Ref.TH.2902-CERN

JET CALCULUS BEYOND LEADING LOGARITHMS

J. Kalinowski ^{*)}, ⁺⁾ and K. Konishi
CERN -- Geneva

and

T. R. Taylor ⁺⁾
Institute of Theoretical Physics
Warsaw University

A B S T R A C T

It is shown that the evolution of hadronic jets produced in hard processes can be studied in terms of a simple parton branching picture, beyond the leading log approximation of QCD. The jet calculus is generalized to any given order of logs (but always to all orders of α_s). We discuss the general structure of the formalism. Universality of jet evolution is discussed. We consider also a jet calorimetry measure and the multiplicity distribution of final states in a form which allows a systematic improvement of approximation. To the next-to-leading order, we prove the finiteness and elucidate the scheme-dependence of parton subprocess probabilities. The physical inclusive cross-section is shown to be scheme-independent : next-to-leading results for $e^+e^- \rightarrow q(\text{non-singlet})$ X agree with those by Curci and others.

^{*)} Present address : Institute of Theoretical Physics, Warsaw University, Warsaw, Poland.

⁺⁾ Partially supported by the Polish Ministry of Sciences, Education and Technology, Problem MR.I.7.

1. INTRODUCTION

In the leading log approximation (LLA) of perturbative Quantum Chromodynamics (QCD), the evolution of jets in hard processes is described by a compact parton-probabilistic algorithm (jet calculus) [1]. This algorithm, besides providing a powerful tool for resumming large radiative corrections and thus proving to be a useful aid for finding numerous quantitative tests of QCD, has led recently via some generalizations, to further insight into the colour and space-time evolution of jets [2].

In this paper we wish to point out that the appearance of a simple probabilistic algorithm for parton branching is not limited to LLA, but is a general feature of contributions associated with collinear mass singularities. Our starting point is the observation that the factorization theorem [3] of mass singularities which has been proved to all orders of α_s and of logs, has an implication wider than generally acknowledged. Proof of the QCD improved parton model is not the only outcome: the diagrammatic derivation of the factorization theorem also implies an important property of unobserved (i.e. summed-over) particles. The universal (= process-independent) fragmentation or structure function is determined essentially by the emission of partons in the direction of the detected (initial) hadron. The properties of these jets of partons are also process-independent and are calculable in terms of the same set of diagrams. Such a knowledge should be useful in understanding the property of actual jets consisting of hadrons.

The aim of this paper is to show how the diagrammatic study of production of jets leads to the jet calculus, generalized to any given order of logs (and always to all orders of α_s). It is shown how a stochastic branching process analogy and the parton picture survive beyond the leading log approximation. The resulting simple formalism can be taken as a basis of improved calculations or of a model-construction of jet evolution.

In this paper we mainly restrict ourselves to the general aspects of jet evolution, leaving a more extensive analysis in the next-to-leading order to a forthcoming work [4]. However, we discuss explicitly, in the next-to-leading

order, some subtle questions about the scheme dependence and finiteness of the elementary parton decay probabilities. We also compare the results for $e^+e^- \rightarrow q$ (non singlet) + X with those of Ref. [5], and show that the scheme dependence cancels in the physical cross-section.

In Section 2, the structure of final states in the e^+e^- annihilation process (as an illustration) is studied with the help of off-shell unitarity equations written in axial gauge QCD. Based on this, in Section 3 we construct the jet calculus which generalizes the one in LLA to any order. Associated probability interpretation and parton branching process analogy are emphasized and a Kolmogorov equation [6] for the generating functional is derived. We consider a jet calorimetry measure (jet energy spread) and the multiplicity distributions in QCD, in a form which allows a systematic improvement of approximation. Section 4 is devoted to the proof of finiteness and the scheme dependence of relevant jet-calculus vertices (and the generalized Altarelli-Parisi kernels), to the next-to-leading order. We make some comments on naive "on-shell" and "off-shell" infrared regularization methods.

2. GENERAL STRUCTURE OF FINAL STATES

2.1 Off-shell unitarity equations and mass singularities

A particularly elegant way to study jet evolution beyond leading log approximation, is to combine the off-shell unitarity equation of Polyakov [7], with the use of an axial gauge in QCD. For illustration we shall consider the e^+e^- annihilation process, although a similar analysis can be done for any hard process. To the lowest order in the electromagnetic coupling constant, the total hadronic cross-section is given by the discontinuity

$$\sigma_t \propto \frac{\alpha^2}{(Q^2)^2} \text{Disc}_{Q^2} \pi(Q^2) = \frac{\alpha^2}{Q^2} \text{Disc}_{Q^2} \tilde{\pi}(Q^2) , \quad (1)$$

where Q is the total energy in the c.m. system of e^+e^- . Following Polyakov, but in QCD in an axial gauge, we write $\text{Disc}_{Q^2} \tilde{\pi}(Q^2)$ as

$$\text{Disc}_{Q^2} \tilde{\pi}(Q^2) = \text{Disc}_{Q^2} \left\{ \dots \circlearrowleft + \dots \circlearrowright + \dots \circlearrowright + \dots \right\} \quad (2)$$

where a crossed line is the imaginary part of the exact propagator of quark or gluon, and the dotted line is a virtual photon. The (uncut) propagators are also exact ones and the vertices are one-particle irreducible (1pI) ones. Equation (2) can be obtained from the ordinary unitarity equation with on-shell cut lines [with $\theta(k_0)\delta(k^2)$], by first decomposing the amplitudes on both sides of the cut into one-particle irreducible (or proper) vertices, inserting them back into the original equation, and then recombining to get the discontinuity of exact propagators.

Each crossed line is a jet of particles: in an axial gauge the entire leading log contribution (and all 2-jet events) come from the first term of Eq. (2). Three-jet events get a contribution from all of the terms shown (and some others).

We shall now perform the following three operations on Eq. (2):

Operation A

The expressions in Eq. (2) imply the integration over the mass of each crossed line. Let us introduce an arbitrary mass scale $M_0 > 0$. For each crossed line, if the corresponding mass $\sqrt{q^2}$ is larger than M_0 , then we write the propagator as

$$\begin{aligned} \text{---} \times \text{---} &= \text{Disc}_{q^2} G(q^2) \mid \sqrt{q^2} \geq M_0 \\ &= \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \dots \end{aligned} \quad (3)$$

in terms of cut exact propagators with smaller masses. Equation (3) is a strict analogy of Eq. (2) for $\text{Disc } \tilde{\pi}$. If $\sqrt{q^2} < M_0$, the line is left as it is. In each of the crossed lines appearing in Eq. (3) the same step should be repeated: if a propagator carries mass larger than M_0 it should be replaced by an expansion of the form of Eq. (3). This operation is repeated in all crossed lines in all terms, until all crossed lines have masses less than M_0 . We obtain in this way a set of diagrams more complicated than the ones we started with. [By dividing the integration region over mass, many graphs are produced from each of the terms in Eq. (2).]

The aim of the next two operations is to make manifest the mass singularity structure of each diagram.

Operation B

Take a graph obtained after the operation A. We define a "minimal 2pI subgraph" as a subgraph

- i) which contains crossed lines;
- ii) which is two-particle irreducible (2pI), that is, which cannot be separated into two disconnected parts by cutting two uncrossed lines^{*)};
- iii) which is connected to the rest of diagram by just two lines (necessarily of equal momenta).

A minimal 2pI subgraph looks like each term of the right-hand side of Eq. (3). On the contrary, subgraphs such as

(4)

are not minimal 2pI subgraphs: (a), (b), and (c) violate the condition (ii); (d) does not satisfy (iii). [Two-particle reducible graphs (2pR) such as (a)-(c), never appear in Eq. (2) nor at each step of operation A in Eq. (3). They do, however, when the operation A is iterated.]

Contract now all the vertices in minimal 2pI subgraphs into two effective vertices, one on the left and one on the right of the cut (crossed lines), so that

(5)

(this formal operation defines the right-hand sides). After this operation is done in all minimal 2pI subgraphs, we proceed to larger subgraphs, always containing

^{*)} For the sake of this definition, the original crossed lines are regarded as connected.

the crossed lines. We identify all subgraphs that *would be minimal 2pI subgraphs, were it not for the ones they already contain*. In other words, in these graphs, only (2) is violated by the presence of minimal 2pI subgraphs they necessarily contain. Graphs (4a) and (4b) are examples of these. Graph (4c) is not. Contract again all the vertices as in Eq. (5), leaving however the effective vertices of the minimal 2pI part intact. Diagrams obtained at this stage will look "doubly nested".

The operation can now be repeated to larger and larger subgraphs, by similar rules, until all the original vertices (including the electromagnetic vertex) are contracted and absorbed into effective ones. Diagrams obtained this way typically look like Fig. 1b (the change from the hatched to black vertices is due to operation C below. The structure of graphs is however the same).

The advantage of the operation B is that, in axial gauges, a collinear mass singularity occurs only if there are two Feynman denominators of the same momenta (appearing as a complex conjugate pair): a 2pI subgraph itself is free of collinear divergences for fixed total momentum q . Each of the graphs in Eq. (5) can give rise to at most *one* collinear divergence upon integration over q^2 .

A 2pR subgraph, however, yields finite contributions as well as collinear singularities. This observation leads us to the third step:

Operation C

Diagrams obtained above are formally diagonal: each off-shell propagator not absorbed into effective vertices, appears in a complex conjugate pair. We insert now, for each pair of uncut propagators, a projection operator [8],

$$1 = (1 - \mathcal{P}) + \mathcal{P} \quad , \quad (6)$$

where $\mathcal{P}$ extracts only the part leading to collinear singularities upon integration over q^2 (q^μ = the momentum of the pair of propagators)*). $\mathcal{P}$ contains the projection on physical spin states. $\mathcal{P}$ also contains the operation of subtracting non-singular

*) In order to define $\mathcal{P}$, we first integrate over all variables except for invariant mass squared q^2 and longitudinal momentum fractions. Resulting effective vertices will be coupled through these two variables.

contributions originating from the kinematical phase-space limitations. (See Ref. [4] for more details and explicit construction of $\mathcal{F}$.)

