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LARGE N, CHIRAL APPROACH TO PSEUDOSCALAR MASSES, MIXINGS AND DECAYS

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A B S T R A C T

Recent results on the combined chiral and large N limit of QCD are used for an improved treatment of the pseudoscalar nonet mass matrix and of η, η' decays. We point out that a few next-to-leading terms in $1/N$ and m/Λ are necessary and sufficient for a reasonable over-all agreement with existing data.

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1. - INTRODUCTION

In spite of the growing evidence that quantum chromodynamics (QCD) can describe the basic, constituent level, strong interactions measured in hard processes ¹⁾, the problem of confinement and, a fortiori, that of predicting hadronic masses and interactions in the low energy domain, remains an outstanding one.

Large N expansion techniques ²⁾ provide an appealing framework in which conventional approaches to soft hadronic physics, such as dual models, dual topological unitarization, Regge-Gribov theories, etc., find a natural place. On the other hand, large N predictions are, in general, of a semi-quantitative nature at best.

An exception to this somewhat unhappy situation comes from the opposite end of the energy spectrum, i.e., from the study of the lightest hadronic system, the pseudoscalar mesons. Predictivity is enhanced here by the assumption that the pseudoscalar nonet is a manifestation of the approximate chiral symmetry of QCD, realized in the (spontaneously broken) Nambu-Goldstone (NG) way. Taking into account some recent development ³⁾ it even looks possible that one can prove the necessity of the NG mechanism in QCD, at least within the $1/N$ expansion framework.

Finally, by combining the chiral and large N limits of QCD, it has been possible to construct ⁴⁾ an explicit solution to the famous $U(1)$ problem of QCD. The results of this effort can be nicely summarized ⁵⁾ in terms of an effective Lagrangian containing a series of terms of increasing order in the quark mass m and/or in $1/N$. As usual, there is a certain degree of arbitrariness encountered in writing down the effective Lagrangian. Nevertheless, as we discuss later on in more detail, those results that follow from current algebra and $1/N$ expansion arguments cannot depend on the actual form used. An amusing application of these ideas has been recently given ⁶⁾ by two of us in the framework of composite Higgs structures that can be fitted on the Glashow-Weinberg-Salam model.

In this paper we continue the study of the spectrum and dynamics (e.g., decays) of the pseudoscalar nonet within the effective Lagrangian framework. One of our purposes is to determine how many terms are needed in the above-mentioned expansion of the effective Lagrangian before achieving a reasonable quantitative picture for the masses, mixing and decays of the pseudoscalar nonet.

In particular, we find that the singlet particle is stable in the chiral limit (apart from the radiative decay that we discuss in the last section) thus explaining qualitatively the narrowness of the η' . In order to improve the agreement with masses and mixing angle and also in order to obtain the momentum dependence in the decay $\eta' \rightarrow \eta 2\pi$ predicted by the Adler zeros and observed experimentally we are forced to add to the Lagrangian of Ref. 5) next-to-leading terms in $1/N$ and in m/N .

Allowing, finally, isospin violations in the mass matrix ($m_u \neq m_d$) we can also compute the decays $\eta \rightarrow 3\pi$ finding reasonably good agreement with the data.

The outline of the paper is as follows. In Section 2 we discuss the linear and non-linear effective Lagrangians found in Ref. 5) and we compute the quadratic and quartic terms in the meson fields. In Section 3 we discuss the mass matrix and the decays $\eta \rightarrow 3\pi$, $\eta' \rightarrow 3\pi$ and $\eta' \rightarrow \eta\pi\pi$ allowing for isospin violations due to $m_u \neq m_d$ and for next-to-leading terms in $1/N$ and m/Λ . Finally in Section 4, we give a more systematic derivation of the non-leading terms and we add a term due to the electromagnetic anomaly that allows to compute the decays π^0 , η , $\eta' \rightarrow 2\gamma$.

2. - THE EFFECTIVE LAGRANGIAN : LEADING TERMS

It has been found ⁵⁾ that the effective Lagrangian describing the low energy dynamics of the scalar and pseudoscalar mesons in the large N expansion ^{*)} of QCD is a kind of σ model ⁷⁾ given by :

$$\begin{aligned}
 L = & \frac{1}{2} \text{Tr} (\partial_\mu U \partial_\mu U^\dagger) - V(U, U^\dagger) + \\
 & + \frac{i}{2} \varrho(x) \text{Tr} [\log U - \log U^\dagger] + \frac{N}{2F_\pi^2} \varrho^2(x) + \\
 & + \frac{F_\pi}{2\sqrt{2}} \text{Tr} (MU + M^\dagger U^\dagger)
 \end{aligned}
 \tag{2.1}$$

^{*)} N is the number of quark colours.

Here U is a 3×3 matrix ^{*)} describing the fields of the scalar and pseudoscalar mesons. The derivative and the potential terms in L are invariant under chiral $U(3) \otimes U(3)$ transformations that act on U as follows :

$$U \rightarrow A U B^\dagger \quad (2.2)$$

where A and B are unitary matrices.

