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A SUPERSPACE APPROACH
TO EXTENDED CONFORMAL SUPERGRAVITY

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ABSTRACT

Extended conformal supergravity is
discussed in a superspace context.

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Conformal supergravity^{1),2)} has proved to be of use in the construction of off-shell Poincaré theories³⁾, at least for the known cases ($N = 1$ and 2), and it is therefore of interest to see whether or not the conformal approach can be employed for $N > 2$. Recently, the full off-shell $N = 4$ conformal theory has been constructed in component form⁴⁾, and it is widely believed that this is the highest value of N for which a consistent action may be constructed (there are fields with spins greater than 2 for $N > 4$). In this note we show how to obtain the component fields for extended conformal supergravity theories in superspace for $N \leq 4$, and the very fact that we use superspace guarantees that we shall obtain a set of fields which will form an (off-shell) representation of supersymmetry. We shall also discuss briefly some problems and possibilities for $N \geq 5$.

We recall that the superconformal algebra $SU(2,2|N)$ contains generators corresponding to translations (P), Lorentz transformations (M), dilations (D), conformal boosts (K), $2N$ supersymmetries (Q and S) and an internal $SU(N)$ [$SU(N)$ is possible only for $N = 4$]. The x space versions of conformal supergravity^{1),2),4)} have been constructed by gauging the entire group and then imposing constraints on the group curvatures, thus eliminating the gauge fields corresponding to M , D , K and S symmetries and ensuring that gauged translations are related to general co-ordinate transformations. In this paper we shall adopt a slightly different viewpoint and explicitly gauge only the subgroup $SL(2, C) \times U(N)$ [although again we will have no independent $SL(2, C)$ gauge field]. P , Q and K transformations will then be subsumed in general superspace co-ordinate transformations whilst dilations and S supersymmetry will appear as additional symmetries as in the superspace version of the $N = 1$ theory⁵⁾ [although there, the local $U(1)$ was not explicitly gauged either]. This approach seems to be more directly connected to the Poincaré theory and has the additional advantage of being almost simple.

We consider a superspace with co-ordinates $Z^M = (x^m, \theta^{\underline{\mu}}, \bar{\theta}^{\dot{\underline{\mu}}})$ where the x^m are four commuting co-ordinates and the $\theta^{\underline{\mu}}$ $2N$ anticommuting co-ordinates with complex conjugates $\bar{\theta}^{\dot{\underline{\mu}}}$. The geometry is describable in terms of the vielbein E_M^A and the connection $\Omega_{M,A}^B$ where the tangent space indices (A, B) are acted upon by $SL(2, C) \times U(N)$ *). Thus a vector X^A transforms as follows

*) Curved (tangent space) indices are taken from the middle (beginning) of the alphabet; small Latin indices are internal, Greek are spinorial and Gothic vectorial; underlined Greek are combined spinorial and internal whilst capitals are used for all types together. For a general review of superspace geometry see, e.g., Ref. 6).

$$\delta X^A = X^B L_B^A \quad (1)$$

where

$$\begin{aligned} L_{\mu}^{\alpha} &= -L^{\mu}_{\alpha} \\ L_{\beta}^{\alpha} &= \delta_{\beta}^{\alpha} L^b_a + \delta_a^b L_{\beta}^{\alpha} \\ L_{\beta}^{\alpha} &= \frac{1}{2} (G^{\mu\nu})_{\beta}^{\alpha} L_{\mu\nu} ; \quad \overline{L}_a^b = -L^b_a \quad (2) \\ L_{\beta}^{\dot{\alpha}} &= -(\overline{L}_{\beta}^{\dot{\alpha}}) \end{aligned}$$

and all other components, e.g., L_{α}^b are zero. The connection transforms inhomogeneously.

$$\delta \Omega_{M,A}^B = -\partial_M L_A^B + \Omega_{M,A}^C L_C^B - (-1)^{M(A+C)} L_A^C \Omega_{M,C}^B \quad (3)$$

and allows one to construct a covariant derivative, e.g.,

$$D_M X^A = \partial_M X^A + (-1)^{MB} X^B \Omega_{M,B}^A \quad (4)$$

The field strengths are the torsion T_{AB}^C and the curvature R_{ABC}^D which are constructed in the usual way⁽⁶⁾ and the Bianchi identities are

$$0 = I_{ABC}^D = \sum_{(ABC)} (R_{ABC}^D - D_A T_{BC}^D - T_{AB}^E T_{EC}^D) \quad (5)$$

$$0 = I_{ABC D}^E = \sum_{(ABC)} (D_A R_{BC D}^E + T_{AB}^F R_{FCD}^E) \quad (6)$$

where the summations are generalized cyclic. It has been shown⁽⁷⁾ that because of the structure of the tangent space group, (5) allows one to obtain R_{ABC}^D in terms of T_{AB}^C and its derivatives, whilst (6) is automatically satisfied if (5) is.