Accordingly we shall contract all the vertices connected by the $1-\mathcal{F}$ part of the propagators into a new effective vertex, e.g.

The result of the operation C is that all the effective vertices are replaced by new ones, and all lines by the $\mathcal{F}$ part of the original off-shell propagators.

Integration over internal masses is done in the simply ordered region,

$$\dots q_1^2 \left\{ \begin{array}{l} \geq q_2^2 \\ \geq q_3^2 \end{array} \right. \left\{ \begin{array}{l} \geq q_4^2 \\ \geq q_5^2 \\ \geq q_6^2 \\ \geq q_7^2 \\ \geq q_8^2 \end{array} \right. \dots \geq M_0^2 \quad \text{for} \quad \dots 1 \dots 2 \dots 3 \dots 4 \dots 5 \dots 6 \dots 7 \dots 8 \dots \quad (8)$$

along the branching chain. By definition, integration over each q_i^2 generates a collinear singularity.

After the operations A to C, we find a sum of diagrams each of which is of the form of square of a tree graph as is shown in Fig. 1b. The operations A to C are of purely formal nature, and the sum of these (squared) tree graphs is equivalent to the right-hand side of the unitarity equation, Eq. (2). The last task required is to relate diagrammatic contributions obtained above to various differential cross-sections in terms of observed hadrons or jets.

2.2 One-hadron inclusive production and QCD improved parton model

This case has been considered in Ref. [3] and the factorization theorem has been proved for it. The cross-section for one-parton production, $e^+e^- \rightarrow a(x) + X$, where $x = 2p \cdot Q/Q^2$ is the energy fraction of the parton a in the c.m. system, is of the form (in the notation of Ref. [8])

$$d\sigma_a(p, Q^2) = \sum_b \int_x^1 dz \Gamma_{ab}(z, M^2) d\sigma_b(p/z; Q^2; M^2) \quad (9)$$

where

$$\Gamma_{ab}(z, M^2) = \sum \cdot z^2 \int d^4k \delta(z - \frac{n \cdot p}{n \cdot k}) [1 - L'']^{-1} \quad (10)$$

(n^2 is a vector such that $n^2 = 0$, $n \cdot k_T = 0$, $n \cdot p \neq 0$). Γ_{ab} contains all the mass singularities at $p^2 = 0$. $d\sigma_b$ is a mass-singularity-free "short-distance" cross-section. M^2 is an arbitrary mass scale introduced in order to write $d\sigma(p, Q^2)$ in an explicitly factorized form above. $(1 - L'')^{-1}$ represents a ladder-like contribution of Fig. 2, and Z the correction on the external line.

The factorization theorem allows the introduction of another mass scale M_0 , and we can write $\Gamma_{ab}(x, M^2)$ as

$$\Gamma_{ab}(x, M^2) = \sum_c \int_x^1 \frac{dz}{z} \Gamma_{ac}(x/z, M_0^2) D_{c,b}(M^2; M_0^2; z). \quad (11)$$

$\Gamma_{ac}(x/z, M_0^2)$ contains all mass singularities, and $D_{c,b}$ is a finite evolution function between M^2 and M_0^2 (M^2 being the upper limit of the parent parton mass squared).

One *hadron* inclusive cross-section for $e^+e^- \rightarrow h(x) + X$ is obtained by convoluting in x Eq. (9) with a non-perturbative fragmentation function. On convolution, the singular function $\Gamma_{ac}(x/z, M_0^2)$ of Eq. (11) is absorbed into a "renormalized" fragmentation function which is to be determined empirically.

Independence of $d\sigma_a(p; Q^2)$ on M^2 ,

$$M^2 (d/dM^2) d\sigma_a(p; Q^2) = 0 \quad (12)$$

leads to an integro-differential equation

$$M^2 \frac{d}{dM^2} \Gamma_{ab}(z, M^2) = \sum_c \int_z^1 \frac{du}{u} \Gamma_{ac}(z/u, M^2) \Pi_{c,b}(M^2, u). \quad (13)$$

This equation defines $\Pi_{c,b}(M^2, u)$, which is a finite function of M^2 and u .

By using Eq. (11), Eq. (13) can be translated into one for D:

$$M^2 \frac{d}{dM^2} D_{b,a}(M^2, M_0^2; x) = \sum_c \int_x^1 \frac{dz}{z} D_{b,c}(M^2, M_0^2; x/z) \Pi_{c,a}(M^2, z) \quad (14)$$

This is a generalized Altarelli-Parisi equation [9].

We have so far not mentioned the renormalization mass scale μ . In fact, the "physical" cross-section $d\sigma_a(p, Q^2)$ is independent both of μ and M . This of course corresponds to the presence of two arbitrary scales -- one for the renormalization of α_s , one for the renormalization of operator matrix element, in OPE. Usually, these scales are taken to be the same.

For the study of jets, however, the distinction is crucial. M is related to the jet width -- it is the upper limit of invariant mass of partons forming the jet. In other words, M (similarly M_0) is a physical quantity. On the other hand, μ is a scale of renormalization of α_s : any physical quantity should be independent of it.

This means that $d\sigma_b(p/z; Q^2; M^2)$, $D_{b,a}(M^2; M_0^2; z)$, and $\Pi_{b,a}(M^2; z)$, are all separately independent of μ^2 . For dimensionless quantities, this implies that μ^2 enters only through the renormalization-group-invariant coupling, $\alpha_s(Q^2)$, $\alpha_s(M^2)$, etc.

In particular, $\Pi_{b,a}(M^2, z)$, being dimensionless and dependent on M^2 , can only be a function of $\alpha_s(M^2)$,

$$\begin{aligned} \Pi_{b,a}(M^2, x) &\equiv \Pi_{b,a}[\alpha_s(M^2); x] \\ &= \frac{\alpha_s(M^2)}{2\pi} P_{ba}^{AP}(x) + \left(\frac{\alpha_s(M^2)}{2\pi}\right)^2 P_{ba}^{NL}(x) + \dots \end{aligned} \quad (15)$$

where P^{AP} and P^{NL} are the usual Altarelli-Parisi and the next-to-leading parton decay probabilities, respectively.

We have explicitly checked, to lowest non-trivial order, the appearance of the renormalization-group-invariant $\alpha_s(M^2)$. For an Abelian subset of diagrams, it is possible to deduce, by using an argument which is an extension of the Appendix of Amati et al. [3], that

$$\int \frac{d^2 q^2}{q^2} [\alpha_s(\mu^2)]^m \xrightarrow{\text{dressing skeletons}} \int \frac{d^2 q^2}{q^2} [\alpha_s(q^2)]^m$$

in the one-loop approximation for the β function.

Choosing $M^2 = Q^2$ and formally solving Eq. (14), the n^{th} moment of the single-parton inclusive cross-section can be written in a familiar form,

$$\begin{aligned} \int dx x^n (d\sigma/dx) &= \sum_i C_{\gamma \rightarrow i}(\alpha_s(Q^2); 1; n) D_{a,i}(Q^2, M^2; n) \\ &= \sum_i C_i(\alpha_s(Q^2); 1; n) \left[\exp \int_{M_0^2}^{Q^2} \frac{d^2 q^2}{q^2} \left\{ \frac{\alpha_s(q^2)}{2\pi} A_n^{(0)} + \left(\frac{\alpha_s(q^2)}{2\pi} \right)^2 A_n^{(1)} + \dots \right\} \right] \end{aligned} \quad (16)$$

where $(A_n^{(i)})_{b,a}$ is the n^{th} moment of the coefficient of $(\alpha_s/2\pi)^{i+1}$ in $\Pi_{b,a}(\alpha_s; x)$, and $C_{\gamma \rightarrow i}(\alpha_s(Q^2); Q^2/M^2; z)$ arises from $d\sigma_b$ in Eq. (9) after angular integrations.

In Eq. (10), M^2 is the upper limit of integration over internal mass which was introduced for the purpose of mass-singularity factorization. This allows us to make a connection with the construction of operations A to C. Let us write

$$\Pi_{b,a}[\alpha_s(M^2); x] = \hat{\Pi}_{b,a}[\alpha_s(M^2); x] + \delta(1-x) \delta_{ba} S, \quad (17)$$

then $\hat{\Pi}_{b,a}$ is given by the sum

$$\hat{\Pi}_{b,a}[\alpha_s(M^2); x] = \begin{array}{c} a \\ | \\ \bullet \text{---} \bullet \text{---} \bullet \\ | \\ b \end{array} + \begin{array}{c} a \\ | \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ | \\ b \end{array} + \begin{array}{c} a \\ | \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ | \\ b \end{array} + \dots, \quad (18)$$

where vertices are defined before, and the full vertex corrections and half of the self-energy corrections on the lines a and b are included. The right-hand side of Eq. (18) is finite at fixed M^2 , and can be computed by setting $p_b^2 = 0$.

The second term of Eq. (17) can be determined by using the charge sum rule to be discussed in Section (3.2), Eq. (31) without direct computation.

3. JET CALCULUS BEYOND LEADING LOGS

3.1 General construction*)

Let us now consider the distribution of particles $a_1(x_1), \dots, a_m(x_m)$ inside a jet, defined to be a group of particles having total invariant mass up to M ,

*) The analysis from now on is independent of the particular hard process chosen.

where $M \ll Q$ [but $M^2 \sim O(Q^2)$]. For fast particles, $x_i \sim O(1)$, this means that they are produced within a cone of opening angle δ , $\delta \approx M/Q$.

We shall discuss mainly multiparton distributions; hadrons will be introduced at the end, in the manner described in Ref. [1] (see Section 3.7).