We assume that the potential $\mathcal{V}(U, U^\dagger)$ breaks spontaneously chiral symmetry : it can be taken, for instance, to be of the form :

$$\mathcal{V}(U, U^\dagger) = \frac{\lambda}{4} \text{Tr} \left\{ (U U^\dagger - F_\pi^2/2)^2 \right\} \quad (2.3)$$

where F_π will be identified with the pion decay constant ($F_\pi \approx 95$ MeV). The terms containing the topological charge density

$$q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \quad (2.4)$$

are required by the axial anomaly and break explicitly the axial $U(1)$ while preserving chiral $SU(3) \otimes SU(3)$. These terms provide the mechanism which gives a mass to the singlet even in the chiral limit, thus solving the $U(1)$ problem. Explicit soft breaking of chiral invariance due to the quark masses is provided by the term containing the mass matrix M . Without loss of generality ^{**)} this can be taken of the real diagonal form:

$$M_{ij} = \mu_i^2 \delta_{ij} \quad (2.5)$$

where μ_i^2 is proportional to the quark masses m_i :

$$\begin{aligned} \mu_i^2 &= -2m_i F_\pi^{-2} \langle \bar{\Psi}_i \Psi_i \rangle, \\ m_\pi^2 &= 1/2 (\mu_u^2 + \mu_d^2), \\ m_{K^+}^2 &= 1/2 (\mu_u^2 + \mu_s^2), \text{ etc.} \end{aligned}$$

^{*)} We take the number L of light flavours to be 3, but our general arguments are quite independent of L .

^{**)} Actually there is another free parameter, the vacuum angle θ , which is related to $\text{Arg det } M$. In the following we consider $\theta = 0$. For effects due to $\theta \neq 0$, see Ref. 8).

In addition to the masses μ_1^2 and to F_π , the Lagrangian (2.1) contains also the parameter a , which is given by :

$$\int d^4y T \langle q(0) q(y) \rangle_{Y.M.} = \frac{1}{2} \frac{a}{N} F_\pi^2 \quad (2.6)$$

where the Green function in the left-hand side is computed in pure Yang-Mills theory (i.e., without quarks) in the limit $N \rightarrow \infty$.

It is well known that "current algebra" results are not dependent on the specific choice of the potential $V(U, U^\dagger)$ and can be obtained easily in the limit where (2.1) becomes a non-linear σ model. This can be achieved by sending λ or, more generally, the mass of the scalar particles to infinity, keeping F_π fixed. In this limit (2.1) becomes a non-linear σ model described by the following Lagrangian⁵⁾ :

$$L = \frac{1}{2} \text{Tr} (\partial_\mu U \partial_\mu U^\dagger) + \frac{F_\pi}{2\sqrt{2}} \text{Tr} [MU + M^\dagger U^\dagger] + \frac{a F_\pi^2}{16N} \left\{ \text{Tr} (\log U - \log U^\dagger) \right\}^2 \quad (2.7)$$

where the algebraic equation of motion for $q(x)$ has been used and the field U satisfies the constraint :

$$UU^\dagger = \frac{F_\pi^2}{2} \mathbb{1} \quad (2.8)$$

Two parametrizations for U have been especially used in the literature :

$$U = \frac{F_\pi}{\sqrt{2}} \frac{1 + if\phi}{1 - if\phi} ; \quad U = \frac{F_\pi}{\sqrt{2}} e^{2if\phi} \quad (2.9)$$

with $f = 1/F_\pi \sqrt{2}$. ϕ is a matrix containing the fields of the pseudoscalar nonet and can be written as :

$$\phi = \pi^i \lambda^i + \frac{1}{\sqrt{3}} S \mathbb{1} \quad (2.10)$$

where the λ 's are the Gell-Mann λ matrices normalized to $\text{Tr}(\lambda_i \lambda_j) = \delta_{ij}$.

However, if one just requires U to be unitary, the most general expansion in terms of ϕ , up to fourth order in the field ϕ , is given by⁹⁾ :

$$U = \frac{F_\pi}{\sqrt{2}} \left(\mathbb{1} + 2if\phi + 2(if\phi)^2 + a_3 (if\phi)^3 + 2(a_3-1)(if\phi)^4 \right) \quad (2.11)$$

where a_3 is an arbitrary parameter. The two expressions for U in (2.9) correspond, in particular, to $a_3 = 2$ and to $a_3 = 4/3$, respectively. Of course, the presence of terms of higher order in the field ϕ in (2.11) will be accompanied by additional arbitrary parameters. It can be shown, however, that there is no dependence on these parameters for S matrix elements (i.e., processes with external on-shell particles) ¹⁰⁾

