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THE PHASE TRANSITION IN THE SU(5) MODEL AT HIGH TEMPERATURES

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A B S T R A C T

Within the minimal GUT model we have studied the nature of the fluctuation induced transition between the SU(5) and the $SU(3)_C \times SU(2) \times U(1)$ phase which occurs at high temperatures. Our analysis is limited to the case when the phase transition occurs outside the critical (fluctuation dominated) region. For this to happen the SU(5) model has to be in a mode analogous to the type I superconductor. This corresponds to having the scalar quartic couplings in the Higgs sector less than the squared gauge coupling. For generic values of the coupling constants the phase transition is found to be weakly first order. As we approach the boundaries for the stability region of the $SU(3)_C \times SU(2) \times U(1)$ phase, however, a strong first-order transition occurs. The SU(5) mode (analogous to the type II superconductor) when the phase transition occurs inside the fluctuation dominated region has been recently studied by Ginsparg. His results together with ours show that there is a continuous merging of the type I mode into the type II mode. Finally our analysis elucidates some aspects of the monopole problem in grand unified theories.

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I. - INTRODUCTION

We consider the minimal GUT, the $SU(5)$ model of Georgi and Glashow¹⁾, where the simple group $G = SU(5)$ undergoes a hierarchy of spontaneous symmetry breaking into successive subgroups: $G \rightarrow H_1 \rightarrow H_2$, with $H_1|_{\text{locally}} = SU(3)_C \times SU(2) \times U(1)$, and $H_2|_{\text{locally}} = SU(3)_C \times U(1)_{\text{em}}$. This is a gauge theory having spontaneous symmetry breakdown at zero temperature. The first breaking is achieved by introducing a Higgs multiplet Φ in the adjoint (24) representation with $\langle 0|\Phi|0\rangle \neq 0$. How does this gauge theory look at finite temperature? Kirzhnits and Linde²⁾ have argued, in analogy with superconductivity, that the symmetry of such a relativistic theory will be restored above a critical temperature T_C . What happens is the following: when the system is coupled to a heat bath, the vacuum expectation value $\langle 0|\Phi|0\rangle$ is replaced by a thermal average $\langle \Phi \rangle_T$ taken with respect to a Gibbs' ensemble. The order parameter of the theory thus becomes temperature-dependent and vanishes at T_C , that is the symmetry is restored for $T > T_C$.

In this paper we are concerned with the phase transition $G \rightarrow H_1$ only. This transition is expected to occur at very high temperatures. Indeed in the weakly coupled $SU(5)$ model $T_C \sim \langle \Phi \rangle_{T=0}$. Since $\langle \Phi \rangle_{T=0} = \langle 0|\Phi|0\rangle \sim$ grand unification scale, it is clear that T_C is very high. If the GUT model under consideration turns out to be a reasonable description of nature, then the transition $G \rightarrow H_1$ might have played an important rôle in the evolution of the Universe. In order to assess possible cosmological implications (magnetic monopole problem, baryon-to-entropy ratio) it is necessary to understand the nature of the phase transition at the strong breaking. To do this it is essential to examine the theory close enough to the critical region. For temperatures too close to T_C , however, standard perturbation expansion breaks down in spite of the smallness of the various couplings³⁾⁻⁵⁾. That is, higher order terms become important in the vicinity of T_C . This is connected with the vanishing of the quasi-particle masses $m(T)$ at $T = T_C$ which leads to infra-red divergences. Recall that at finite temperatures one uses bosonic propagators $\sim ((2\pi nT)^2 + \vec{k}^2 + m^2(T))^{-1}$ with summation over the frequency modes, i.e., summation over the integer n . It is clear that the zero frequency mode $n = 0$ is plagued with infra-red divergences. Moreover these divergences are those expected in a super-renormalizable (3D theory) rather than an ordinary renormalizable theory.

We shall be concerned with an $SU(5)$ theory broken down to $SU(3)_C \times SU(2) \times U(1)$ in a mode which is the analogue of the type I superconductor. This corresponds to having $m_X, m_Y \gg m_H$ (m_X, m_Y are the masses of the superheavy X and Y

vector bosons, and m_H is the mass of the physical Higgs boson). In general we shall take the scalar quartic couplings to be less than the squared gauge coupling and we shall allow the former to approach the latter strictly from below. The phase transition in this case occurs outside the critical (fluctuation dominated) region, where scalar field fluctuations become important. One-loop calculations then suffice to establish the existence of a first-order phase transition. For generic values of the coupling constants the transition is weakly first order. As we approach the boundaries for the stability region of the $SU(3)_C \times SU(2)_L \times U(1)_Y$ phase, however, a strongly first-order transition occurs. As a result the discontinuity in the order parameter $\langle \Phi \rangle_T$ is comparable with its zero temperature value.

The fluctuation dominated region, where higher order terms in perturbation expansion are important, has been studied recently by Ginsparg⁶⁾. This is the analogue of the type II superconductor. In this mode the scalar quartic couplings are of the order g^2 . The phase transition is again found to be first order, but weakly first order so that it may be indistinguishable from a second-order transition for most phenomenological purposes.

These results support the intuition we have for gauge theories undergoing phase transition drawn from 3D superconductors. The superconducting phase transition for both type I and type II materials is predicted to be weakly first order⁷⁾.

In our analysis we shall use a Higgs potential $V(\Phi)$ with a negative mass squared term to drive the breaking and without a cubic term. In the $SU(5)$ theory a gauge invariant cubic term exists and it can be added to $V(\Phi)$ at the cost of introducing a dimensional cubic coupling⁸⁾. The presence of a strong cubic term easily induces a first-order phase transition. Nevertheless we shall refrain from introducing such a term at the tree level. We believe, somehow, that it is the temperature which should play the rôle of the cubic coupling mentioned above. Indeed at finite temperatures terms proportional to $T\langle \Phi \rangle_T^3$ could be present in the effective potential and it is these terms which signal the presence of a first-order transition. We are interested therefore in fluctuation-induced first-order transitions starting with a Higgs potential free of cubic terms at the tree level.