The problem is therefore to find constraints on the torsion which are consistent with the Bianchi identities and which reduce the representation furnished by E_M^A and Ω_{MA}^B to an irreducible one without obtaining equations of motion.

It transpires that (for $N \neq 4$) the only non-conventional constraints^{*)} we need impose are at dimension zero where we choose

$$\begin{aligned} T_{\alpha\beta, \gamma\delta} &= -2i \delta_{\beta}^{\alpha} \epsilon_{\alpha\gamma} \epsilon_{\beta\delta} \\ T_{\alpha\beta, \gamma\dot{\delta}} &= 0 \end{aligned} \quad (7)$$

At dimension one-half (mass) we have

$$T_{\alpha\beta}{}^{\gamma}{}_{\dot{\delta}} = T_{\alpha\beta}{}^{\dot{\gamma}}{}_{\delta} = T_{\alpha\beta}{}^{\gamma\delta} = 0 \quad (8)$$

and

$$T_{\alpha\beta}{}^{\dot{\gamma}}{}_{\delta} = \epsilon_{\alpha\beta} \bar{\chi}^{abc} \delta^{\dot{\gamma}}{}_{\delta} \quad (9)$$

where $\bar{\chi}^{abc\dot{\gamma}}$ is totally skew on its internal indices and is a superfield whose leading component (for $N \geq 3$) corresponds to the dimension one-half spinors in the theory. We observe that these equations are identical to those for on-shell extended Poincaré supergravity in superspace⁹⁾. At dimension one we then find (having imposed $T_{\alpha\beta}{}^{\gamma}{}_{\dot{\delta}} = 0$)

$$T_{\alpha\dot{\alpha}, \beta\dot{\beta}}{}^{\gamma\dot{\gamma}}{}_{\delta\dot{\delta}} = i \epsilon_{\alpha\beta} \bar{M}_{\dot{\beta}\dot{\delta}}{}^{\gamma\dot{\gamma}} + i \epsilon_{\dot{\alpha}\dot{\beta}} (\epsilon_{\alpha\gamma} S^{\beta\delta} + N_{\alpha\beta}{}^{\gamma\delta}) \quad (10)$$

and

$$T_{\alpha\dot{\alpha}, \beta\dot{\beta}}{}^{\gamma\dot{\gamma}}{}_{\delta\dot{\delta}} = -i (\epsilon_{\beta\gamma} G_{\alpha\dot{\alpha}}{}^{\dot{\beta}\dot{\delta}} + \epsilon_{\alpha\gamma} G_{\beta\dot{\alpha}}{}^{\dot{\gamma}\dot{\delta}} + \epsilon_{\alpha\beta} H_{\dot{\alpha}\dot{\beta}}{}^{\gamma\dot{\gamma}}{}_{\delta\dot{\delta}}) \quad (11)$$

Here, M and N are skew on their internal indices and symmetric on their spinor indices, S is symmetric and G and H are Hermitean. For $N > 3$, M is not an independent field,

$$M_{\alpha\beta}{}^{ab} = \frac{1}{(N-2)} D_{(\alpha}{}^c \chi_{\beta) abc} \quad (12)$$

whilst for $N = 2$ it must be regarded as a new field (there being no dimension one-half spinors in the theory). We still have a constraint corresponding to $\Omega_{\alpha\beta}{}^b{}_c$ left and it is convenient to choose

^{*)} By conventional constraints we mean those that may be imposed to solve algebraically for parts of the connection and vielbein; for a discussion see Ref. 8).

$$H_{\alpha\beta}^a = \zeta_{\alpha\beta}^a + \frac{(N-3)}{12} \delta_b^a \bar{\chi}_{\beta}^{cde} \chi_{\alpha cde} \quad (13)$$