In order to check this explicitly in one example, we restrict ourselves to the expansion in (2.11), and we insert it into (2.7), to get :

$$\begin{aligned} L = & \frac{1}{2} \text{Tr} (\partial_\mu U \partial_\mu U^\dagger) - \frac{1}{2} \text{Tr} (M \phi^2) - \frac{a}{2N} (\text{Tr} \phi)^2 + \\ & + \frac{1}{4F_\pi^2} \left\{ \text{Tr} [\partial_\mu \phi^2 \partial_\mu \phi^2 - a_3 \partial_\mu \phi^3 \partial_\mu \phi] + \right. \\ & \left. + (a_3-1) \text{Tr} (M \phi^4) - a (4/3 - a_3) \text{Tr} \phi \text{Tr} \phi^3 \right\} \end{aligned} \quad (2.12)$$

where we have neglected terms containing powers of ϕ higher than four. However, if we are interested in using (2.12) to compute on-shell processes, we can use the equation of motion for ϕ :

$$\square \phi + \frac{1}{2} (M \phi + \phi M) + \frac{a}{N} (\text{Tr} \phi) \mathbb{1} = F(\phi, \partial_\mu \phi) \quad (2.13)$$

where the right-hand side contains at least the third power of ϕ and hence can be neglected when dealing with four-point interactions.

It is very easy to check that the use of (2.13) with $F(\phi, \partial_\mu \phi) = 0$ implies that the coefficient of a_3 in Eq. (2.12) vanishes. One finds therefore the nice result that the on-shell four-point amplitudes do not depend on the value of a_3 .

If one inserts in (2.12) the value $a_3 = 4/3$ corresponding to the exponential form in (2.9) one finds the following expression for the piece of the effective Lagrangian (2.12) that describes the on-shell four-meson couplings :

$$L_4 = \frac{1}{4F_\pi^2} \left\{ \text{Tr} (\partial_\mu \phi^2 \partial_\mu \phi^2 + \frac{4}{3} \phi^3 \square \phi) + \frac{1}{3} \text{Tr} (M \phi^4) \right\} \quad (2.14)$$

Notice that this interaction conserves exactly the OZI rule since it corresponds to leading order in $1/N$. Its consequences for η, η' decays will be discussed in Section 3.

3. - THE EFFECTIVE LAGRANGIAN : NEED FOR NON-LEADING TERMS

The first important consequence of Lagrangian (2.7) is that the singlet is a stable particle in the chiral limit. In fact when $M = 0$, using the exponential form in (2.9), one gets from (2.7) :

$$L = \frac{1}{2} \text{Tr} (\partial_\mu V \partial_\mu V^\dagger) + \frac{1}{2} (\partial_\mu S)^2 - \frac{3}{2} \frac{a}{N} S^2 \quad (3.1)$$

where

$$V = \frac{F_\pi}{\sqrt{2}} \exp \left(i \frac{\sqrt{2}}{F_\pi} \pi^i \lambda^i \right)$$

does not contain the singlet.

Equation (3.1) shows that, in the chiral limit, the singlet is an eigenstate of the mass matrix and decouples from the massless non-singlet mesons. This suggests itself as an "explanation" for the narrowness of the η' and is an important consequence of having broken the axial $U(1)$ in a way consistent with the anomaly and not arbitrarily as was done in the early days of current algebra ¹¹⁾.

In order to compute the mass matrix and the S matrix for processes involving four pseudoscalar mesons, we can restrict ourselves to the quadratic and quartic terms in the field ϕ . One gets :

$$L_2 = \frac{1}{2} \text{Tr} (\partial_\mu \phi \partial_\mu \phi) - \frac{1}{2} \text{Tr} (M \phi^2) - \frac{a}{2N} (\text{Tr} \phi)^2 \quad (3.2)$$

and from (2.14) :

$$L_4 = \frac{1}{4F_\pi^2} \left\{ -\frac{4}{3} \pi^i \pi^j \partial_\mu \pi^k \partial_\mu \pi^l f_{ikl} f_{jlm} + \right. \\ \left. + \frac{1}{3} \text{Tr} (M \phi^4) \right\} \quad (3.3)$$

where the structure constants of SU(3) are defined through the relation :

$$2i f_{abc} \lambda^c = [\lambda^a, \lambda^b] \quad (3.4)$$

Equation (3.3) shows that the derivative coupling is absent for all processes involving only elements of the Cartan subalgebra of SU(3) (e.g., $\eta' \rightarrow \eta \pi^0 \pi^0$, $\eta \rightarrow 3\pi^0$, etc.) whose amplitudes are therefore independent of momenta. For such processes, (3.3) reduces to

$$L_{\text{(Cartan subalgebra)}} = \frac{1}{12F_\pi^2} \text{Tr} (M \phi^4) \quad (3.5)$$

In the following, we will use L_2 and L_4 of Eqs. (3.2), (3.3) to compute the mass matrix and hadronic decays of the pseudoscalar mesons. In order to get good agreement with the experimental data, we will need to introduce next-to-leading terms in $1/N$ and m/N .

A. - Pseudoscalar spectrum and mixing

In order to motivate the phenomenological need for an additional non-leading quadratic term in L , we recall that the Lagrangian (3.2) gives qualitatively (and we can even say semiquantitatively) correct results for the spectrum and mixing pattern of the pseudoscalar nonet.