In Section 2 we examine the model at zero temperature for special values of the quartic couplings $\ll g^2$. The symmetry breaking is then dominated by one-loop gauge field radiative corrections and the system is in the Coleman-Weinberg⁹⁾ mode (but with a negative mass squared term in the Lagrangian). Section 3 presents a finite temperature one-loop calculation of the order parameter when the $SU(5)$

system is in a mode corresponding to the type I superconductor. The one-loop calculation is reliable and suffices to establish the existence of a first-order transition ^{3),10)}. In Section 4 we turn to the characterization of the nature of this first-order phase transition. Finally in Section 5 we discuss the relevance of our results to the monopole problem in grand unified theories and draw some conclusions.

2. - DYNAMICAL RESTORATION OF SYMMETRY AT $T = 0$

In this section we examine the $SU(5)$ model in the Coleman-Weinberg ⁹⁾ mode. This mode is characterized by the fact that the quartic couplings in the Higgs sector $\ll$ the squared gauge coupling.

Let $\hat{\mathcal{L}}$ be the Lagrangian density of the $SU(5)$ gauge theory without fermions. We follow the notation of the authors of Ref. 11) and write

$$\hat{\mathcal{L}} = -\frac{1}{4} \text{Tr} (G_{\mu\nu} G^{\mu\nu}) + \frac{1}{2} \text{Tr} (D_\mu \Phi)^2 - V(\Phi) \quad (2.1)$$

$$V(\Phi) = -\frac{1}{2} \mu^2 \text{Tr} \Phi^2 + \frac{1}{4} a (\text{Tr} \Phi^2)^2 + \frac{1}{2} b \text{Tr} \Phi^4 \quad (2.2)$$

with the Higgs multiplet Φ in the adjoint representation. Thus

$$G_{\mu\nu} = \frac{\lambda^i}{\sqrt{2}} G_{\mu\nu}^i, \quad W_\mu = \frac{\lambda^i}{\sqrt{2}} W_\mu^i, \quad \Phi = \frac{\lambda^i}{\sqrt{2}} \Phi^i$$

and

$$D_\mu \Phi = \partial_\mu \Phi - \frac{i}{\sqrt{2}} g [W_\mu, \Phi] \quad (2.3)$$

where λ^i ($i = 1, \dots, 24$) are the $SU(5)$ generators normalized to give $\text{Tr} \lambda^i \lambda^j = 2\delta^{ij}$. They satisfy the following (anti)-commutation relations

$$\begin{aligned} [\lambda^i, \lambda^j] &= 2i c^{ijk} \lambda^k \\ \{\lambda^i, \lambda^j\} &= \frac{4}{5} \delta^{ij} \mathbf{1} + 2 d^{ijk} \lambda^k \end{aligned}$$

Consider now $\mathcal{L} = \hat{\mathcal{L}} + \text{gauge fixing terms} + \text{F.P.} + \text{C.T.}$ To $\mathcal{L}$ we add a linear gauge invariant coupling of Φ to an external source J :

$$\mathcal{L} \longrightarrow \mathcal{L} + \text{Tr}(J\Phi) \quad (2.4)$$

J is a 5×5 matrix transforming according to the adjoint representation of the group.

If $b > 0$, $15a + 7b > 0$ ¹²⁾, then the $SU(5)$ breaks spontaneously to $SU(3)_c \times SU(2) \times U(1)$ by the vacuum expectation value of Φ ¹¹⁾. After the shift $\Phi = H + \Phi_0$ with

$$H = \left(\begin{array}{c|cc} H_8 + 2\sqrt{\frac{1}{30}} H^0 \mathbb{I} & \underline{H}_X & \underline{H}_Y \\ \hline \underline{H}_Y^\dagger & \sqrt{\frac{1}{2}} H_Z - 3\sqrt{\frac{1}{30}} H^0 & H^+ \\ \underline{H}_X^\dagger & H^- & -\sqrt{\frac{1}{2}} H_Z - 3\sqrt{\frac{1}{30}} H^0 \end{array} \right) \quad (2.5)$$

$$\Phi_0 = \langle 0 | \Phi | 0 \rangle = \phi_c \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ & & & & -\frac{3}{2} \end{pmatrix}$$

$\text{Tr}(J\Phi)$ yields $\sqrt{15/2} \phi_c J^0$.

Let $V(\phi_c)$ be the effective potential of the theory. For a minimum to occur at $\phi_c = \sigma \neq 0$ and for vanishing external current we must have

$$\frac{\sqrt{2}}{15} \left. \frac{dV(\phi_c)}{d\phi_c} \right|_{\phi_c=\sigma} = J^0 = 0 \quad (2.6)$$

$$\frac{2}{15} \left. \frac{d^2V(\phi_c)}{d\phi_c^2} \right|_{\phi_c=\sigma} = 2\mu^2 > 0 \quad (2.7)$$

Conditions (2.6) and (2.7) imply that the position of the minimum of $V(\phi_c)$ at $\phi_c = \sigma$ and the curvature of the effective potential at this point are fixed at the values obtained in the classical approximation. Indeed the effective potential at the tree level, $V_0(\phi_c)$, is given by

$$V_0(\phi_c) = \frac{15}{2} \left[-\frac{1}{2} \mu^2 \phi_c^2 + \frac{1}{8} (15a+7b) \phi_c^4 \right] \quad (2.8)$$

Hence

$$\left. \frac{dV_0}{d\phi_c} \right|_{\phi_c=\sigma} = 0$$

gives

$$\mu^2 = \frac{1}{2} (15a+7b) \sigma^2$$

and consequently

$$\frac{2}{15} \left. \frac{d^2 V_0}{d\phi_c^2} \right|_{\phi_c=\sigma} = 2\mu^2.$$

We consider the case when the scalar quartic couplings $a, b \ll g^2$.