As was done for LLA [1], we start from branching-process type graphs obtained in Section 2 (such as Fig. 1b), and simply select m partons $a_1(x_1), \dots, a_m(x_m)$ out of the final crossed lines, where x_i 's denote the longitudinal momentum fractions with respect to the total jet, and a_i 's the parton species. We shall assume that any pair of partons from $(a_1, \dots, a_m)$ form an invariant mass larger than M_0 ; a pair of partons with invariant mass less than M_0 are not regarded as distinct ones.

Summing over all the choices of selecting m partons out of a given final state, integrating over all unobserved partons, and finally summing over different tree diagrams, one finds a set of effective tree graphs. Such effective graphs for $j \rightarrow a_1 + a_2 + a_3 + X$ ($m = 3$) are illustrated in Fig. 3. The "propagator" of our (generalized) jet calculus rules is just the evolution function D ,

$$a \xrightarrow{q_1^2} b(x) \xleftarrow{q_2^2} \equiv D_{b,a}(q_1^2, q_2^2; x) \quad (19)$$

The vertex

$$V_{i \rightarrow \{a\}}^{(l)}(\alpha_s(q^2); x_i) \sim i \xrightarrow{q^2} \begin{array}{c} a_1(x_1) \\ \vdots \\ a_l(x_l) \end{array} \quad (20)$$

describing the *inclusive* process $i \rightarrow a_1 a_2 \dots a_l + X$, is defined in a way analogous to the hard subprocess (short-distance) cross-section. As in LLA, the iteration of all possible insertions of one-parton inclusive vertices $\Pi_{a,i}[\alpha_s(q^2), x] \equiv V_{i \rightarrow a}^{(1)}[\alpha_s(q^2), x]$ in a (generalized) ladder, leads to the propagator $D_{b,a}$. In Eq. (20), q^2 plays the role of large momentum (similar to Q^2 of the virtual photon). All the collinear mass singularities related to the final partons, have been factorized into D functions. $V^{(l)}$ is thus free of mass singularities and can be computed by setting the final external lines on-mass-shell (with $m^2 = 0$). Soft

singularities are absent^{*}). These statements have been checked explicitly in the next-to-leading approximation, as will be discussed in Section 4.1. The method of calculation for $V^{(\ell)}$ is thus identical to that for hard subprocess cross-sections.

The separation of the collinear singularity is not unique, and hence $V^{(m)}$ and D separately depend on the method of infrared regularization (sections 3.3 and 4.2). For instance, to next-to-leading order, $V_{Q \rightarrow ab}^{(2)}$ and $V_{Q \rightarrow abc}^{(3)}$ get the contributions from the diagrams shown in Fig. 4.

The q^2 dependence of $V^{(m)}$ occurs, as in $\Pi[\alpha_s(q^2), x]$, only through $\alpha_s(q^2)$. We shall consider therefore an approximation scheme in which all $V^{(m)}$ and Π are computed up to a given order of $\alpha_s(q^2)$.

To a given, say the L^{th} order in $\alpha_s(q^2)$, there exist obviously only a finite number of vertex functions $V^{(m)}[\alpha_s(q^2); x_1 \dots x_m]$ with m limited by

$$m \leq L + 1 \quad . \quad (21)$$

At a given order of approximation, then, the multiparton spectra are described by branching process type graphs such as Fig. 3 with a finite number of vertex functions $V^{(m)}$, and the evolution function $D_{a,i}(q_1^2; q_2^2; x)$. Obviously, there is an iterative structure in the whole process; the same vertices $V^{(m)}$ and evolution function $D_{a,i}$ appear everywhere, as a consequence of scale invariance of the theory. The result is that the generating functionals for the jet, $i \rightarrow a_1 + \dots + a_m + X$, satisfy a simple (Kolmogorov type [6]) integro-differential equation. Define the generating functional G^i for an i -jet as

$$G^i(u; \phi; t) \equiv \sum_{m=1}^{\infty} \sum_{\eta_i + \eta_j = m} \left[\frac{1}{\eta_i! \eta_j!} \right] \int \frac{\pi du_i}{u^{m-1}} \cdot D_{\{a_j\},i}^{(m)}(t; \frac{u_i}{u}) \quad (22) \\ \times \pi \phi_{a_j}(u_j) \delta(\sum u_j - u) \quad ,$$

where $D_{\{a_j\},i}^{(m)}$ is the m parton inclusive distribution for $i \rightarrow a_1 + \dots + a_m + X$; $\phi_a(u)$ an arbitrary sink function; u = the total energy; and $t = \log M^2$, then G^i satisfies the equation

^{*}) The general argument for Bloch-Nordsieck cancellation applies only for $(1/\sigma)d\sigma/dx_1 \dots dx_m$. The proof of Ref. [8] can be applied here to show that V and P are separately free of soft singularities.

$$\begin{aligned}
 \frac{d}{dt} G^i(u; \phi; t) &= \sum_{\ell=2}^L \frac{1}{\ell!} \sum_{a_j} \int \frac{\pi du_j}{u^{\ell-1}} V_{i \rightarrow \{a\}}^{(\ell)}(t; u_j/u) \\
 &\quad \cdot \delta(\sum_j u_j - u) \cdot \prod_{j=1}^{\ell} G^{a_j}(u_j; \phi; t) + \\
 &\quad + \int_0^u \frac{dx}{u} \pi_{i,i}(t, \frac{x}{u}) G^i(x; \phi; t)
 \end{aligned} \tag{23}$$

This equation can be obtained as follows. The only $t(M^2)$ dependence of G^i occurs from the upper limit of the integration over q_1^2 and from the argument of the first evolution function: G^i is of the form (see Fig. 5),

$$\begin{aligned}
 G^i &= \int_{M_0^2}^{M^2} \frac{dq_1^2}{q_1^2} D_{a,i}(M^2; q_1^2; x) \cdot [V_{a \rightarrow b_1 b_2}^{(2)} D_{c_1, b_1} D_{c_2, b_2} \dots \\
 &\quad + V_{a \rightarrow b_1 b_2 b_3}^{(3)} D_{c_1, b_1} D_{c_2, b_2} D_{c_3, b_3} \dots \\
 &\quad + \dots]
 \end{aligned} \tag{24}$$

Differentiation with respect to $t = \log M^2$ leads immediately to Eq. (23), after use of

$$D_{a,i}(M^2, M^2; x) = \delta_{a,i} \delta(1-x) \tag{25}$$

[which follows from Eq. (11)] and the generalized Altarelli-Parisi equation, Eq. (14).

Equations for $D_{\{a\},i}$ are obtained by taking functional derivatives of Eq. (24) with respect to ϕ and setting $\phi = 0$ [10]. The formal solution is the generalized jet calculus rules for $D_{\{a\},i}^{(m)}$:

- i) Write all possible tree graphs with m final partons, allowing vertices $V^{(n)}$ with $2 \leq n \leq L$.
- ii) Write $V^{(n)}[\alpha_s(q^2); x_i]$ for a vertex and $D_{b,a}(q_1^2, q_2^2, x)$ for a propagator.
- iii) Integrals over all q^2 's. q^2 's are simply ordered along the tree from the initial towards the final lines.
- iv) Sum over all internal indices, and over all tree graphs.

In LLA, the above rules reduce to those of Ref. [1], which have been exploited for finding various quantitative tests of QCD.

It is to be noticed that all the Q^2 dependence of the jet evolution (generating functional G) comes through $M^2 = \delta^2 Q^2$ (and not Q^2 and δ separately).

Distributions $D_{\{a_j\},i}(M^2; \{x_j\})$ computed as above, should be convoluted in x with a "hard subprocess cross-section",

$$C_{\gamma^* \rightarrow i}(\alpha_s(Q^2); Q^2/M^2; x)$$

(or an analogous function in a general hard process), which contains all the contributions from 2pI graphs (which have no mass singularities) and 2pR graphs with intermediate masses larger than M and/or with intermediate lines with $1 - \beta$.

Note on the M_0 dependence

As already mentioned, the M_0 -dependence brought in by the operation A of Section 2 cancels completely (as it should) for unobserved particles, in a summation such as

$$\ell \rightarrow \times + \text{loop} = \text{loop}_{(q^2 < M_0^2)} + \text{loop}_{(q^2 \geq M_0^2)} + \dots, \quad (26)$$

where the right-hand side is written in terms of the ordinary, on-shell external lines with $\theta(k_0)\delta(k^2)$.

Consequently, the M_0 -dependence only remains as the scale for distinguishing two-parton from one-parton states (i.e. only those having invariant mass larger than M_0 are counted as two-parton systems), and as the lower limit of the evolution function $D_{ba}(M^2; M_0^2; x)$ -- the evolution below M_0 is lumped into a non-perturbative fragmentation function.

3.2 Probabilistic interpretation

We have already (somewhat carelessly) referred to V as the parton subprocess probability and mentioned a branching process analogy. Appearance of the probabilistic picture in LLA is well understood, but is more subtle beyond it. That it is possible to choose a convention such that the probability interpretation is exact to all order of logs and α_s is a consequence of the Kinoshita-Lee-Nauenberg (KLN) theorem [11] and of the factorization theorem [3].

To illustrate the point, let us consider a simple ϕ_6^3 diagram shown in Fig. 6a. Cuts indicated in Fig. 6a are some of the contributions to the total decay rate of the quantum a . The sum of the cuts A , A' , and B , according to KLN theorem, is free of collinear singularities related to the lines b , d , and e . Thus the order α_s correction to the total decay probability of b is finite. This finite correction can be absorbed by definition into the subprocess probability for three-body decay (Fig. 6b) or two-body decay (Fig. 6c), according to whether it comes from the cut B or $A + A'$, respectively. After this redefinition, the probability for the parton b to do something contains only singular contributions which sum up to 1 [$O(\alpha_s)$ corrections cancel exactly], for the total probability. Higher order corrections can be treated similarly: if the bubble d - e in Fig. 6a is replaced by a sum of higher order graphs, all *finite* contributions from various cuts can be absorbed into subprocess probabilities (such as Fig. 6b, 6c or other types). After this reshuffling, the total probability for the parton b to do anything will be exactly unity to all orders in α_s , corresponding to the complete cancellation of collinear singularities.