Using as inputs the pion and kaon masses and neglecting small isospin mixing (to be discussed in B), one arrives at a two-by-two matrix having a as its single unknown parameter¹²⁾. The experimental quantities to be fitted are three : the η , η' masses and the pseudoscalar mixing angle ϕ . As shown in Ref. 12), one can determine a from the sum of the η and η' squared masses :

$$a = m_{\eta}^2 + m_{\eta'}^2 - 2m_K^2 \simeq_{\text{exp.}} 0.726 \text{ GeV}^2 \quad (3.6)$$

The resulting values for $(m_{\eta}^2, -m_{\eta}^2)$ and for φ turn out to be (our convention for φ is : $\eta = \cos\varphi \eta_8 + \sin\varphi \eta_1$ etc.)

$$\Delta_{m^2} \equiv (m_{\eta'}^2 - m_{\eta}^2) \simeq 0.68 \text{ GeV}^2 ; \quad \varphi \simeq 14^\circ \quad (3.7)$$

which are neither bad nor too precise ($\Delta_{m^2}^{\text{exp}} = 0.62 \text{ GeV}^2$; $\varphi^{\text{exp}} \simeq 10^\circ$).

For phenomenological applications it is natural to ask for better precision, otherwise theoretical predictions turn out to depend too much on the theoretical relations used before inserting the experimental numbers.

The leading form of the mass matrix obtained from (3.2) has entries which are either of order μ_1^2 (N independent pseudoscalar masses) or of order Λ^2/N (the strength of the anomaly, i.e., a/N). Next-to-leading corrections can be of order $\mu_1^2 \mu_j^2 / \Lambda^2$, of order Λ^2/N^2 or, finally, of order μ_1^2/N . The control of terms of order μ^4 goes beyond the scope of any current algebra scheme. Although we are aware that the strange quark mass is not much smaller than Λ , we shall proceed under the working assumption that terms of order m_K^4/Λ^2 are negligible. Terms of order $1/N^2$ come from diagrams with higher topological complexity but should not have a direct effect on the mass matrix (apart from a redefinition of a). We are left with terms of order μ_1^2/N which can be of practical importance in the case of strange quarks.

The possibility that such terms appear in the flavourless sector of the mass matrix can be supported by (anomalous) Ward identity arguments. The anomaly contribution $-\frac{1}{2}a/N [\text{Tr}(\Phi)]^2$ in Eq. (2.12) can be seen to follow from saturating a Ward identity with double Goldstone boson poles ^{*)}. There are corrections to these double pole contributions coming from single poles. These are of the form $c/N (m_i + m_j)$ ($c = O(\Lambda)$) for the i, j element of the mass matrix. A simple way to rewrite such a contribution is to add in the Lagrangian a quadratic term of the form

$$-\frac{1}{2} \frac{\varepsilon}{N} \text{Tr}(\Phi) \text{Tr}(M\Phi) \quad (3.8)$$

^{*)} See, for instance, G. Veneziano, Ref. 3).

with ϵ a pure number of order 1 (in $1/N$). Making the obvious approximation of neglecting terms in $m_{u,d}$ (but not in m_s), it is very easy to write the modified mass matrix in the presence of the extra term (3.8). In the octet-singlet basis one finds ($N = 3$)

$$\mathcal{M}^2 = \begin{pmatrix} \frac{1}{3}(4K-\pi) & -\frac{2\sqrt{2}}{3}(K-\pi) - \frac{\sqrt{2}}{3}\epsilon K \\ \text{symm.} & \frac{1}{3}(2K+\pi) + \frac{2}{3}\epsilon K + a \end{pmatrix} \quad (3.9)$$

where each particle symbol stands for the corresponding squared mass. Equation (3.9) reduces to the expression of Ref. 12) for $\epsilon = 0$. A trivial diagonalization then shows that, if we take

$$\epsilon = -1.08 \quad ; \quad a = 0.90 \text{ GeV}^2 \quad (3.10)$$

the η and η' masses as well as the experimental mixing angle $\varphi \approx 10^\circ$ are reproduced.

Remembering that ϵ and a should be of order one and Λ^2 , respectively, we consider the values in Eq. (3.10) as very reasonable (at least as not surprisingly small or large). A non-leading term in the Lagrangian which can give rise to a quadratic term of the form (3.8) is

$$\frac{i\epsilon}{2\sqrt{2}F_\pi} \frac{1}{a} q(x) T_r (M(U-U^\dagger)) \quad (3.11)$$

Upon elimination of $q(x)$ through its equations of motion one recovers a term like (3.8) plus higher order terms in the fields which induce new, non-leading interactions. A more complete discussion of the non-leading structure of L_{eff} is given in Section 4.

B. - Decays $\eta, \eta' \rightarrow 3\pi$

A mass difference between the u and d quarks implies isospin violations that, among other effects, will induce the decay $\eta \rightarrow 3\pi$ ¹³⁾. Indeed this process can be used to obtain an estimate of the ratio of the light quark masses, since electromagnetic contributions to it are estimated to be small¹⁴⁾.