$V(\phi_c)$ is now dominated by the one-loop radiative corrections from the gauge fields.

After the shift $\mathcal{L}$ contains a term $\frac{1}{2} W_\mu^i M_{ij}(\phi_c) W_\mu^j$, where

$$M_{ij}(\phi_c) = -\frac{1}{4} g^2 \text{Tr} \{ [\lambda^i, \Phi] [\lambda^j, \Phi] \}.$$

The matrix M is in fact proportional to the projection operator, P which projects onto the X, Y subspace of the vector boson mass matrix ¹¹⁾, i.e.,

$M = (25/8) g^2 \phi_c^2 P$. Working in the Landau gauge and using Λ as a momentum cut-off we obtain the following expression for the effective potential ⁹⁾ including the one-loop radiative corrections arising from the gauge fields :

$$V(\phi_c) = V_0(\phi_c) + V_{1,G}(\phi_c)$$

where

$$\begin{aligned} V_{1,G}(\phi_c) = \frac{36}{32\pi^2} \left\{ \frac{25}{8} g^2 \phi_c^2 \Lambda^2 + \frac{1}{2} \left(\frac{25}{8} g^2 \right)^2 \phi_c^4 \left[\ln \frac{25 g^2 \phi_c^2}{\Lambda^2} - \frac{1}{2} \right] \right\} \\ + \frac{1}{2} B \phi_c^2 + \frac{1}{4} C \phi_c^4 \end{aligned} \quad (2.9)$$

The last two terms in (2.9) come from the counter terms in $\mathcal{L}$.

Following Linde ^{13),3)} we use conditions (2.6) and (2.7) to renormalize the theory. Standard calculations lead to the effective potential

$$V(\phi_c) = V_0(\phi_c) + \frac{5625}{1024\pi^2} g^4 \left[\phi_c^4 \ln \frac{\phi_c^2}{\sigma^2} - \frac{3}{2} \phi_c^4 + 2\sigma^2 \phi_c^2 \right] \quad (2.10)$$

It follows from (2.10) that

$$\begin{aligned} \left. \frac{dV}{d\phi_c} \right|_{\phi_c=0} &= 0 \\ \frac{2}{15} \left. \frac{d^2 V}{d\phi_c^2} \right|_{\phi_c=0} &= -\mu^2 + \frac{375}{64\pi^2} g^4 \frac{\sigma^2}{2} \end{aligned}$$

Hence for $15a + 7b < (375/64\pi^2)g^4$ the effective potential acquires a new minimum at $\phi_c = 0$. Moreover the condition $V(\sigma) - V(0) > 0$ shows that the minimum at $\phi_c = 0$ is deeper than the minimum at $\phi_c = \sigma$ for $15a + 7b < (375/128\pi^2)g^4$. That is, the local minimum at $\phi_c = 0$ becomes the absolute minimum and we have restoration of symmetry. Thus symmetry breaking occurs only if

$$15a + 7b > \frac{375}{128\pi^2} g^4 \quad (2.11)$$

This result may be interpreted in the following way. For $a, b \ll g^2$ we have a strong first-order phase transition with relatively low critical temperature T_c . Moreover, $T_c \rightarrow 0$ as $15a + 7b \rightarrow (375/128\pi^2)g^4$ so that the symmetry is restored even at zero temperature. (Further support for this state of affairs comes from the finite temperature calculations in Section 3.) We remark also that the above calculation is a trivial generalization of what happens in the Abelian Higgs model ^{13),3)}.

It is of interest to express the condition (2.11) on the couplings ^{*)} in terms of the masses $m_X = m_Y$ of the superheavy gauge bosons and the mass m_{H_0} of

^{*)} Strictly speaking a , b and g^2 are not the coupling constants. To clarify this point we take $b = 0$. Then the constant $a > 0$ defined by (2.6)-(2.7) has the meaning of $2u^2/15\sigma^2$, but is not equal to the coupling constant a_{int} , of the scalar fields on mass shell. In the tree approximation $a = a_{int}$, but in the single loop approximation we consider (for $a \ll g^2$) we have

$$a_{int} = \left(\frac{2}{15}\right)^2 \frac{1}{6} \left. \frac{d^4 V}{d\phi_c^4} \right|_{\phi_c=\sigma} = a + O(g^4).$$

Similarly $g_{int}^2 = g^2 + O(g^4) \approx g^2$.

one of the physical Higgs fields. To this end we recall that, at the tree level ¹¹⁾, $m_{H^0}^2 = (15a+7b)\sigma^2$ and $m_X^2 = m_Y^2 = (25/8)g^2\sigma^2$. The inequality (2.11) now implies that

$$\frac{m_{H^0}^2}{m_X^2} > \frac{15}{4\pi} \alpha_{GUM} \quad (2.12)$$

where $\alpha_{GUM} = (g^2/4\pi)$ is the "fine structure constant" of the SU(5) theory at the grand unification scale.

3. - FINITE TEMPERATURE CALCULATIONS

First we determine the stability region ($m_i^2 > 0$, where m_i are the scalar field masses) of the disordered phase $\phi_c = 0$. The state with $m_i^2 > 0$ is locally stable although possibly metastable. To this end we work in the Landau gauge and we consider the contributions to the polarization operators $\Pi_i(T, k)$ coming from the diagrams of Fig. 1. Recall that in the Landau gauge there is no coupling between the scalar and ghost fields. Moreover derivative couplings arising from (2.3) give a zero contribution to $\Pi_i(T, 0)$.