Coming back to QCD, consider the n^{th} moment of Eq. (9) for $e^+e^- \rightarrow a(x) + X$, with respect to x . Taking $n = 1$ and summing over a , one finds

$$\sum_a \int_0^1 d\sigma_a \cdot x = \sum_{a,b} \Gamma_{a,b}^{(1)}(M^2) \cdot C_b^{(1)}(\alpha_s(Q^2); Q^2/M^2) \quad (27)$$

where $d\sigma_b(p; Q^2; M^2) \equiv C_b[\alpha_s(Q^2); Q^2/M^2; x] dx$ after angular integrations, and superscripts are used to indicate the moment taken. $\sum_a \Gamma_{a,b}^{(1)}(M^2)$ corresponds to the total decay for $b \rightarrow \text{anything}$, and by KLN theorem is a finite number (depending on M^2). This number can be absorbed in $C_b^{(1)}[\alpha_s(Q^2); Q^2/M^2]$ by a redefinition of Γ and C , and thus one gets

$$\sum_a \Gamma_{a,b}^{(1)}(M^2) = 1 \quad (28)$$

to all orders of α_s . By analogous arguments, using the charge sum rule, one finds (e_a = charge of a)

$$\sum_a e_a \Gamma_{a,b}^{(0)}(M^2) = e_b \quad (29)$$

Let us recall now that, because of independence of Eq. (9) of M^2 , $\Gamma_{ab}(z, M^2)$ satisfies Eq. (13),

$$M^2 \frac{d}{dM^2} \Gamma_{a,b}(z, M^2) = \sum_c \int_z^1 \frac{du}{u} \Gamma_{a,c}\left(\frac{z}{u}, M^2\right) \Pi_{c,b}(M^2, u). \quad (13)$$

It is then easy to find^{*}), by combining Eqs. (28), (29), and (13), that

$$\sum_b \int_0^1 dx \times \Pi_{b,a}(M^2, x) = 0 \quad (30)$$

and

$$\sum_b e_b \int_0^1 dx \Pi_{b,a}(M^2, x) = 0. \quad (31)$$

Equations (30) and (31) express the conservation of energy and charge during the parton branching. It follows from Eqs. (30) and (31) that D_{ba} satisfies:

$$\begin{aligned} \sum_b \int_0^1 dx \times D_{b,a}(M^2, M_0^2; x) &= 1 \\ \sum_b e_b \int_0^1 dx D_{b,a}(M^2, M_0^2; x) &= e_a \end{aligned} \quad (32)$$

[the normalization follows from $D_{b,a}(M^2; M^2; x) = \delta_{ba} \delta(1-x)$, which in turn results from Eq. (11)].

The vertex functions $V_{j \rightarrow b_1 + \dots + b_m}^{(m)}$ have been defined as short-distance cross-sections, by factorizing and dividing out all collinear singularities associated with final partons $b_1, b_2, \dots, b_m$. $V_{j \rightarrow b_1 + \dots + b_m}^{(m)}$ are then computed by summing over exclusive processes (given by various Feynman diagrams), and hence, by construction, satisfy the inclusive sum rules among themselves,

$$\begin{aligned} \sum_{b_{m+1}} \int dx_{m+1} x_{m+1} V_{j \rightarrow b_1 + \dots + b_{m+1}}^{(m+1)} [\alpha_s(q^2), x_1 \dots x_{m+1}] &= \left(1 - \sum_{i=1}^m x_i\right) V_{j \rightarrow b_1 + \dots + b_m}^{(m)} [\alpha_s(q^2), x_1 \dots x_m], \quad (33) \\ \Pi_{b,a} [\alpha_s(q^2); x] &= \Pi_{ba}(q^2, x) \equiv V_{a \rightarrow b}^{(1)} [\alpha_s(q^2); x] \end{aligned}$$

The set of equations (32) and (33) show that the parton branching described by the jet calculus can be formally regarded as a stochastic process.

^{*}) Γ_{ab} is an infrared singular function. We regularize Γ by some method (e.g. dimensional regularization), take appropriate moment and sum in Eq. (13), use Eqs. (28) and (29) and then remove the regularization.

3.3 Regularization and renormalization scheme dependence

Our prescription [Eq. (15), Eq. (20)] for $V_{i \rightarrow \{a\}}$ consisted in computing these quantities at a fixed value q_i^2 , as a coefficient of $1/q^2$. They are "hard subprocess" cross-sections with q^2 playing the role of the large mass scale. Collinear singularities of a subsystem either cancel (for unobserved systems) or are explicitly subtracted (for iterated subprocesses); soft singularities cancel in the inclusive sum.

It is well-known that in the operator product expansion (OPE) approach to the deep inelastic structure functions, the anomalous dimensions and the coefficient function computed at $\mu^2 = Q^2$, are separately dependent on the renormalization scheme, beyond LLA [12]. Only the product of the two is a physical quantity and independent of the convention. The equivalent in our diagrammatic mass singularity approach, is that Π , V , and C (hard subprocess cross-section containing no mass singularities), separately depend on the way the collinear and soft singularities are regularized. This scheme dependence must cancel in the physical combination,

$$\sigma \sim \sum_i C_i(\alpha_s(Q^2); \frac{Q^2}{M^2}; x) \otimes D_{\{a\},i}(M^2, M_0^2; \{x\}) \quad (34)$$

To the next-to-leading order, this means that the quantity

$$c_n = \frac{1}{2\pi b} A_n^{(1)} \quad (35)$$

where c_n and $A_n^{(1)}$ are the coefficient of $(\alpha_s/2\pi)$ in $C[\alpha_s(Q^2); 1; n]$ and of $(\alpha_s/2\pi)^2$ in $\Pi[\alpha_s(M^2), n]$, respectively, must be scheme-independent. The combination (35) corresponds to the renormalization prescription independent object

$$B_n = \frac{1}{2} \frac{\gamma_1^n}{\beta_0}$$

(in the notation of Bardeen et al. [13]) in OPE.

In Section 4.3 we shall illustrate this scheme dependence by explicitly comparing our scheme with the one in Ref. [5] to the next-to-leading order. We find that the scheme-dependent piece contains a term behaving as

$$\sim \alpha_s(Q^2) \log^2 n$$

in the large moment of single parton inclusive cross-section. The combination of Eq. (34) will however be found to be scheme-independent.

Another source of non-uniqueness of the answer beyond LLA is the lack of a natural choice of α_s in QCD, in contrast to QED. Even the physical cross-section, Eq. (34), depends on the renormalization prescription, i.e. on the definition of α_s (MS, $\overline{\text{MS}}$, MOM, etc.). Several methods to maximize the usefulness of perturbative QCD have been proposed [13], and a similar idea should be supplemented before a formula such as Eq. (34) is seriously compared with the data.

3.4 Jet energy spread (jet calorimetry, partonometry)*)

As a first application, let us consider the following calorimetry measurement (this has been considered in Ref. [1] and the leading log formula has been obtained). Cover the whole solid angle of e^+e^- (for definiteness) event by calorimeters of half-opening angle δ . Let E_j be the total hadronic energy deposited in the j^{th} calorimeter. Define

$$x_j = E_j / \sum_k E_k = E_j / Q \quad \left(\sum x_j = 1 \right)$$

The quantity

$$C_{e^+e^-}(Q^2; \delta^2; N) \equiv \left\langle \sum_j x_j^N \right\rangle \quad (36)$$

is the N^{th} moment of the inclusive cross-section of a mini-jet depositing an energy fraction x in a cone of half-opening angle δ . The inclusive production of a parton having virtual mass larger than δQ , can be expressed by the evolution function $D(Q^2; \delta^2 Q^2; x)$ and the hard subprocess cross-section $C_{\gamma^* \rightarrow i}[\alpha_s(Q^2); 1; z]$ in the usual form. On the other hand, integrations with precise cone- and energy-constraints are needed for the decay of a parton with virtual mass less than δQ , for a kinematical reason.

As a result we get, to all orders of α_s and logs, a simple formula

$$C_{e^+e^-}(Q^2; \delta^2; N) = \int_0^1 dx x^N \tilde{C}_{e^+e^-}(Q^2; \delta^2; x) [1 + O(\delta^2)] \quad (37)$$

with

*) We thank S. Gupta and A.H. Mueller for discussions related to the subjects of this subsection.

$$\begin{aligned}
 & \widetilde{C}_{e^+e^-}(Q^2; \delta^2; x) \\
 &= \sum_{i,a} \int C_{\gamma^* \rightarrow i}(\alpha_s(Q^2); 1; z) D_{a,i}(Q^2; \delta^2 Q^2; u/z) \theta(z-u) \cdot \\
 & \quad \cdot J_a(\delta Q; w; u) \delta(uw-x) du dw dz / z .
 \end{aligned} \tag{38}$$

The combination $C_{\gamma^* \rightarrow i} \cdot D_{a,i}$ is the usual inclusive distribution of a single parton of mass δQ , and can be computed in any method, e.g. either in our method, in a dimensional regularization -- RG method [5], or in a cut-vertex method [15].

The function $J_a(\delta Q; w; u)$ is computed as follows:

$$J_a(\delta Q; w; u) = \sum_{\substack{m=1 \\ n=0}}^{\infty} \int d\Omega_{m,n} \left\{ \begin{array}{c} \text{Diagram: A square blob with incoming momentum } q \text{ and outgoing momenta } p_1, \dots, p_m, k_1, \dots, k_n \end{array} \right\} \tag{39}$$

where the square blob means the full amplitude for $a(q) \rightarrow p_1 + \dots + p_m + k_1 + \dots + k_n$, and $d\Omega_{m,n}$ is the $m+n$ body phase volume with the constraints*)
 $(p_i^2 = k_j^2 = 0, \text{ all } i \text{ and } j)$:

$$\left. \begin{aligned}
 \sum_{i=1}^m p_i^{(0)} &= w q^{(0)} = w u Q , \\
 \theta_{p_1}, \dots, \theta_{p_m} &\leq \delta , \\
 \theta_{k_1}, \dots, \theta_{k_n} &> \delta , \\
 \text{and } q^2 &\leq M_0^2 = \delta^2 Q^2
 \end{aligned} \right\} \tag{40}$$

(The fact that the constraints are imposed in the c.m. system of e^+e^- , while $C_{\gamma^* \rightarrow i} D_{a,i}$ is often calculated in an infinite momentum frame, does not cause a problem, because only the region $u \neq 0$ matters in the positive moment and because $u_{(\text{IMF})} = u_{(\text{cm})}$ for $u \gg \delta^2$.)