The amplitude for $\eta \rightarrow 3\pi^0$ receives only a contribution from the piece (3.5) of the Lagrangian (3.3). One gets in a straightforward way :

$$A(\eta \rightarrow 3\pi^0) = \frac{\mu_u^2 - \mu_d^2}{\sqrt{6} F_\pi^2} \left(\frac{\cos \varphi}{\sqrt{2}} + \sin \varphi \right) \quad (3.12)$$

where φ is the mixing angle between the octet and the singlet defined through :

$$\begin{aligned} \pi_1 &= \cos \varphi \eta' + \sin \varphi \eta \\ \pi_8 &= \cos \varphi \eta - \sin \varphi \eta' \end{aligned} \quad (3.13)$$

The observed decay width :

$$\Gamma(\eta \rightarrow 3\pi^0) \simeq 0.25 \text{ Kev} \quad (3.14)$$

is then obtained for

$$m_u/m_d = \mu_u^2/\mu_d^2 = 0.43 \quad (3.15)$$

in reasonable agreement with other estimates ¹⁵⁾.

The result (3.14) is not significantly altered if one includes full mixing among π^0 , η , η' , which is present once isospin is violated.

The π^0 - η mixing is instead relevant for the charged decay $\eta \rightarrow \pi^+ \pi^- \pi^0$, and through it this process gets also a contribution from the derivative terms in (3.3).

Neglecting second-order corrections to isospin violations which are, however, very small, the amplitude turns out to be :

$$\begin{aligned} A(\eta \rightarrow \pi^+ \pi^- \pi^0) &= \frac{\mu_u^2 - \mu_d^2}{6\sqrt{3} F_\pi^2} (\cos \varphi + \sqrt{2} \sin \varphi) \cdot \\ &\cdot \left\{ 1 - \frac{6\sqrt{3} m_\pi^2 \alpha_1 \beta_1}{(\mu_u^2 - \mu_d^2)(\cos \varphi + \sqrt{2} \sin \varphi)} \frac{s - s_0}{m_\pi^2} \right\} \end{aligned} \quad (3.16)$$

where $s = (p_\eta - p_{\pi^0})^2 = (p_{\pi^+} + p_{\pi^-})^2$ and $s_0 = \frac{1}{3}(m_\eta^2 + 3m_\pi^2)$.

The mixing coefficients α_1 and β_1 are defined by

$$\pi_3 = \alpha_1 \pi^0 + \beta_1 \eta + \gamma_1 \eta' \quad (3.17)$$

and are determined by diagonalizing the quadratic part of the Lagrangian. To first order in $(\mu_u^2 - \mu_d^2)$ they are given by

$$\begin{aligned} \alpha_1 &= 1 + O((\mu_u^2 - \mu_d^2)^2) \\ \beta_1 &\simeq - \frac{1}{\sqrt{3} m_\eta^2} \frac{\mu_u^2 - \mu_d^2}{2} (\cos \varphi + \sqrt{2} \sin \varphi) \end{aligned} \quad (3.18)$$

Inserting these values in (3.16) one gets

$$A(\eta \rightarrow \pi^+ \pi^- \pi^0) = \frac{(\mu_u^2 - \mu_d^2)}{3\sqrt{6} F_\pi^2} \left[\frac{1}{\sqrt{2}} \cos \varphi + \sin \varphi \right] \left\{ 1 - \sigma \frac{s - s_0}{m_\pi^2} \right\} \quad (3.19)$$

with

$$\sigma = - \frac{3 m_\pi^2}{m_\eta^2} \simeq - 0.2 \quad (3.20)$$

The expression (3.19) is known to reproduce the observed behaviour of the Dalitz plot ¹⁶⁾ with a value of σ well given by (3.20).

It is interesting to notice that the presence of the parameter a , due to the axial anomaly which explicitly breaks the axial $U(1)$ affects the decays of $\eta \rightarrow 3\pi$ only through the determination of the mixing angle. Equation (3.19) agrees with the analogous amplitude computed in a recent paper by Kawarabayashi and Ohta ¹⁸⁾ that we received while completing this work ¹⁷⁾.

Proceeding in an analogous way, one can also compute the decay of the η' in 3π . From Lagrangian (3.5) one finds, for the neutral mode :

$$A(\eta' \rightarrow 3\pi^0) = \frac{\mu_u^2 - \mu_d^2}{\sqrt{6} F_\pi^2} \left(\cos \varphi - \frac{\sin \varphi}{\sqrt{2}} \right) \quad (3.21)$$