The scalar masses, $m_i(T)$, as functions of temperature ^{*)} are determined by

$$\Pi_i(T, k) \Big|_{\substack{\vec{k}=0 \\ k_0^2 = -m_i^2(T)}} = -\mu^2 - m_i^2(T) + \Pi_i^R(T, 0) = 0 \quad (3.1)$$

To calculate Π_i^R we need the scalar quartic couplings and the four-vertex coupling from (2.3). These are given, respectively, by

$$\frac{a}{4} \left(\sum_i (H^i)^2 \right)^2 + \frac{b}{8} \sum_{i,j,k,\ell} H^i H^j H^k H^\ell \left(\frac{4}{5} \delta^{ij} \delta^{kl} + 2 \sum_n d^{ijn} d^{kln} \right) \quad (3.2)$$

*) For a discussion of what is meant by a particle mass at finite temperature we refer to Ref. 3).

and

$$-\frac{1}{16} g^2 W_\mu^i W_\mu^k H^i H^k \text{Tr}([\lambda^i, \lambda^j][\lambda^k, \lambda^l]) \quad (3.3)$$

Taking only leading terms ($\sim T^2$) in $\Pi_i^R(T, 0)$ we get an expression for Π_i^R which is independent of i , namely

$$\Pi_i^R(T, 0) = \left(\frac{15}{2} g^2 + 13a + \frac{141}{15} b \right) \frac{T^2}{6} \quad (3.4)$$

Hence, (3.1) gives

$$m_i^2(T) = \frac{\frac{225}{2} g^2 + 13(15a + 7b) + 50b}{90} (T^2 - T_1^2) \quad (3.5)$$

where

$$T_1^2 = \frac{90\mu^2}{\frac{225}{2} g^2 + 13(15a + 7b) + 50b} \quad (3.6)$$

The above formulae hold provided $T \gg m_i(T)$, $m_A(T)$ where m_A denotes the mass of the vector fields which are essentially massless.

Now we analyze the ordered phase. From this analysis it turns out that T_1 is not a critical temperature, but rather the limit of metastability of the disordered phase. As in Section 2 we perform the shift $\Phi \rightarrow H + \Phi_0$, with $\langle H \rangle = 0$ and $\langle \Phi_0 \rangle = \phi_c \text{diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2})$. Here $\langle \psi \rangle$ denotes the thermal average of ψ over a Gibbs ensemble at temperature T rather than the vacuum expectation value. Consequently the ordered parameter ϕ_c becomes temperature dependent. It is convenient also to introduce a mass term, with $m_1^2 > 0$, for every Higgs scalar subtracting at the same time the same terms from the interaction Lagrangian ³⁾.

After the shift we have four-particle, three-particle, and one-particle vertices. The term

$$-\mu^2 \text{Tr}(\Phi_0 H) + a (\text{Tr} \Phi_0^2) (\text{Tr}(\Phi_0 H)) + 2b \text{Tr}(\Phi_0^3 H)$$

gives rise to a one-particle vertex with coupling $\sqrt{15/2} \phi_c [-\mu^2 + \frac{1}{2}(15a+7b)\phi_c^2]$. Observe that this coupling is equal to $\sqrt{2/15}(dV_0/d\phi_c)$, as it should be. The three-particle vertices with two gauge fields, see Fig. 2, arising from the covariant derivative (2.3) give a coupling $\sqrt{2/15}(25/8)g_c^2 \delta_{\mu\nu} \delta_{ij}$ for indices $i, j = 13, \dots, 24$ which label the twelve superheavy gauge bosons. The term of $V(H+\Phi_0)$ given by

$$a \text{Tr}(H^2) \text{Tr}(\Phi_0 H) + 2b \text{Tr}(\Phi_0 H^3)$$

gives rise to three-scalar field vertices, see Fig. 2, with couplings

$$\begin{aligned} \sqrt{\frac{2}{15}} \left(\frac{15}{2} a + 6b \right) \phi_c \delta_{ij} &, \quad i, j = 1, \dots, 8 \\ \sqrt{\frac{2}{15}} \left(\frac{15}{2} a + \frac{27}{2} b \right) \phi_c \delta_{ij} &, \quad i, j = 9, 10, 11 \\ \sqrt{\frac{2}{15}} \left(\frac{15}{2} a + \frac{7}{2} b \right) \phi_c &, \quad i, j = 0 \\ \sqrt{\frac{2}{15}} \left(\frac{15}{2} a + \frac{7}{2} b \right) \phi_c \delta_{ij} &, \quad i, j = 13, \dots, 24 \end{aligned}$$

The indices taking values $1, \dots, 8$ label the eight Higgs fields that correspond to $SU(3) \subset SU(5)$, and those taking values $9, 10, 11$ go with $SU(2) \subset SU(5)$. The index 0 corresponds to H^0 and the remaining values $13, \dots, 24$ label the twelve unphysical Higgs fields H_X, H_Y and their Hermitian conjugates.

In the ordered phase we consider the equilibrium equation

$$J^0 = \sqrt{\frac{2}{15}} \frac{dV}{d\phi_c} = 0 \quad (3.7)$$

This has the diagrammatic representation shown in Fig. 3. Thus, we get

$$\sqrt{\frac{15}{2}} \phi_c \left[-\mu^2 + \frac{1}{2} (15a+7b) \phi_c^2 \right] + \Gamma_{(T)}^R = 0 \quad (3.8)$$

where $\Gamma_{(T)}^R$ is the renormalized sum of one-tail diagrams at temperature T . In the lowest order of perturbation theory $\Gamma_{(T)}$ is given by

$$\begin{aligned} \Gamma_{(T)} = & \sqrt{\frac{2}{15}} \left\{ \sum_{i=1}^8 \left(\frac{15}{2} a + 6b \right) \phi_c G_{(T, m_i^2)} + \sum_{i=9}^{11} \left(\frac{15}{2} a + \frac{27}{2} b \right) \phi_c G_{(T, m_i^2)} \right. \\ & + 3 \left(\frac{15}{2} a + \frac{7}{2} b \right) \phi_c G_{(T, m_0^2)} + \sum_{i=13}^{24} \left(\frac{15}{2} a + \frac{7}{2} b \right) \phi_c G_{(T, m_i^2)} \\ & \left. + \frac{225}{2} g^2 \phi_c G_{(T, \frac{25}{8} g^2 \phi_c^2)} \right\} \end{aligned} \quad (3.9)$$

where

$$G_{(T, m^2)} = \frac{T}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \int \frac{d^3 \vec{k}}{(2\pi n T)^2 + \vec{k}^2 + m^2}$$