*) $d\Omega_{m,n}$ contains also the statistical factors to avoid double counting of the same events.

Because of the KLN theorem, all the mass singularities cancel in J_a and, hence, J_a can depend on δ and Q only through $\alpha_s(\delta Q)$. [J_a can then be computed to a given order of $\alpha_s(\delta Q)$].

Because of the cone constraints of the integration, J_a depends upon u [energy, $q^{(0)}$] as well as upon w , and this prevents the u and w integrations in Eq. (38) from being of a simple convolution type. However, this does not present a problem, since the quantity $C_{\gamma^* \rightarrow i} D_{a,i}$ corresponds to a purely perturbative evolution up to the scale δQ and can be computed to any given order absolutely.

In the leading log approximation, we find

$$C_{\gamma^* \rightarrow i}(\alpha_s(Q^2); 1; z) = \delta(1-z) \delta_{i, \text{quark}}$$

and

$$J_a(\delta Q; w; u) = \delta(1-w)$$
(41)

and Eq. (38) coincides with the result in Ref. [1]. To the next-to-leading order, $\Pi_{a,i}$ in D should be computed up to $[\alpha_s(q^2)]^2$, $C_{\gamma^* \rightarrow i}$ up to $\alpha_s(Q^2)$, and J_a up to terms of order $\alpha_s(\delta^2 Q^2)$. Equation (38) is then valid in the region

$$\Lambda^2/Q^2 \ll \delta^2 \ll 1$$
(42)

A related measure, in somewhat closer form to the original Sterman-Weinberg formula, has recently been studied by Duncan, Gupta and Mueller [14], using a renormalization group approach.

3.5 Universality of jet evolution

As already mentioned in the Introduction, the factorization theorem implies the universal properties of jets. It is such universal (= process-independent) properties of jets that we have been mainly interested in so far. But to what degree are the results really universal?

It has been assumed that only 2pR diagrams are important in determining the properties of a jet (see Fig. 7). To all orders in α_s and in logs, the 2pR contribution to the normalized distribution $i \rightarrow a_1 + \dots + a_m$ is (trivially) process-independent by factorization. Only the 2pI "hard subprocess cross-section" is

responsible for the process dependence. Since 2pI graphs are free of collinear singularities, the contamination in the evolution inside a jet due to particles from the hard subprocess is of the order of relevant phase space, i.e.

$$\sim \int_0^{\delta^2} d\theta^2 = \delta^2 \approx M^2/Q^2 \ll 1 \quad (43)$$

For general distribution, $i \rightarrow a_1 + \dots + a_m + X$, where $a_1, \dots, a_m$ are produced inside a jet of cone (δ) with $x_i \neq 0$, the process-dependent contribution will have at least one factor of δ^2 . We thus conclude that the normalized distribution inside a jet is determined by universal branching processes described by the jet calculus, up to a power correction of Eq. (43).

In particular, there are cases in which the parent parton of a jet is dominated by a single species of quanta: e.g. for large x or large moments in hard processes where the hard vertex is electromagnetic (or weak), the quark dominates. In such a case, one finds (at parton level) ($\sum_{i=1}^m n_i \gg 1$),

$$\begin{aligned} & \int dx_1 \dots dx_m x_1^{n_1} \dots x_m^{n_m} \left(\frac{1}{\sigma} \frac{d\sigma}{dx_1 \dots dx_m} \right)_{\text{jet}}^{\text{inside}} / \int dx x^{n_1+\dots+n_m} \left(\frac{1}{\sigma} \frac{d\sigma}{dx} \right)_{\text{jet}}^{\text{inside}} \\ & \approx D_{\{a\},q}(M^2, n_1, \dots, n_m) / D_{a,q}(M^2, n_1+\dots+n_m) \cdot \{1 + O(\delta^2) + O(\text{mixing})\} \end{aligned} \quad (44)$$

Since the non-perturbative fragmentation, partons $\rightarrow$ hadrons, is also universal, the above relation can be used for the direct test of universality of jets. The point is that the error in Eq. (44) is either of δ^2 -- which can be chosen to be reasonably small experimentally -- or due to the mixing of parent parton whose smallness can be checked to any given order case by case, but neither of the accuracies is logarithmic. Equation (44) holds to all orders of α_s and logs (with the assumption of dominance of a single species of parent parton), and does not even depend upon the scheme of regularization*). The ratio of Eq. (44) is a new kind of physical quantity which can be computed with perturbative QCD and can be used to test QCD.

*) $C_i(n_1 + \dots + n_m) \cdot D_{a_1, \dots, a_m, i}$ and $C_i(N) D_{a, i}(N)$ are both independent of the regularization scheme, and C_i cancels in the ratio of Eq. (44).

3.6 Multiplicity distribution in QCD

The simplicity of diagrammatic structure of the unitarity equation obtained in Section 2 suggests a possibility of formulating the problem of average multiplicity and multiplicity-moments in QCD. A reservation to be added immediately is that a naive treatment with on-shell quarks and gluons leads to divergent quantities, and one is tempted to conclude that the problem is beyond the scope of a perturbative approach.

However, the multiplicity distributions of *hadrons* are probably more closely related to those of off-shell quanta at a typical confinement mass scale. In the absence of a complete theory of hadron production, we shall search for a measure of proliferation and correlation in terms of off-shell quarks and gluons.

Such a measure should satisfy several conditions, to say the least. Firstly, it should be well-defined and computable systematically in perturbative QCD, up to any desired order. Dealing with off-shell quanta, one should be particularly aware of the danger of double-counting. Secondly, the calculated Q^2 dependence, and the dependence on a finite mass scale put in by hand, must factorize. In other words, the result for the Q^2 dependence should not depend on which finite mass scale we choose for counting off-shell partons. Thirdly, the measure should be gauge-invariant.

A possible way to formulate the problem is to define a *cluster* as a minimal 2pI subgraph introduced in Section 2, which, by definition, corresponds to a group of on-shell quanta having an invariant mass

$$\sqrt{q^2} \geq M_0 \quad . \quad (45)$$

The average multiplicity/moments can then be defined in terms of these clusters.

Let us concentrate on the average multiplicity of the clusters. Starting from a diagram such as shown in Fig. 1b, we compute one-cluster inclusive cross-section, normalized to σ_{tot} . The integral of this cross-section is the average multiplicity.

For the moment, consider only the subset of clusters having mass between M_0 and $M_0 + \Delta M_0$. Then the average multiplicity of these is given, from the diagrammatic analysis of Sections 2 and 3, by

$$\langle \Delta n(Q^2, M_0^2) \rangle = \sum_{i,a} \int_{M_0^2}^{Q^2} \frac{dq^2}{q^2} G_i(Q^2; q^2) E_{a,i}(q^2; M_0^2) \left(\frac{\Delta M_0^2}{M_0^2} \right) R_a(M_0^2), \quad (46)$$

$$E_{a,i}(q^2; M_0^2) = \sum_b \int_{M_0^2}^{q^2} \frac{dq'^2}{q'^2} V_{bi}(q^2; q'^2) E_{a,b}(q'^2; M_0^2) + 1 \quad (47)$$

where

$$G_i(Q^2; q^2) = \int_{q^2/Q^2}^1 dz C_{g \rightarrow i}(Q^2; q^2; z), \quad (48)$$

$$V_{a,i}(q^2; q'^2) = \int_{q'^2/q^2}^1 dx \Pi_{a,i}[\alpha_s(q^2); x] \quad (49)$$

and $R_a(M_0^2)$ is the total decay probability of the parton a with off-shell mass M_0 , the analogue of R in e^+e^- annihilation.

The integral equation (47) determines the increase of cluster multiplicity due to parton branchings discussed in Section 2.

The above problem was solved in LLA, for e^+e^- annihilation

$$\begin{aligned} \Pi_{a,i}[\alpha_s(q^2); x] &= \frac{\alpha_s(q^2)}{2\pi} P_{a,i}^{AP}(x), \\ C_{g \rightarrow i}(Q^2; M^2; z) &= \delta(1-x) \delta_{i, \text{quark}}, \end{aligned} \quad (50)$$

and

$$R_a(M_0^2) = 1,$$

in which case Eqs. (47) and (49) lead to the known results [16,17]

$$\langle \Delta n(Q^2; M_0^2) \rangle \underset{Q^2 \gg M_0^2}{\sim} \exp c(\sqrt{\log Q^2/\Lambda^2} - \sqrt{\log M_0^2/\Lambda^2}) \quad (51)$$

($c \equiv 2\sqrt{C_A/\pi b} \approx 2.4$). For multiplicity moments, analogous recursion formulae can be solved and give [1,16]

$$\begin{aligned} \langle n(n-1) \rangle &\simeq (4/3) \langle n \rangle^2 \quad (\text{gluon jet}) \\ &\simeq \frac{3C_F + C_A}{3C_A} \langle n \rangle^2 = \frac{7}{4} \langle n \rangle^2 \quad (\text{quark jet}) \end{aligned} \quad (52)$$

Does the average multiplicity defined by Eqs. (46) to (49) satisfy the conditions mentioned before? The structure of the integral equation (47) suggests that the dominant Q^2 -dependence (coming from the large q^2 -region) does factorize. Once the factorization is found in $\langle \Delta n(Q^2, M_0^2) \rangle$, i.e. if

$$\langle \Delta n(Q^2; M_0^2) \rangle \simeq f\left(\frac{Q^2}{\Lambda^2}\right) g\left(\frac{M_0^2}{\Lambda^2}\right) \cdot \frac{\Delta M_0^2}{M_0^2} R(M_0^2) \quad (53)$$

then $\langle n(Q^2) \rangle_{\text{cluster}}$ will also factorize:

$$\langle n(Q^2) \rangle_{\text{clusters}} \underset{Q^2 \rightarrow \infty}{\simeq} f\left(\frac{Q^2}{\Lambda^2}\right) \cdot \left(\begin{array}{c} \text{a function} \\ \text{of } M_0^2 \end{array} \right). \quad *) \quad (54)$$

As for the gauge invariance of the above measure, it has a good chance of being so, because the diagrammatic definition of the minimal 2pI graphs is really a definition by the collinear mass singularity structure. On the contrary, a naive "off-shell-mass quark" has no gauge-invariant meaning.