Neglecting electromagnetic contributions, one can predict the ratio of the rates $\eta' \rightarrow 3\pi^0$ to $\eta \rightarrow 3\pi^0$. The latter being given by Eq. (3.4), one predicts

$$\Gamma(\eta' \rightarrow 3\pi^0) \simeq 1.7 \text{ Kev} \quad (3.22)$$

The present experimental upper limit to this decay is ¹⁸⁾ :

$$\Gamma_{\text{exp.}}(\eta' \rightarrow 3\pi^0) < 10 \text{ Kev} \quad (3.23)$$

It is interesting to observe that even if a is not explicitly present in Eq. (3.21), if we had put no anomaly in the Lagrangian (i.e., $a = 0$), the diagonalization of the quadratic fact would have given "ideal" mixing ($\tan \varphi = \sqrt{2}$) and the amplitude (3.21) would have been zero. For the charged decay $\eta' \rightarrow \pi^+ \pi^- \pi^0$ there is also a derivative coupling due to $\eta' - \pi^0$ mixing. To first order in isospin breaking, one finds :

$$A(\eta' \rightarrow \pi^+ \pi^- \pi^0) = \frac{\mu_u^2 - \mu_d^2}{3\sqrt{6} F_\pi^2} \left[\cos \varphi - \frac{\sin \varphi}{\sqrt{2}} \right] \left\{ 1 - \sigma' \frac{s' - s'_0}{m_\pi^2} \right\} \quad (3.24)$$

where $s' = (p_{\eta'} - p_{\pi^0})^2$, $s'_0 = \frac{1}{3}(m_\eta^2 + 3m_\pi^2)$ and $\sigma' = -(3m_\pi^2/m_\eta^2) = -0.06$. One gets a slope σ' that is about three times smaller than in the case of the decay $\eta \rightarrow \pi^+ \pi^- \pi^0$ [Eq. (3.20)]. Notice that this is just the result from the leading order in $1/N$. Non-leading corrections to be discussed below, although inessential for $\eta \rightarrow 3\pi$, give a sizeable contribution to $\eta' \rightarrow 3\pi$.

C. - $\eta' \rightarrow \eta \pi \pi$

The amplitude for the decay $\eta' \rightarrow \eta \pi^0 \pi^0$ can be easily obtained from Lagrangian (3.5). One finds :

$$A(\eta' \rightarrow \eta \pi^0 \pi^0) = \frac{m_\pi^2}{3 F_\pi^2} \left(\sqrt{2} \cos 2\varphi + \frac{1}{2} \sin 2\varphi \right) \quad (3.25)$$

where we have neglected isospin violations. Consequently :

$$A(\eta' \rightarrow \eta \pi^+ \pi^-) = A(\eta' \rightarrow \eta \pi^0 \pi^0) \quad (3.26)$$

Inserting the value for the mixing angle $\varphi \approx 10^\circ$ one gets the following partial widths :

$$\Gamma(\eta' \rightarrow \gamma \pi^+ \pi^-) = 2 \Gamma(\eta' \rightarrow \gamma 2\pi^0) \simeq 2.4 \text{ Kev} \quad (3.27)$$

which imply :

$$\Gamma(\eta' \rightarrow \gamma 2\pi) \simeq 3.6 \text{ Kev} \quad (3.28)$$

This is much smaller than the recently observed ¹⁹⁾ :

$$\Gamma^{\text{exp}}(\eta' \rightarrow \gamma 2\pi) = 180 \pm 60 \text{ Kev} \quad (3.29)$$

Moreover, the amplitude (3.25) is constant over phase-space, while the experimental data are fitted by an amplitude of the form :

$$A(\eta' \rightarrow \gamma 2\pi) = A (1 - \sigma_1 T_\eta) \quad (3.30)$$

where T_η is the η kinetic energy and A and σ_1 are constants. Experimentally, the slope parameter is ²⁰⁾ :

$$\sigma_1^{\text{exp.}} = - (0.48 \pm 0.18) m_\pi^{-1} \quad (3.31)$$

Actually the vanishing of $A(\eta' \rightarrow \eta \pi \pi)$ when the mass of the quarks are zero and its dependence of the η kinetic energy can easily be understood from the requirement of Adler zeros.

The most general low energy expression for $A(\eta' \rightarrow \eta \pi \pi)$ symmetric under the exchange of the two pions is given by :

$$A(\eta' \rightarrow \eta \pi \pi) = \alpha_1 + \alpha_2 (s+t) + \alpha_3 u + \text{higher orders in } s, t, u. \quad (3.32)$$

where

$$s = (p_{\pi}^{(1)} + p_{\eta})^2, \quad t = (p_{\pi}^{(2)} + p_{\eta})^2, \quad u = (p_{\pi}^{(1)} + p_{\pi}^{(2)})^2$$

In a world where $m_u = m_d = m_s = 0$ (i.e., $m_{\pi} = m_{\eta} = 0$) one must require the vanishing of $A(\eta' \rightarrow \eta \pi \pi)$ at the following values of s, t and u :

$$\begin{aligned} 1) \quad & u = s = 0, \quad t = m_{\eta'}^2 \\ 2) \quad & s = t = 0, \quad u = m_{\eta'}^2 \\ 3) \quad & t = u = 0, \quad s = m_{\eta'}^2 \end{aligned} \quad (3.33)$$

Those requirements imply the vanishing of the lowest order term in (3.32).

If instead we assume that $m_u = m_d = 0$ while $m_s \neq 0$, one must require the vanishing of (3.32) only when $s = m_{\eta}^2, u = 0$ and $t = m_{\eta}^2$, or $s \leftrightarrow t, u = 0$. This implies that

$$A(\eta' \rightarrow \eta \pi \pi) = \text{const.} \cdot p_{\pi}^{(1)} \cdot p_{\pi}^{(2)} \quad (3.34)$$

where the constant is vanishing in the limit of a vanishing strange quark mass.

