coming from the S matrix elements, can easily be expressed in terms of x_i 's and p_{iT} 's, and $d^{N-2} p_{iT}$ integrations can be explicitly performed, leaving only $dq^2 dx_i$ integrations.

APPENDIX B - THREE-BODY QUARK DECAY PROBABILITIES

We list here the complete set of three-body quark decay probabilities.
They are :

$$\begin{aligned} V_{q_i \rightarrow q_i(x_3) + G(x_1) + G(x_2)}^{(3)} &\equiv V_{q_i \rightarrow q_i G G}^{(3)}(1-x_1, -x_2, x_1, x_2) \delta(1-x_1-x_2-x_3) \\ &= C_F^2 \{ A(x_1, x_2) + B(x_1, x_2) \} + \frac{1}{2} C_F C_A \{ C(x_1, x_2) + D(x_1, x_2) - B(x_1, x_2) \}, \end{aligned} \quad (B.1)$$

$$\begin{aligned} V_{q_i \rightarrow q_i(x_1) + q_j(x_2) + \bar{q}_j(x_3)}^{(3)} &\equiv V_{q_i \rightarrow q_i q_j \bar{q}_j}^{(3)}(x_1, x_2, 1-x_1-x_2) \delta(1-x_1-x_2-x_3) \\ &= C_F (C_F - C_A/2) \delta_{ij} E(x_1, x_2) + C_F T_R \{ F(x_1, x_2) + \delta_{ij} F(x_2, x_1) \} \end{aligned} \quad (B.2)$$

where $T_R = \frac{1}{2}$, $C_F = \frac{4}{3}$, $C_A = 3$, and

$$\begin{aligned} A(x_1, x_2) &= -\frac{1}{x_1+x_2} \left\{ \frac{1}{x_1} + \frac{1}{x_2} \right\} \left\{ [1+(1-x_2)^2] \left[1 + \left(\frac{1-x_1-x_2}{1-x_2} \right)^2 \right] + [1+(1-x_1)^2] \left[1 + \left(\frac{1-x_1-x_2}{1-x_1} \right)^2 \right] \right\} \\ &\quad + (1-x_1-x_2) \left[\frac{1}{(1-x_1)^2} + \frac{1}{(1-x_2)^2} \right] + \left[\hat{P}\left(\frac{x_1}{1-x_2}\right) \right]_+ \frac{\hat{P}(x_2) \log(1-x_2)}{1-x_2} \\ &\quad + \left[\hat{P}\left(\frac{x_2}{1-x_1}\right) \right]_+ \frac{\hat{P}(x_1) \log(1-x_1)}{1-x_1}, \end{aligned} \quad (B.3)$$

$$\begin{aligned} B(x_1, x_2) &= \frac{1}{x_1+x_2} \left\{ \frac{1}{x_1} + \frac{1}{x_2} \right\} \left\{ 3(1-x_1-x_2)^2 - 2x_1x_2 + 3 + (1-x_1-x_2-x_1x_2) \left(\frac{1-x_2}{1-x_1} + \frac{1-x_1}{1-x_2} \right) \right\} \\ &\quad - \frac{2}{x_1x_2} \left[1 + (1-x_1-x_2)^2 \right] \log \frac{(1-x_1)(1-x_2)}{1-x_1-x_2}, \end{aligned} \quad (B.4)$$

$$\begin{aligned}
 c(x_1, x_2) = & 4 \hat{p}(x_1, x_2) \left[\frac{1}{x_1} + \frac{1}{x_2} - \frac{1}{x_1 + x_2} \right] - 2(1 + x_1 + x_2) \left[\frac{1}{1-x_1} + \frac{1}{1-x_2} \right] + 2 \\
 & + 4 \hat{p}(x_1, x_2) \left\{ \frac{1}{(x_1 + x_2)} \left[\left(\frac{\log x_1 / (x_1 + x_2)}{x_1 / (x_1 + x_2)} \right) + \left(\frac{\log x_2 / (x_1 + x_2)}{x_2 / (x_1 + x_2)} \right) \right] \right. \\
 & \left. - \frac{1}{x_1} \log \frac{1-x_2}{1-x_1-x_2} - \frac{1}{x_2} \log \frac{1-x_1}{1-x_1-x_2} \right\} \\
 & + 2 \left[1 - 2 \frac{x_1}{x_1 + x_2} + \frac{1 + (1-x_2)^2}{x_1(x_1 + x_2)} \right] \log \frac{x_2}{(1-x_1)(x_1 + x_2)} \\
 & + 2 \left[1 - 2 \frac{x_2}{x_1 + x_2} + \frac{1 + (1-x_1)^2}{x_2(x_1 + x_2)} \right] \log \frac{x_1}{(1-x_2)(x_1 + x_2)}, \quad (B.5) \quad *)
 \end{aligned}$$