Coming back to the question of hadronic multiplicity, should one expect Eq. (59) or its extension Eq. (54) to describe $\langle n \rangle_{\text{hadrons}}$? We have no strong arguments supporting such a connection. Nevertheless, we stress that it is very difficult to understand if $\langle n \rangle_{\text{hadrons}}$ and $\langle n \rangle_{\text{clusters}}$ have different Q^2 -dependence at $Q^2 \rightarrow \infty$, especially in view of the consistent over-all view of jet evolution which emerges from various analyses (preconfinement, space-time evolution, and rapidity picture) [2]. In such a case, none of the perturbative calculations would be relevant for describing hadronic final states in hard processes.

3.7 Hadrons

Hadrons can be introduced following Ref. [1]. A set of non-perturbative fragmentation functions $d_a^{h_1}(M_0^2, z)$, $d_a^{h_1 h_2}(M_0^2; z_1, z_2)$, ... need to be introduced. They represent the whole effects of transition from a parton with off-shell mass $\leq M_0$ into hadrons. They satisfy among themselves inclusive sum rules. Two-hadron distributions, for example, can be constructed as

*) In Eq. (54) we have assumed that the integration over M_0 up to Q would not produce an extra Q^2 -dependence.

$$\sigma e^+e^- \rightarrow h_1 + h_2 + X$$

$$= C^{e^+e^-i} \left[D_{a,i} V_{a \rightarrow b_1 b_2} (D_{c_1 b_1} d_{c_1}^{h_1}) (D_{c_2 b_2} d_{c_2}^{h_2}) + D_{a,i} d_a^{h_1 h_2} \right] \quad (55)$$

For $M_0^2 \ll (P_{h_1} + P_{h_2})^2 \ll M^2$, only the first term in Eq. (55) should be important.

Multihadron distributions constructed this way can be proved to be correctly normalized [1].

4. SOME RESULTS IN THE NEXT-TO-LEADING ORDER

4.1 Cancellation of soft divergences and of extra (spurious) divergences of $n^2 = 0$ gauges

Actual calculation of vertex functions $V^{(m)}$ and Π (or D) is most conveniently done in a light-like axial gauge (for which the diagrammatic simplification of Section 2 applies), $n \cdot A = 0$, $n^2 = 0$. Both general axial gauges ($n^2 \neq 0$) and covariant gauges seem to present considerable computational problems.

An $n^2 = 0$ gauge presents, on the other hand, another problem: there appear divergences which are neither of usual ultraviolet nor of infrared origin. To the next-to-leading order, the integrals in question are of the form,

$$J_p = \int d^4k \frac{1}{n \cdot k \ k^2 (p+k)^2} \quad (56)$$

where $1/n \cdot k$ comes from the gluon propagator

$$i \rho^{\mu\nu}(k)/k^2 = i \left\{ -g^{\mu\nu} + \frac{n^\mu k^\nu + n^\nu k^\mu}{n \cdot k} \right\} / k^2 \quad (57)$$

in this gauge. Though finite by a naive power counting in both ultraviolet and infrared (for $p^2 \neq 0$) domains, the integral is actually divergent. In $4 - 2\epsilon$ dimensions ($d^4k \rightarrow d^{4-2\epsilon}k$), it gives

$$J_p \propto \Gamma(\epsilon) \int_0^1 \frac{dx}{x} [p^2 x(1-x)]^{-\epsilon} \quad (58)$$

We have, however, found that the problem is actually absent. In Table 1, we show how integrals of this type cancel *algebraically* (hence there is no need to specify how to regularize them), if a gauge-invariant set of graphs are summed

first. It can be seen that by use of Table 1 the spurious divergences due to the $n^2 = 0$ gauge drop out completely to next-to-leading order of logs but to all order of α_s . In view of the computational advantage of the light-like gauge, this is very satisfactory.

For the C_F^2 part, the above cancellation can easily be seen as follows. The self-energy correction of Fig. 8a contains

$$\begin{aligned} & -\frac{1}{2} \int d^4k \frac{\not{n}(\not{q}+\not{k})\not{k} + \not{k}(\not{q}+\not{k})\not{n}}{n \cdot k \ k^2 (q+k)^2} \\ & = +\frac{1}{2} \int d^4k \frac{\not{n}(\not{q}+\not{k})\not{n} + \not{n}(\not{q}+\not{k})\not{n}}{n \cdot k (q+k)^2 k^2} = \not{n} q^2 J_q + \text{other terms} \end{aligned} \quad (59)$$

The vertex diagram of Fig. 8b has a term,

$$-\int d^4k \frac{\not{n}(\not{p}+\not{k})\gamma^\rho(\not{q}+\not{k})\not{k} + \not{k}(\not{p}+\not{k})\gamma^\rho(\not{q}+\not{k})\not{n}}{n \cdot k \ k^2 (q+k)^2 (p+k)^2} \quad (60)$$

Writing $(\not{q} + \not{k})\not{k} = (q + k)^2 - (\not{q} + \not{k})\not{q}$ and $\not{k}(\not{p} + \not{k}) = (p + k)^2 - \not{p}(\not{p} + \not{k})$, it becomes

$$-\not{n}\not{p}\gamma^\rho J_p - \gamma^\rho \not{q}\not{n} J_q + \text{other terms} \quad (61)$$

Upon insertion into a larger graph, the J_q part in Eqs. (59) and (61) clearly cancels. The J_p part cancels similarly with a self-energy graph on line p^μ . Cancellation in the $C_F C_A$ part is slightly more complicated, but can be explicitly checked.

After the cancellation of integrals such as Eq. (56), there remain still infrared (soft or collinear) divergences which should also be cancelled in inclusive quantities. Consider the $C_F C_A$ part, which is genuinely non-Abelian. The remaining infrared divergences are all regularized by going to $4 - 2\epsilon$ dimensional space-time. We first keep $\epsilon > 0$ to regularize the ultraviolet poles, subtract them upon renormalization (we use the minimal subtraction scheme), then continue to $\epsilon < 0$. All the remaining poles of infrared origin are tabulated in Table 2. It is seen that all infrared divergences (corresponding to unobserved soft or

parallel quanta) completely cancel. The C_F^2 part is also shown to be finite. This proves the finiteness of $P_{Q,Q}(x)$ for $x \neq 1$, and $V_{Q \rightarrow QG}^{(2)}(q^2; x_1, x_2)$ at $x_1 + x_2 \rightarrow 1$.

Note on the double soft divergence in QCD

We note that the double divergence of the graph 3c in Table 2 is a purely soft divergence. It is easy to check that it in fact arises from the region, $k^\mu \approx 0$ of the gluon A (see Fig. 4c) attached to two external particles. In axial gauges, of course, this diagram is free of collinear divergences for fixed q^2 . (Other graphs such as Fig. 4d contain both types of divergences.) Such a strong soft divergence in a non-Abelian graph is an interesting indication that soft properties of QCD are indeed quite different from those in electrodynamics.

This type of enhanced contribution, non-leading in collinear singularity but doubly divergent in the soft limit, can have direct physical consequences. For example, a study of the integral equation (47) shows that the appearance of these terms in the singlet anomalous dimension [or $P^{NL}(x)$] behaving as $\sim \alpha_s^2/n^3$ for small n , would modify the leading log prediction, for the average multiplicity, Eq. (51).

4.2 Some sample calculations and an example of scheme dependence of $P^{NL}(x)$

In order to give a concrete idea as to how the calculation goes, let us consider simple examples. As a check consider first the leading log contribution to $\hat{\Pi}_{b,a}$, shown in Fig. 9. We go to an infinite momentum frame and parametrize

$$\begin{aligned} p &= (x q^* + p_T^2/2x q^*, p_T, x q^*) , \\ k &= ((1-x) q^* + p_T^2/2(1-x) q^*, -p_T, (1-x) q^*) , \\ q &= p + k , \quad (q^* \rightarrow \infty) \end{aligned} \quad (62)$$

while the phase volume in this frame is

$$\frac{d^3 p}{2p(2\pi)^3} \frac{d^3 k}{2k(2\pi)^3} \{ d^4 q \delta^4(q-p-k) \} = \frac{d^3 q}{2q_0(2\pi)^3} d(q^2) d\Phi_2 \quad (63)$$

with

$$d\Phi_2 = \frac{1}{4(2\pi)^2} \frac{dx}{x(1-x)} dp_T^2 \delta(q^2 - \frac{p_T^2}{x(1-x)}) = \frac{dx}{4(2\pi)^2}$$

Thus at fixed q^2 and x , the integration is over the azimuthal angle, which is trivial. We find, from the coefficient of $1/q^2$ ($x \neq 1$),

$$\hat{\Pi}_{Q,Q}(\alpha_s; x) = \hat{\Pi}_{Q,Q}(\alpha_s; 1-x) = \left(\frac{\alpha_s}{2\pi}\right) C_F (1+x^2)/(1-x) \quad (64)$$

$$\hat{\Pi}_{Q,G}(\alpha_s; x) = \hat{\Pi}_{\bar{Q},G}(\alpha_s; x) = \left(\frac{\alpha_s}{2\pi}\right) N_f \{ x^2 + (1-x)^2 \}$$

and

$$\hat{\Pi}_{G,G}(\alpha_s; x) = \left(\frac{\alpha_s}{2\pi}\right) \cdot 2C_A \left\{ (1-x)/x + x/(1-x) + x(1-x) \right\}$$

which are the known results [9]. The use of Eq. (31) to determine the $\delta(1-x)$ part leads to the usual Altarelli-Parisi function, as in the original derivation.