Summation over n yields

$$G_{(T, m^2)} = \frac{1}{4\pi^2} \int_0^{\infty} \frac{k^2 dk}{(k^2 + m^2)^{3/2}} + F_{(T, m^2)}$$

with the temperature dependent term $F_{(T, m^2)}$ given by

$$F_{(T, m^2)} = \frac{1}{2\pi^2} \int_0^{\infty} \frac{k^2 dk}{(k^2 + m^2)^{3/2} \left[e^{\frac{(k^2 + m^2)^{1/2}}{T}} - 1 \right]} \quad (3.10)$$

The temperature-independent term is divergent and gets renormalized by the counter terms of the "cold" theory.

In fact to renormalize $\Gamma_{(T)}$ we note that according to (3.8) $\Gamma_{(0)}^R = 0$ because $\phi_c(T=0) = \sigma$ and $\mu^2 = \frac{1}{2}(15a+7b)\sigma^2$. Hence we have

$$\begin{aligned} \Gamma_{(T)}^R &= \Gamma_{(T)} - \Gamma_{(0)} \\ &= \sqrt{\frac{2}{15}} \left\{ \sum_{i=1}^8 \left(\frac{15}{2} a + 6b \right) \phi_c F_{(T, m_i^2)} \right. \\ &\quad \left. + \sum_{i=9}^{11} \left(\frac{15}{2} a + \frac{27}{2} b \right) \phi_c F_{(T, m_i^2)} + 3 \left(\frac{15}{2} a + \frac{7}{2} b \right) \phi_c F_{(T, m_0^2)} \right\} \end{aligned}$$

$$+ \sum_{i=13}^{24} \left(\frac{15}{2}a + \frac{7}{2}b \right) \phi_c F_{(T, m_i^2)} + \frac{225}{2} g^2 \phi_c F_{(T, \frac{25}{8} g^2 \phi_c^2)} \} \quad (3.11)$$

The integral $F_{(T, m^2)}$ [see Eq. (3.10)] can be expanded as a series in terms of m/T ⁵⁾. Then we have

$$F_{(T, m^2)} = T^2 \left[\frac{1}{12} - \frac{1}{4\pi} \frac{m}{T} + O\left(\left(\frac{m}{T}\right)^2, \left(\frac{m}{T}\right)^2 \ln \frac{m}{T}\right) \right] \quad (3.12)$$

In the limit $T \gg m_i$, $(5\sqrt{2}/4)g\phi_c(T)$ only the leading term in F contributes. Consequently we obtain

$$\phi_c^2(T) = \frac{\frac{225}{2} g^2 + 13(15a+7b) + 50b}{45(15a+7b)} (T_1^2 - T^2)$$

The order parameter $\phi_c(T)$ vanishes at $T = T_1$ which is also the boundary of the region where the disordered phase is stable. Of course, this is the typical picture of a second order phase transition. This conclusion is valid, however, only if we neglect the masses in computing $F_{(T, m^2)}$ in Eq. (3.11), and if T is not too close to the critical temperature.

Now the masses m_i included in the Lagrangian are regarded as fixed and independent of T . After the calculation of $\Gamma_{(T)}^R$ is carried out, m_i is identified with $m_i(T)$, the quasi-particle mass at temperature T [see Ref. 3] for details]. The equilibrium values of $m_i^2(T)$ are defined by looking at the polarization operators in the ordered phase. When the $SU(5)$ theory is in a type I mode we may contemplate a situation where $m_i(T) < T \lesssim T_G$, where T_G is the Ginzburg temperature (to be defined later). The right-hand side of the inequality ensures that T is not too close to the critical temperature and consequently the one-loop calculations at temperature T are reliable. The left-hand side indicates that terms of $O(m/T)$ could be comparable to the $O(1)$ term in (3.12). Taking such terms into account the expression for $\Gamma_{(T)}^R$ now becomes :

$$\begin{aligned}
 \Gamma_{(T)}^R = & \sqrt{\frac{2}{15}} \left[\sum_{i=1}^8 \left(\frac{15}{2} a + 6b \right) \phi_c \left(\frac{T^2}{12} - \frac{m_i T}{4\pi} \right) \right. \\
 & + \sum_{i=9}^{11} \left(\frac{15}{2} a + \frac{27}{2} b \right) \phi_c \left(\frac{T^2}{12} - \frac{m_i T}{4\pi} \right) \\
 & + 3 \left(\frac{15}{2} a + \frac{7}{2} b \right) \phi_c \left(\frac{T^2}{12} - \frac{m_0 T}{4\pi} \right) \\
 & + \sum_{i=13}^{24} \left(\frac{15}{2} a + \frac{7}{2} b \right) \phi_c \left(\frac{T^2}{12} - \frac{m_i T}{4\pi} \right) \\
 & \left. + \frac{225}{2} g^2 \phi_c \left(\frac{T^2}{12} - \frac{\frac{5\sqrt{2}}{2} g \phi_c T}{4\pi} \right) \right] \quad (3.13)
 \end{aligned}$$