In order to obtain an amplitude of the form (3.34) we must add ²¹⁾ to Lagrangian (2.7) non-leading terms in the $1/N$ expansion, that may be very important because of the proportionality of the leading terms to the tiny pion mass. It is easy to check that the next-to-leading term is given by:

$$\frac{\beta}{a^2 F_{\pi}^4} q^2(x) \text{Tr} (\partial_{\mu} U \partial_{\mu} U^{\dagger}) \quad (3.35)$$

This term can be added to the term (3.11) already introduced to improve the mass matrix, to give the following contribution to the fourth order term:

$$\left\{ \frac{\beta}{2N^2 F_{\pi}^2} (\text{Tr} \phi)^2 + \frac{\epsilon \beta}{2a F_{\pi}^2 N^2} (\text{Tr} \phi) \text{Tr} (M \phi) \right\} \text{Tr} (\partial_{\mu} \phi \partial_{\mu} \phi) \quad (3.36)$$

Keeping only contributions proportional to the kaon mass and taking $N = 3$, one gets the following amplitude for the decay $\eta' \rightarrow \eta \pi \pi$

$$A(\eta' \rightarrow \eta \pi \pi) = - \frac{\beta}{6 F_\pi^2} \left\{ \sin 2\varphi + \frac{2\epsilon m_K^2}{3a} (\sin 2\varphi - \sqrt{2} \cos 2\varphi) \right\} \cdot 2 P_\pi^{(1)} P_\pi^{(2)} \quad (3.37)$$

Inserting in (3.37) the values for a , ϵ and φ determined from the mass spectrum (subsection 3A) one gets :

$$A(\eta' \rightarrow \eta \pi \pi) \simeq -1.6 \beta \left\{ 1 - \frac{2 m_{\eta'}}{(m_{\eta'} - m_\eta)^2} T_\eta \right\} \quad (3.38)$$

We find therefore a slope parameter

$$\sigma_1 = - \frac{2 m_{\eta'}}{(m_{\eta'} - m_\eta)^2} \quad (3.39)$$

which is the one given by current algebra and is β independent, since we have neglected terms independent of the momenta and of order m_π^2 .

One should point out here that our result for σ_1 becomes of order $1/m_\eta$ in the chiral limit ($m_\eta \gg m_\pi$). One might think therefore that the predicted value will be too small compared to the experimental number (3.31). However, by inserting the experimental values of the masses, σ_1 turns out to be numerically $-1.6 m_\pi^{-1}$ hence correct in sign, but a factor of three too large. We are not too worried about this discrepancy, which, as we pointed out, comes from the Adler zero structure which has been used over a wide range of momenta.

Neglecting the dependence of the amplitude from phase space (which is, however, small) we can determine β from the known width (3.29) obtaining a reasonable value for β :

$$\beta \simeq 4.5 \quad (3.40)$$

It is a straightforward exercise to see that, with the values of ϵ and β just determined the new terms added to the leading $1/N$ Lagrangian do not alter the predictions for $\eta \rightarrow 3\pi$ (the effect here is less than 10%). For $\eta' \rightarrow 3\pi$ the effect is sizeable but the experimental upper bound (3.23) is still widely satisfied.

4. - IMPROVED EFFECTIVE LAGRANGIAN

In the previous sections we have shown that good agreement with the experimental data requires the introduction, in the effective Lagrangian discussed in Section 1, of additional terms that are non-leading in the combined chiral and large N limit.

In this section we want to give a systematic discussion of the additional terms up to the order $1/N^2$ and μ_1^2/N .

This effective Lagrangian L must be of the form

$$L = L_{inv} + L_M \quad (4.1)$$

where L_{inv} is a function of $q(x)$ and $U(x)$ that is invariant under $SU(3) \otimes SU(3)$, while L_M contains the effect of the breaking of $SU(3) \otimes SU(3)$ due to the mass matrix M .

If we require no more than two derivatives of the field U , L_{inv} can be written as follows

$$L_{inv} = L_{inv}^{(0)} + L_{inv}^{(1)} + L_{inv}^{(2)} \quad (4.2)$$

up to the order $1/N^2$, where

$$L_{inv}^{(0)} = \frac{1}{2} \text{Tr} (\partial_\mu U \partial_\mu U^\dagger) + \frac{N}{a F_\pi^2} q^2(x) \quad (4.3)$$

$$L_{inv}^{(1)} = \frac{i}{2} q(x) \text{Tr} [\log U - \log U^\dagger] + \frac{\alpha}{F_\pi^2} (\text{Tr} [U^\dagger \partial_\mu U])^2 \quad (4.4)$$

$$L_{inv}^{(2)} = \frac{\beta q^2}{F_\pi^4 a^2} \text{Tr} [\partial_\mu U \partial_\mu U^\dagger] + \frac{\gamma}{a^2 F_\pi^2} q^4(x) \quad (4.5)$$

It is easy to check that $L_{inv}^{(1)}$ gives terms of the right order in $1/N^1$ if α, β, γ are constants of $O(1)$.