The probability functions to the next-to-leading order $O[\alpha_s^2(q^2)]$ can be computed similarly, apart from the fact that there are now non-trivial integrations over p_T 's, and apart from several technical points which will be presented in full detail in another work. Our results on the non-singlet parton density $\Pi[\alpha_s(q^2), x]$, to order $\alpha_s^2(q^2)$, agree with those of Curci, Furmanski and Petronzio [5], except for a few terms which are explicitly scheme-dependent. For the purpose of illustrating this scheme dependence, let us concentrate on these terms here, for the process $e^+e^- \rightarrow q$ (non singlet) X.

These terms arise from the graphs of Figs. 4i and j and analogous gluon loop diagrams Figs. 4d and e. For definiteness, we shall be explicit only for the contribution proportional to $N_f C_F$ of Figs. 4i and j.

The diagram with a cut quark loop Fig. 4i gives, before q^2 -integration and in $4 - 2\epsilon$ space-time dimensions [apart from an over-all $(q^2)^{-1}$ factor],

$$\frac{N_f C_F (\alpha_s)^2}{2(2\pi)^2} (q^2)^{-2\epsilon} \cdot \frac{2}{3} \frac{1+x^2}{1-x} \left[-\frac{1}{\epsilon} + \log^2(1-x)x - \frac{5}{3} - 2\zeta \right] \quad (65)$$

where $\zeta = \log 4\pi - \gamma_E$, as a contribution to the α_s^2 term in

$$\Pi_{Q,Q}[\alpha_s(q^2); x] = \frac{\alpha_s(q^2)}{2\pi} P_{Q,Q}^{AP}(x) + \left[\frac{\alpha_s(q^2)}{2\pi}\right]^2 P_{Q,Q}^{NL}(x) + \dots \quad (66)$$

Similarly the graph of Fig. 4j gives after the minimal subtraction of the UV pole,

$$\frac{N_f C_F (\frac{\alpha_s}{2\pi})^2 (q^2)^{-\epsilon}}{2} \cdot \frac{2}{3} \left[\frac{1+x^2}{1-x} \left\{ \frac{1}{\epsilon} - \log x(1-x) + \zeta \right\} - (1-x) \right] \quad (67)$$

By adding these two terms at fixed q^2 and setting $\epsilon \rightarrow 0$, we find (for the non-singlet parton density),

$$\frac{N_f C_F (\frac{\alpha_s}{2\pi})^2 (1+x^2)}{3(1-x)} \log q^2/\mu^2 + (\frac{\alpha_s}{2\pi})^2 \hat{P}_{Q,Q}(x) \quad (68)$$

with

$$\hat{P}_{Q,Q}(x) = \frac{N_f C_F}{3} \left[\frac{1+x^2}{1-x} \left\{ \log(1-x) - \frac{5}{3} - \zeta \right\} - (1-x) \right] \quad (69)$$

The first term in Eq. (68) is obviously the correct $\alpha_s^2 \log q^2/\mu^2$ term, which is a part of dressing of α_s into the renormalization-group-invariant coupling constant $\alpha_s(q^2)$ in the leading log term of Eq. (66). (We find, of course, a similar piece $\propto C_F C_A$ from the diagrams of Figs. 4d and e.)

If instead we integrate first over q^2 [with the over-all $(q^2)^{-1}$], and pick up the simple pole term, the answer is

$$-\frac{1}{2\epsilon} (\frac{\alpha_s}{2\pi})^2 \hat{P}_{Q,Q}^{CFP}(x) + \frac{1}{\epsilon^2} \text{terms} + \text{non singular terms} \quad (70)$$

with

$$\hat{P}_{Q,Q}^{CFP}(x) = \frac{N_f C_F}{3} \left[\frac{1+x^2}{1-x} \left(-\log x - \frac{5}{3} \right) - 2(1-x) \right] \quad (71)$$

in agreement with CFP [8].

There are analogous contributions proportional to $C_A C_F$ from diagrams in Figs. 4d and e. The two-loop quark self-energy corrections can be determined by use of the charge sum rule, Eq. (31). In this way we find, for the full difference,

$$P_{Q,Q}(x) - P_{Q,Q}^{CFP}(x) = -2\pi b C_F \left[\frac{1+x^2}{1-x} \left\{ \log x(1-x) - \zeta \right\} + (1-x) \right]_+ \quad (72)$$

where $b = (11N_c - 2N_f)/12\pi$ and $[f(x)]_+ \equiv f(x) - \delta(1-x) \int_0^1 dy f(y)$.

It is not difficult to find out which terms in the "coefficient function at $\mu^2 = Q^2$ " or the hard subprocess cross-section, $C_{Y*}[\alpha_s(Q^2), 1; x]$, compensate the difference, Eq. (72). Such a term in C_{Y*} can be traced back to the contribution of Fig. 10. In the convention of Refs. [5] and [18], $C_{Y*}[\alpha_s(Q^2); 1; x]$ is defined to be the finite part *after* q^2 -integration; in our convention, $C_{Y*}[\alpha_s(Q^2), Q^2/M^2; x]$ contains only the contribution from $M^2 \leq q^2 \leq Q^2$ for the mass singular terms $\propto dq^2/q^2$ (and all contributions from non-singular terms). We find that the only difference between CFP [5] and our scheme comes from the finite terms CFP pick up after the q^2 -integration, $(-1/\epsilon) \cdot [O(\epsilon) \text{ terms in the numerator}]$. The only $O(\epsilon)$ terms at finite x are $-\epsilon\{(1+x^2)/(1-x)\}[\log x(1-x) - \zeta]$ from the phase volume in $4 - 2\epsilon$ dimensions and $-\epsilon \cdot (1-x)$ from the gamma matrix algebra. After adding the contributions proportional to $\delta(1-x)$, we get

$$C_{Y*} - C_{Y*}^{\text{CFP}} = -C_F \left[\frac{1+x^2}{1-x} \left\{ \log x(1-x) - \zeta \right\} + (1-x) \right]_+ \quad (73)$$

which cancels the effect of $P_{Q,Q} - P_{Q,Q}^{\text{CFP}}$, Eq. (72), in the physical combination of Eq. (34) and Eq. (35).

From the above discussion it is clear that the piece behaving as $\alpha_s^2 \log^2 n$ at large n of the inclusive cross-section comes from the parallel emission ($q^2 = 0$) of a gluon: it is a universal (process-independent) contribution, and in our scheme is contained in $P_{Q,Q}^{\text{NL}}(x)$, which describes the physical jet evolution*). On the contrary, in the scheme of Ref. [5], the separation between C_{Y*} and $\exp \int A_n$ parts does not correspond always to a physical distinction of final states.

This is of course the reason why, in studying the jet evolution, the simpler recipe of Ref. [5] is not entirely satisfactory.

In any event, as far as the *single parton inclusive* spectra are concerned, the separation between jet evolution and hard subprocess is arbitrary beyond LLA, and we confirm the results of Ref. [5] for the non-singlet one-particle inclusive cross-section for $e^+e^- \rightarrow q(x) + X$.

*) This agrees with the arguments of Ref. 19.

4.3 Two comments on naive on-shell and off-shell regularization methods

i) "On-shell method" with non-zero gluon mass

We have tried the calculations with a non-zero gluon mass m_g put in by hand in the Feynman denominators, keeping external particles on-mass-shell. It was found that, after the cancellation of integrals of the type of J_p , the non-zero gluon mass is sufficient to regularize all the loop and phase-space integrations relevant to P and V up to next-to-leading order. However, the resulting $\log m_g$'s do not cancel in $\hat{P}_{QQ}(x)$, $x \neq 1$, and the problem seems to arise precisely from the non-Abelian graph with a double soft divergence. We conclude that putting a non-zero mass by hand in the Feynman denominators is not a correct procedure in QCD, at least in a class of gauges including the light-like axial gauge. This is perhaps not surprising as a non-zero gluon mass put by hand breaks the gauge invariance, and its consequence is known to be more serious in QCD [20] than in QED*).

ii) "Off-shell method" of infrared regularization

The "off-shell method" of infrared regularization (i.e. keeping external particle lines off-shell to avoid the infrared divergences) gives in general an incorrect answer beyond LLA, because it violates gauge invariance. For a cut off-shell line, a naive off-shell method is simply wrong: the cut propagator is not given by $\theta(k_0 - p_0)\delta(k^2 - p^2)$, but by

$$\theta(k_0)\delta(k^2) + O(\alpha_s) \quad (74)$$

To take account of $O(\alpha_s)$ corrections will require computation of graphs of order α_s with on-shell cut lines. Hence one is forced, as a result, back to the on-shell regularization.

It remains, of course, true that intermediate lines appearing in the parton decay chain, are off-shell and represent physical jets. At any given order of collinear singularities, however, the final partons $\{a_j\}$ can be set to be on-mass-shell and massless in computing $v_{i \rightarrow \{a_j\}}^{(m)}[\alpha_s(q^2); \{x_j\}]$: the effects of non-zero off-shell masses are taken into account by going to higher order.

*) We thank B. Lautrup for discussions on this point.

5. CONCLUSION

The main result of this paper has been the extension of the jet calculus beyond LLA. To any given order of logs and to all orders in α_s , the problem has been reduced to the computation of a finite number of parton decay probability functions $V_{i \rightarrow \{a_j\}}^{(\ell)}[\alpha_s(q^2), \{x_j\}]$, and, in terms of these, the jet evolution is described in a simple probabilistic algorithm. (For instance, $V_{i \rightarrow a}^{(1)}[\alpha_s(q^2); x] \equiv \Pi_{ai}[\alpha_s(q^2); x]$ is the kernel of the generalized Altarelli-Parisi equation.)