Consider, now, the polarization operators in the ordered phase. Due to the presence of three-particle vertices there are many more diagrams to be considered. First there are the diagrams which occur also in the disordered phase. Moreover there are the diagrams shown in Fig. 4. These can be estimated to be small compared with the leading contribution arising from the loop-diagrams of Fig. 1. Also the tadpole diagrams of the type shown in Fig. 5 give a zero contribution because of Eq. (3.8). The tree contribution to the polarization operators in the ordered phase arises from the terms

$$\begin{aligned}
 & a \left[(\text{Tr } \Phi_0 H)^2 + \frac{1}{2} (\text{Tr } \Phi_0^2) (\text{Tr } H^2) \right] \\
 & + b \left[2 \text{Tr } (\Phi_0^2 H^2) + \text{Tr } (\Phi_0 H \Phi_0 H) \right]
 \end{aligned}$$

which yield the following vertices

$$\begin{aligned}
 & \left(\frac{15}{4} a + \frac{6}{2} b \right) \phi_c^2, \quad i = 1, \dots, 8 \\
 & \left(\frac{15}{4} a + \frac{27}{4} b \right) \phi_c^2, \quad i = 9, 10, 11 \\
 & 3 \left(\frac{15}{4} a + \frac{7}{4} b \right) \phi_c^2, \quad i = 0 \\
 & \left(\frac{15}{4} a + \frac{7}{4} b \right) \phi_c^2, \quad i = 13, \dots, 24
 \end{aligned} \quad (3.14)$$

Gathering everything together we have the following equations

$$-\mu^2 - m_i^2 + \frac{1}{2}(15a + 12b)\phi_c^2 + \Pi_i^R(T, 0) = 0, \quad i=1, \dots, 8 \quad (3.15)$$

$$-\mu^2 - m_i^2 + \frac{1}{2}(15a + 27b)\phi_c^2 + \Pi_i^R(T, 0) = 0, \quad i=9, 10, 11 \quad (3.16)$$

$$-\mu^2 - m_0^2 + \frac{3}{2}(15a + 7b)\phi_c^2 + \Pi_0^R(T, 0) = 0 \quad (3.17)$$

$$-\mu^2 - m_i^2 + \frac{1}{2}(15a + 7b)\phi_c^2 + \Pi_i^R(T, 0) = 0, \quad i=13, \dots, 24 \quad (3.18)$$

Π_i^R ($i = 0, 1, \dots, 12, \dots, 24$) is computed by taking only the loop diagrams of Fig. 1. The leading term in Π_i^R is $\sim T^2$ and is independent of the masses. So taking into consideration only such leading contributions and putting the masses of the unphysical Higgs particles equal to zero, that is $m_i^2(T) = 0$, $i = 13, \dots, 24$, the system of equations (3.15)-(3.18) yields

$$m_i^2(T) = \begin{cases} \frac{5}{2} b \phi_c^2(T), & i=1, \dots, 8 \\ 10b \phi_c^2(T), & i=9, 10, 11 \\ (15a + 7b)\phi_c^2(T), & i=0 \\ 0, & i=13, \dots, 24 \end{cases} \quad (3.19)$$

Substituting (3.19) into (3.13) and using (3.8) we finally get

$$\begin{aligned} & \frac{15}{4}(15a + 7b)\phi_c^2 \\ & - \frac{T}{4\pi} \left[\frac{7}{2}(15a + 7b)(10b)^{1/2} + 4(10b)^{3/2} + \frac{3}{2}(15a + 7b)^{3/2} + \frac{1125\sqrt{2}}{8}g^3 \right] \phi_c \\ & + \frac{T^2}{12} \left[\frac{225}{2}g^2 + 13(15a + 7b) + 50b \right] - \frac{15}{2}\mu^2 = 0 \end{aligned} \quad (3.20)$$

It follows directly from (3.20) that $\phi_c = 0$ at $T = T_1$. However, in $T_1 \leq T \leq T_2$, where

$$T_2^2 = T_1^2 \left\{ 1 - \frac{\left[\frac{7}{2}(15a+7b)(10b)^{\frac{1}{2}} + 4(10b)^{\frac{3}{2}} + \frac{5}{2}(15a+7b)^{\frac{3}{2}} + \frac{1125\sqrt{2}}{8}g^3 \right]^2}{20\pi^2(15a+7b)\left[\frac{225}{2}g^2 + 13(15a+7b) + 50b \right]} \right\}^{-1} \quad (3.21)$$

there are two solutions $\phi_c^{(1,2)} \neq 0$, $\phi_c^{(1)} \geq \phi_c^{(2)}$. The solution $\phi_c^{(2)}$ is unstable and corresponds to a maximum of the effective potential. The temperature T_2 is the limit of metastability of the ordered phase. The phase transition occurs at $T = T_c$ ($T_1 \leq T_c \leq T_2$) where T_c satisfies the equation

$$V(\phi_c^{(1)}, T_c) = V(0, T_c) \quad (3.22)$$

In the above calculations we have neglected higher order perturbation corrections. This is justified provided we stay at temperatures which lie outside a certain critical region. This region is determined by the Ginzburg temperature T_G , which is defined as follows^{14),15)}: let λ be the largest scalar coupling of the theory normalized so that $\lambda^{\frac{1}{2}}\phi_c(T)$ is the largest scalar mass at temperature T . Then

$$T_G \approx \frac{4\pi\phi_c(T_G)}{\lambda^{\frac{1}{2}}} \quad (3.23)$$

and $|T_c - T_G|$ defines a fluctuation dominated region known as the critical region around T_c . So, in the ordered phase, large fluctuations are not important for temperatures below T_G . For such temperatures the Ginzburg criterion

$$\frac{T\lambda^{\frac{1}{2}}}{4\pi} \lesssim \phi_c(T) \quad (3.24)$$

holds.