Analogously, the requirements of absence of derivatives and of $U(2) \otimes U(2)$ chiral invariance in the limit $m_\pi^2 \rightarrow 0$, $m_K^2 \neq 0$ imply the following expression for L_M :

$$L_M = L_M^{(0)} + L_M^{(1)} \quad (4.6)$$

where

$$L_M^{(0)} = \frac{F_\pi}{2\sqrt{2}} \text{Tr} (MU + M^\dagger U^\dagger) \quad (4.7)$$

$$L_M^{(1)} = i q(x) \frac{\epsilon}{2\sqrt{2} a F_\pi} \text{Tr} [M(U - U^\dagger)] \quad (4.8)$$

with $L_M^{(1)} \sim O(m/N^1)$. Notice that an obvious term of the type $\text{Tr}(U - U^\dagger) \text{Tr}(M(U - U^\dagger))$ would violate the previously mentioned $U(2) \otimes U(2)$ invariance.

Since we are interested in evaluating processes involving at most one η' , we can neglect the term q^4 in (4.5). We can then eliminate the field $q(x)$ by means of its equation of motion and we are led to the following Lagrangian

$$\begin{aligned} L = & \frac{1}{2} \text{Tr} (\partial_\mu U \partial_\mu U^\dagger) + \frac{\alpha}{F_\pi^2} \text{Tr} [U^\dagger \partial_\mu U] \text{Tr} [U^\dagger \partial_\mu U] + \\ & + \frac{F_\pi}{2\sqrt{2}} \text{Tr} (MU + M^\dagger U^\dagger) + \frac{a F_\pi^2}{4} \left\{ \frac{1}{2} \text{Tr} [b_\mu U - b_\mu^\dagger U^\dagger] + \right. \\ & \left. + \frac{\epsilon}{2\sqrt{2} a F_\pi} \text{Tr} [M(U - U^\dagger)] \right\}^2 \cdot \left\{ 1 - \frac{\beta}{F_\pi^2 a N} \text{Tr} (\partial_\mu U \partial_\mu U^\dagger) \right\} \end{aligned} \quad (4.9)$$

where we have neglected terms containing more than two derivatives.

It is easy to see that Lagrangian (4.9) contains just the terms (3.11) and (3.35) that we have added in Section 3 to (3.1) in order to reproduce the pseudoscalar masses and the decay $\eta' \rightarrow \eta \pi \pi$. In addition we also have a term of order $1/N$ proportional to a new parameter α that breaks nonet symmetry. If we call F_1 and $F_8 = F_\pi$ the singlet and octet decay constant, respectively, one finds that

$$F_1/F_8 = \sqrt{1+3\alpha} \quad (4.10)$$

The previous Lagrangian (4.9) contains only the decays due to strong interactions. In order to compute the radiative decays of the pseudoscalar mesons one must couple (4.9) to electromagnetism. This can be easily done with the following substitution :

$$\partial_\mu U \rightarrow \partial_\mu U + ie [Q, U] \quad (4.11)$$

where Q is the charge operator. With our normalizations one has:

$$Q = \frac{1}{\sqrt{2}} \left(\lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right) \quad (4.12)$$

In addition, in analogy with the procedure followed in the case of the $U(1)$ anomaly, one must add a term that reproduces the electromagnetic anomaly. This is given by :

$$\mathcal{L}_{\text{e.m. anomaly}} = \frac{i}{2} G \text{Tr} [Q^2 (\log U - \log U^\dagger)] \quad (4.13)$$

where

$$G(x) = \frac{e^2 N}{32 \pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \quad (4.14)$$

and $F_{\mu\nu}$ is the field strength of the electromagnetic field. From (4.13) one can easily compute the decays of the π^0, η and η' into 2γ and their amplitude is given by :

$$\begin{aligned} A(\pi^0 \rightarrow 2\gamma) &= \frac{e^2 N}{12 \pi^2 F_\pi} I \\ A(\eta \rightarrow 2\gamma) &= \frac{e^2 N}{12 \pi^2 F_\pi} \frac{1}{\sqrt{3}} (\cos \varphi + 2\sqrt{2} \sin \varphi) I \\ A(\eta' \rightarrow 2\gamma) &= \frac{e^2 N}{12 \pi^2 F_\pi} \frac{1}{\sqrt{3}} (2\sqrt{2} \cos \varphi - \sin \varphi) I \end{aligned} \quad (4.15)$$

where $I = \epsilon_{\mu\nu\rho\sigma} k_1^\mu \epsilon_{1\nu} k_2^\rho \epsilon_{2\sigma}$, which agree with the standard current algebra result ²²⁾.

Notice that the correct treatment of the $U(1)$ axial anomaly affects the amplitudes for the radiative decays of η and η' (which are a consequence of the electromagnetic anomaly) only by fixing the mixing angle φ . Lagrangian (4.13) can be easily extended to a non-Abelian case. This has been done in Ref. 6).

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