We have elucidated the scheme dependence of $\Pi_{a,i}$ to next-to-leading order, and shown that $\Pi_{a,i}$'s and V 's defined in this paper do correspond to the jet evolution inside a narrow cone, in contrast to $\Pi_{a,i}$ in other works. Nevertheless, one-parton inclusive cross-section in $e^+e^- \rightarrow a(x) + X$ has been shown to be scheme-independent.

The crucial idea exploited in this paper is that M -- arbitrary mass scale introduced to make the factorization of mass singularities explicit -- has a physical meaning related to the width of associated jets (a cone in which fast particles are emitted): M is arbitrary in the sense that it can be chosen at our will, experimentally. The property of factorization, and the diagrammatic method in axial gauge QCD, can then be used to study detailed structure of jets.

This is to be contrasted with the situation in a more conventional problem such as the single-particle inclusive production, where M can be conveniently chosen to be equal to μ -- the scale of renormalization for α_s . This renders the problem closely analogous to those treated with OPE. If the formal similarity between the mass singularity factorization and the multiplicative renormalization is a basis of simplified treatments in the problems of single-particle inclusive productions (or similar), the recognition of differences between their physics contents is the starting point of the analysis of jets.

We have to admit that the whole work as presented here is not yet quite at the stage where one can start doing phenomenology. Nonetheless, we hope that the existence of simple "jet calculus" rules and the parton probability interpretation, shown to persist beyond LLA, will be useful as a basis for further study of jets and for model-constructions, and also as a frame for improved calculations.

Some more explicit calculations in the next-to-leading order, and their possible applications, will be discussed in a forthcoming paper [4].

Acknowledgements

We thank Peter Scharbach for his collaboration in the early stage of this work. We are grateful to S. Pokorski, J.C. Taylor and G. Veneziano for many illuminating discussions. We profited also from discussions with D. Duke, M. Teper, R. Petronzio, W. Furmański, E. Floratos and many other colleagues. Finally, we wish to thank the Theory Divisions at Rutherford Laboratory and CERN for warm hospitality extended to us at various stages of this work.

Table 1

Cancellation of integrals of the type of Eq. (53)

Graph	Colour coefficient	Integrals
$\frac{1}{2}$	C_F^2	$-J_q$
$C_F(C_F - C_A/2)$	$J_q + J_p$
$\frac{1}{2}$	C_F^2	$-J_p$
$C_F C_A/2$	$J_q + J_p + J_l$
$\frac{1}{2}$	$C_F C_A/2$	$-J_l$
C_F	$J_q + J_r$
$\frac{1}{2}$	C_F	$-J_q$
$\frac{1}{2}$	C_F	$-J_r$

Table 2

Cancellation of soft and collinear divergences ($C_F C_A$ part).

$$\text{Common factor} = \frac{C_A C_F}{2} \left(\frac{\alpha_s}{2\pi} \right)^2 (q^2)^{-1-2\epsilon} \left\{ \frac{1+x^2}{1-x} - \epsilon(1-x) \right\}$$

Fig.	$\frac{1}{\epsilon^2}$	$\frac{1}{\epsilon} \log x$	$\frac{1}{\epsilon} \log (1-x)$	$\frac{\zeta}{\epsilon}$	$\frac{1}{\epsilon}$
4b	0	0	0	0	4
4c	-2	4	4	4	-8
4d	4	-8	-4	-8	$4 + \frac{11}{3}$
4e	0	0	0	0	$-\frac{11}{3}$ a)
4f	-4	6	4	8	-2
4g	2	-2	-4	-4	2
Sum	0	0	0	0	0

a) This entry actually contains $(q^2)^\epsilon$. See Eq. (64) of the text.

REFERENCES

- [1] K. Konishi, A. Ukawa and G. Veneziano, Nucl. Phys. B157 (1979) 45; Phys. Lett. 78B (1978) 243; Phys. Lett. 80B (1979) 259.
K. Konishi, Physica Scripta 19 (1979) 195, Proc. Int. Conf. on High Energy Physics, Tokyo 1978 (Physical Society of Japan, Tokyo), p. 241.
- [2] D. Amati and G. Veneziano, Phys. Lett. 83B (1979) 87.
A. Bassetto, M. Ciafaloni and G. Marchesini, Nucl. Phys. B163 (1980) 477, and Phys. Lett. 83B (1979) 207.
L. Caneschi and A. Schwimmer, Phys. Lett. 86B (1979) 179.
K. Konishi, CERN TH. 2853, to be submitted to Survey in High Energy Physics.
- [3] D. Amati, R. Petronzio and G. Veneziano, Nucl. Phys. B146 (1978) 29.
R.K. Ellis, H. Georgi, M. Machacek, H.D. Politzer and G.G. Ross, Phys. Lett. 78B (1978) 281 and Nucl. Phys. B152 (1979) 285.
J. Frenkel, M.J. Shailer and J.C. Taylor, Nucl. Phys. B148 (1979) 228.
S.B. Libby and G. Sterman, Phys. Lett. 78B (1978) 618, and Phys. Rev. D 18 (1978) 3252; 4737.
A.H. Mueller, Phys. Rev. D 18 (1978) 3705.
- [4] J. Kalinowski, K. Konishi, P. Scharbach and T.R. Taylor, in preparation.
- [5] G. Curci, W. Furmański and R. Petronzio, preprint CERN TH. 2815 (1980).
- [6] See, for example, T.H. Harris, The theory of branching processes (Springer, Berlin, 1963).
M.S. Bartlett, An introduction to stochastic processes (Cambridge Univ. Press, Cambridge, 1978).
- [7] A.M. Polyakov, Zh. Eksp. Teor. Fiz. 59 (1970) 542 [Sov. Phys. JETP 32 (1971) 296] and Sov. Phys. JETP 33 (1971) 850.
This type of unitarity equation has also been studied by G. Paffuti, Thesis at Scuola Normale Superiore, Pisa (1975).
- [8] R.K. Ellis et al. in Ref. [3].
- [9] G. Altarelli and G. Parisi, Nucl. Phys. B126 (1977) 298.
G. Parisi, Proc. 11th Rencontre de Moriond, 1976 (ed. J. Tran Thanh Van).
L.N. Lipatov, Yad. Fiz. 20 (1974) 181.

- [10] This way of deriving the jet calculus in LLA was given by:
- I. Andreev, Proc. 14th Rencontre de Moriond, Les Arcs, 1979 (ed. J. Tran Thanh Van).
- P. Cvitanovic, P. Hoyer and K. Konishi, Phys. Lett. 85B (1979) 413.
- R. Kirschner, Phys. Lett. 84B (1979) 266.
- Related ideas are further explored in A. Giovannini, Nucl. Phys. B161 (1979) 429.
- [11] T. Kinoshita, J. Math. Phys. 3 (1962) 650.
- T.D. Lee and M. Nauenberg, Phys. Rev. 133 (1964) 1549.
- [12] See, for example, D. Gross, Lectures in Proc. Les Houches Summer School (1976).
- E.G. Floratos and D.A. Ross and C.T. Sachrajda, Nucl. Phys. B129 (1977) 66.
- [13] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D 18 (1978) 3998.
- W. Celmaster and R.J. Gonsalves, Phys. Rev. D 20 (1979) 1420.
- R. Barbieri, L. Caneschi, G. Curci and E. d'Emilio, Phys. Lett. 81B (1979) 207.
- M.R. Pennington and G.G. Ross, Phys. Lett. 86B (1979) 371.
- [14] A. Duncan, S. Gupta and A.H. Mueller, Columbia Univ., CU-TP-179 (1980).
- [15] E.G. Floratos, R. Lacaze and C. Kounnas, Saclay preprint DPh.T. 77/80 (1980).
- [16] A. Bassetto et al., in Ref. 2.
- K. Konishi, Rutherford Lab. RL-79-035 (1979).
- [17] W. Furmański, R. Petronzio and S. Pokorski, Nucl. Phys. B155 (1979) 253.
- [18] See, for example, G. Altarelli, R.K. Ellis and G. Martinelli, Nucl. Phys. B157 (1979) 461.
- [19] G. Curci and M. Greco, preprint CERN TH. 2786 (1979).
- D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini and G. Veneziano, preprint CERN TH. 2831 (1980).
- [20] H. van Dam and M. Veltman, Nucl. Phys. B22 (1970) 397.

Figure captions

- Fig. 1 : An ordinary Feynman diagram (a) and the diagram (b) obtained from (a) after operations A to C.
- Fig. 2 : Factorization for one-parton inclusive cross-section $d\sigma_a$ (Notation follows that of Ellis et al. [3]).
- Fig. 3 : Jet calculus graphs for $i \rightarrow a_1(x_1) + a_2(x_2) + a_3(x_3) + X$.
- Fig. 4 : Diagrams in an axial gauge contributing to the next-to-leading vertices.
- Fig. 5 : Symbolic representation of the Kolmogorov equation for the generating functional G .
- Fig. 6 : a) A diagram of the fourth order in ϕ_6^3 with several cuts.
b) An effective vertex for the subprocess $a \rightarrow d + e + c$.
c) An effective vertex for the subprocess $a \rightarrow b + c$.
- Fig. 7 : Universality of the jet evolution as represented by the jet calculus.
- Fig. 8 : Some of the diagrams containing integrals of the type Eq. (56).
- Fig. 9 : Diagrams to the order α_s , contributing to $P_{b,a}^{AP}(x)$ in LLA.
- Fig. 10 : A diagram contributing to $C_Q[\alpha_s(Q^2); x]$.

Fig. 1a

Fig. 1b

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig.6a

Fig. 6b

Fig. 6c

Fig.7

(a)

(b)

Fig.8

Fig. 9

Fig. 10