In Ref. 3) there is an estimate of the critical region, both for the ordered and disordered phases, in a simple model with one neutral self-interacting scalar field with coupling λ . In the disordered phase a factor $\lambda T/m(T)$ appears in higher orders of perturbation theory. It is clear that near T_c , where $m(T) \rightarrow 0$, the rôle of higher orders can be significant even for small couplings λ . When T satisfies the Ginzburg criterion (3.24), however, $\lambda T/m(T)$ is bounded by a constant and higher order contributions can be neglected. In the

the ordered phase the analysis is more complicated since additional vertices appear with coupling $\lambda\phi_c(T)$. [We have already encountered this situation in the SU(5) model, when we calculated the polarization operators in the ordered phase - see the diagrams of Fig. 4 and Fig. 5.] In this phase also for temperatures below T_G higher order contributions are negligible.

We conclude this section with the following remarks. The one-loop calculations in the ordered phase for the SU(5) model under consideration are reliable provided

$$m(T) \equiv \max\{m_i(T), m_{X,Y}(T)\} < T \lesssim T_G \quad (3.25)$$

In particular T_1 must lie within the region specified by (3.25). This ensures that the phase transition (with supercooling) occurs outside the critical region.

4. - THE NATURE OF THE PHASE TRANSITION

In this section we shall examine the nature of the phase transition $SU(5) \rightarrow SU(3)_c \times SU(2) \times U(1)$. To this end we introduce two parameters θ and ζ which measure the strength of the quartic couplings a and b in units of g^2 . Thus we define

$$\begin{aligned} 15a + 7b &= \theta g^2 \\ b &= \zeta g^2 \end{aligned} \quad (4.1)$$

The one-loop calculations and approximations in Section 3, in the region where they are valid, show that the phase transition at the grand unified scale is necessarily first order. It remains to examine whether it is weakly or strongly first order. In a weak transition the discontinuity in the expectation value of the order parameter at $T = T_c$ is small compared with its value at $T = 0$.

It follows from (3.23) that the estimates of Section 3 are reliable in a region of temperature satisfying

$$m(T) < T \lesssim T_G \quad (4.2)$$

In particular T_1 , the temperature of the limit of metastability of the disordered phase, must satisfy (4.2). It follows from (3.6), (3.19), (3.23) and (4.1) that

$$T_1^2 = \frac{45 \phi_c^2(0)}{\frac{225}{2\theta} + 13 + 50 \frac{\zeta}{\theta}} \quad (4.3)$$

and

$$T_G^2 \approx \frac{16 \pi^2 \phi_c^2(T_G)}{[\max\{\sqrt{10\zeta}, \sqrt{\theta}\}]^2 g^2} \quad (4.4)$$

It is clear from (4.4) that, for fixed values of ζ , T_G is a decreasing function of θ . The inequality $T_1 \lesssim T_G$ imposes, therefore, an upper bound for θ provided an estimate of $\phi_c(T_G)/\phi_c(0)$ can be given. This ratio is expected to be $\lesssim 0(10^{-1})$ for a weak first order transition, whereas for a strong one $(\phi_c(T_G)/\phi_c(0)) \sim 1$.

In order to investigate the validity of (4.2) we need to introduce a measure for the strength of the transition. We recall that a strong first order phase transition is always associated with a big hysteresis loop. We can adopt, therefore, the following measure : the extend to which the transition differs from a continuous transition (i.e., second order) is measured by the quantity Δ , where

$$\Delta = \frac{T_2 - T_1}{T_1} \quad (4.5)$$

It follows from (3.21), at least whenever (3.21) is valid, that this quantity depends solely on the parameters θ, ζ and g^2 of the theory. Of course, some caution is required in using an expression such as Eq. (3.21). This equation is valid provided the calculations of Section 3 are reliable. Our intention is to use (3.21) in order to get an expression for Δ and then check a posteriori the reliability of the calculations.

Consider the expression $T_2 = T_1 [1 - F(\theta, \zeta, g^2)]^{\frac{1}{2}}$ obtained from (3.21) and (4.1). This relation is valid provided $m(T_1) < T_1 \lesssim T_G$. For fixed ζ and g^2 let θ^* satisfy $F(\theta^*, \zeta, g^2) = 1$. The value $\theta = \theta^*$ is not attainable because the left-hand side of the above inequality is violated. In the vicinity of θ^* and for $\Delta\theta = \theta - \theta^*$ we estimate Δ to be

$$\Delta \approx \left[- \frac{\partial F}{\partial \theta} (\theta^*, \zeta, g^2) \right]^{-\frac{1}{2}} (\Delta \theta)^{-\frac{1}{2}} - 1$$

It turns out that, for $\frac{1}{32} \lesssim \zeta \lesssim \frac{1}{2}$, $g^2 = 0.3$ (inferred from the best value of α_{GUM}) and $\Delta \theta \sim O(2 \cdot 10^{-1})$, $\Delta \sim O(1)$. Hence, if γ is the curve in the a - b plane defined by $F(\theta^*, \zeta, g^2) = 1$, $\frac{1}{32} \lesssim \zeta \lesssim \frac{1}{2}$, we can come close to γ while we maintain $\Delta \sim O(1)$. For a fixed value of ζ in the interval $\frac{1}{32} \lesssim \zeta \lesssim \frac{1}{2}$ and $g^2 = 0.3$ let $\theta_{\min}$ be the value of θ closest to θ^* for which the inequality $m(T_1) < T_1 \lesssim T_G$ is satisfied while $\Delta \lesssim O(1)$. For the same fixed values for ζ and g^2 the inequality $T_1 \lesssim T_G$ imposes an upper bound, $\theta_{\max}$, for θ . It turns out that $\theta_{\max} - \theta^* \gg O(10^{-1})$. We now change ζ and for the same value of g^2 we estimate $\theta_{\min}$ and $\theta_{\max}$ by repeating the above procedure. In this way we span a certain region in the a - b plane shown in Fig. 6. Inside this region $\Delta \lesssim O(1)$ and the phase transition is weakly first order. The lower boundary of this region, labelled by A , and the straight lines $b = 0$, $15a + 7b = 0$ specify another region where the phase transition is expected to be strongly first order. The main characteristic of this region is that the mass of one or more of the scalar fields at temperature T_1 is much smaller than T_1 , whereas the rest of the particles have masses much bigger than T_1 .