$$\begin{aligned}
 D(x_1, x_2) = & -4 \left\{ \left(\frac{x_1}{x_2} + \frac{x_2}{x_1} \right) \frac{1 + (1-x_1-x_2)^2}{(x_1 + x_2)^2} + \frac{[x_2 - x_1(1-x_1-x_2)][x_1 - x_2(1-x_1-x_2)]}{(x_1 + x_2)^4} \right. \\
 & \left. + 2 \frac{x_1 x_2 (1-x_1-x_2)}{(x_1 + x_2)^4} \right\} \\
 & + 2 \frac{1+x_1^2}{1-x_1} \left[\frac{1}{x_1} + \frac{1}{x_2} - \frac{2}{x_1 + x_2} + \frac{x_1 x_2}{(x_1 + x_2)^3} \right] \log(1-x_1) \\
 & + 2 \frac{1+x_2^2}{1-x_2} \left[\frac{1}{x_1} + \frac{1}{x_2} - \frac{2}{x_1 + x_2} + \frac{x_1 x_2}{(x_1 + x_2)^3} \right] \log(1-x_2), \quad (B.6)
 \end{aligned}$$

$$\begin{aligned}
 E(x_1, x_2) = & 2 \left[-2 + \frac{1+x_1}{1-x_2} + \frac{1+x_2}{1-x_1} - \frac{2x_1}{(1-x_2)^2} - \frac{2x_2}{(1-x_1)^2} \right. \\
 & \left. + \left\{ 2 - \frac{1+x_2}{1-x_1} - \frac{1+x_1}{1-x_2} + \frac{2}{(1-x_1)(1-x_2)} \right\} \log \frac{(1-x_1)(1-x_2)}{1-x_1-x_2} \right], \quad (B.7)
 \end{aligned}$$

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*) The distribution $((\log x_1 / (x_1 + x_2)) / (x_1 / (x_1 + x_2)))_+$ should be understood as

$$\int_0^1 dz \left(\frac{\log z}{z} \right)_+ \delta \left(z - \frac{x_1}{x_1 + x_2} \right).$$

$$\begin{aligned}
 F(x_1, x_2) = & \frac{2x_1}{(1-x_1)^2} - \frac{(1+x_1)^2}{(1-x_1)^2} \left\{ \left(\frac{1-x_1-x_2}{1-x_1} \right)^2 + \left(\frac{x_2}{1-x_1} \right)^2 \right\} + \frac{4x_1x_2(1-x_1-x_2)}{(1-x_1)^4} \\
 & + \frac{\hat{P}(1-x_1)}{1-x_1} \left\{ \left(\frac{x_2}{1-x_1} \right)^2 + \left(\frac{1-x_1-x_2}{1-x_1} \right)^2 \right\} \log(1-x_1), \quad (B.8)
 \end{aligned}$$

$$\hat{P}(x) = \frac{1 + (1-x)^2}{x} \quad (B.9)$$

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FIGURE CAPTIONS

- Figure 1   Rainbow diagram contributing to the evolution of a quark of momentum q ; z_1 and z_2 are the momentum fractions of gluons of momentum ℓ, k with respect to the quark momenta q and r , respectively.
- Figure 2   Diagrams contributing in the light-like gauge to $v_{Q \rightarrow a_1 a_2}^{(2)}$ and $v_{Q \rightarrow a_1 a_2 a_3}^{(3)}$.
- Figure 3   Jet calculus diagram for the two-particle transverse momentum correlation computed in Section 4.
- Figure 4   Non-leading corrections to the two-particle transverse momentum correlation within a jet, computed with the $\overline{MS}$ and $\overline{MS}$ subtraction schemes. The results are normalized to the leading log prediction.
- Figure 5   Kinematics for the process $\gamma^* \rightarrow p+q, q \rightarrow k_1+k_2+k_3$.

fig. 1

Fig. 2

fig.3

fig. 4

fig. 5