Let us examine the case $a, b \ll g^4$. This corresponds to a small region right at the corner of the wedge defined by the lines $b = 0$, $15a + 7b = 0$. In this case it can be checked that $m_i(T) \ll T_1 \ll m_X(T_1), m_Y(T_1)$ where $m_i(T_1)$ are the scalar field masses and $m_X(T_1), m_Y(T_1)$ the gauge field masses at T_1 . Let T_c be the critical temperature for the phase transition. It is clear that $T_c \gg m_i(T_c)$. If at the same time $T_c \ll m_X(T_c), m_Y(T_c)$, then the contribution to the effective potential coming from the gauge bosons is exponentially suppressed at $T = T_c$. In this case (3.22) yields

$$\begin{aligned} & \frac{15}{2} \left[\frac{\mu^2}{2} \phi_c^{(1)2}(T_c) - \frac{1}{8} (15a + 7b) \phi_c^{(1)4}(T_c) \right] + \frac{48}{90} \pi^2 T_c^4 \\ &= \frac{72}{90} \pi^2 T_c^4 \end{aligned} \quad (4.6)$$

The factor $48\pi^2 T_c^4 / 90$ on the left-hand side of (4.6) arises from the 24 scalar and 12 gauge bosons which are relativistic at $T = T_c$. Let T_2 be the limit of metastability of the ordered phase. If $T_c \ll T_2$, then we can approximate $\phi_c^{(1)}(T_c) \sim \phi_c(0) = \sqrt{2\mu^2 / (15a + 7b)}$. Equation (4.6) yields

$$T_c = \left[\frac{225}{16\pi^2(15a+7b)} \right]^{1/4} \mu \quad (4.7)$$

It can be checked a posteriori that $T_c \ll m_X(T_c), m_Y(T_c)$. Moreover $T_1 \ll T_c$ and this is an indication for a strong first-order transition with a big hysteresis loop.

5. - CONCLUSION AND DISCUSSION

Within the minimal GUT model we have studied the phase transition from the disordered $SU(5)$ phase into the ordered $SU(3)_c \times SU(2) \times U(1)$ phase. Our study is limited to the case where the transition occurs outside the fluctuation dominated region. For generic values of the coupling constants the transition is weakly first order. The stability region of the $SU(3)_c \times SU(2) \times U(1)$ phase is defined by $b > 0, 15a + 7b > 0$. As we approach the boundaries of this region, a strong first-order transition occurs. The results of Ginsparg⁶⁾, in the fluctuation dominated region, when applied to the $SU(5)$ model show that there is a continuous merging of the type I mode into the type II mode.

The gross features of the fluctuation induced phase transition in the $SU(5)$ model are expected to hold in other grand unified models. At least the asymptotically free $SU(N)$ and $O(N)$ gauge theories coupled to scalars belonging to M vector, adjoint, second rank tensor, adjoint plus one vector representations, are expected to have a weak first-order phase transition⁶⁾. For these theories our analysis is also expected to go through, whenever the phase transition occurs outside the critical region. Equation (3.21) would then show that, for scalar couplings $\sim g^2$, $T_2^2 = T_1^2 [1 - O(g^2)]^{-1}$. As a result $\Delta \sim O(10^{-1})$ and the phase transition is weakly first order. So, in the perturbative regime, a rule of thumb is that the phase transition at the grand unification scale is weakly first order for generic values of the coupling constants. A strong first-order transition is not impossible, but it requires in general some of the self-couplings in the Higgs sector to be $\ll$ the squared gauge coupling.

In the analysis of the monopole problem Preskill¹⁵⁾ has assumed a second-order phase transition to occur at the grand unification scale. It is clear that the situation will not change drastically if, instead of dealing with a second-order transition, one deals with a weak first-order one. It is of interest to see whether the monopole problem could be resolved within the framework of a weak, fluctuation induced, first-order transition. There has been some progress¹⁶⁾ in this direction recently.

On the other hand, if the resolution of the monopole problem requires a large degree of supercooling, i.e., a strong first-order transition, then two possibilities may arise. Either we have to choose a model with a very weak Higgs sector, or we may introduce (if gauge invariance allows it) a strong cubic term in the Higgs potential. For the minimal $SU(5)$ model Guth and Tye⁸⁾ have introduced such a cubic term. They argue that, in the presence of the cubic term, the phase transition $SU(5) \rightarrow SU(3)_C \times SU(2) \times U(1)$ probably goes through an intermediate $SU(4) \times U'(1)$ phase. This phenomenon is interesting and deserves further analysis.

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FIGURE CAPTIONS

In Fig. 1 - Fig. 5 solid lines refer to any one of the scalar fields and wavy lines to gauge fields.

Figure 1 Polarization operator diagrams.

Figure 2 Cubic couplings in the ordered phase.

Figure 3 Diagrammatic representation of the equilibrium equation (3.7).

Figure 4, Figure 5 Extra polarization operator diagrams in the ordered phase.

Figure 6 Inside the stability region ($b > 0$, $15a+7b > 0$) of the $SU(3)_c \times SU(3) \times U(1)$ phase there exists a region bounded by A and B where the phase transition is weakly first order. The phase transition continues to be weakly first order beyond B, but this region corresponds to a fluctuation dominated phase transition. In the region bounded by A and the boundaries of the stability region of the $SU(3)_c \times SU(2) \times U(1)$ phase a strongly first-order transition is expected.

Fig. 1

Fig. 2

$$H^0 \text{ tadpole} + \sum \text{tadpole with loop} H^0 + 3 \sum \text{tadpole with wavy loop} H^0 + \dots = 0$$

Fig. 3

Fig. 4

Fig. 5

Fig. 6