We remark that (13) does not determine Ω_a^b as a space time derivative of some field, but merely locates it in the supervielbein θ expansion. The dimension 3/2 torsion is then given by

$$T_{\alpha\beta, \gamma\delta}^c = \epsilon_{\alpha\beta} \bar{\psi}_{\gamma\delta}^c + \epsilon_{\gamma\delta} \psi_{\alpha\beta}^c + \epsilon_{\alpha\beta} (\epsilon_{\gamma\delta} \bar{\rho}_{\beta}^c + \epsilon_{\gamma\delta} \bar{\rho}_{\alpha}^c) \quad (14)$$

where ψ , φ and ρ correspond to the $SL(2, C)$ irreducible parts of the gravitino field strength and for $N > 1$ are expressible as derivatives of the fields already present. (For $N = 1$, ψ is just the Wess-Zumino W field¹⁰⁾). Thus we know T_{AB}^C and can therefore construct R_{ABC}^D using (5) so that the entire supergeometry is described by the fields χ (or M or ψ for $N = 2, 1$), S , N and G which are themselves subject to further constraints which arise from (5). For example

$$D_{(\alpha} \chi_{\beta) c d} = 3 \delta_{[\alpha}^a M_{c d] \alpha \beta} \quad (15)$$

However, there are still too many component fields and the solution to this problem lies in the additional scale invariance, which leaves the constraints (7) and (8) invariant. The vielbein transformation is

$$\begin{aligned} H_{\mu}^{\alpha} &= 2 \delta_{\mu}^{\alpha} W \\ H_{\alpha}^{\beta} &= \delta_{\alpha}^{\beta} W \\ H_{\alpha}^{\bar{\beta}} &= 2i (\sigma_{\alpha})^{\beta \bar{\beta}} D_{\beta} W \end{aligned} \quad (16)$$

where

$$H_A^B = E_A^M \delta E_M^B$$

The corresponding connection transformations may be obtained from the torsion constraints. In (16), W is a real unconstrained scalar superfield whose leading component corresponds to x space scale transformations and whose second component corresponds to S supersymmetry. This may be seen if we make the standard identifications¹⁰⁾

$$\begin{aligned} E_m^a &= e_m^a(x) + \dots \\ E_m^\alpha &= \psi_m^\alpha(x) + \dots \end{aligned} \quad (17)$$

where e_m^a is the vierbein and ψ_m^α the gravitino field. One then has under (16)

$$\delta \psi_m^\alpha = W|_{\theta=0} \psi_m^\alpha - 2i e_m^\mu (\sigma_\mu)^{\alpha\bar{\alpha}} \bar{D}_{\bar{\alpha}} W|_{\theta=0} \quad (18)$$

The role of the rest of the parameters in W is to gauge away unwanted components of the fields N , S and G in the conformal theory. To see this, it is sufficient to consider the linearized results and from (10) and (16) one finds

$$\begin{aligned} \delta S^{ab} &= 2 \epsilon^{\alpha\beta} D_\alpha^a D_\beta^b W \\ \delta N_{\alpha\beta}^{ab} &= -4 D_{(\alpha}^a D_{\beta)}^b W \end{aligned} \quad (19)$$

so that the leading components of S and N may be set to zero. It is not difficult to see that $G_{\alpha\beta}^a(\theta=0)$ may be likewise accounted for by $[D_\alpha^a, \bar{D}_{\beta}^b] W$. Thus the dimension one fields present in the conformal theory are $M_{\alpha\beta}^{ab}$, $\epsilon^{\alpha\beta} D_\alpha^a \chi_{\beta bcd}$ and $D_\alpha^a \chi_{\beta}^{bcd}$ (all at $\theta=0$). M is the (non-gauge) antisymmetric tensor field whilst from the $I_{\alpha\beta\gamma}^{\delta}$ identity one finds

$$D_\alpha^a \bar{\chi}_{\dot{\beta}}^{bcd} = D_\alpha^a [\bar{\chi}_{\dot{\beta}}^{bcd}] \quad (20)$$

and is therefore only present in $N=4$ where it is related to the derivative of the dimension zero (complex) scalar field as we shall see below. The field $\epsilon^{\alpha\beta} D_\alpha^a \chi_{\beta bcd}$ corresponds to the dimension one scalars of the theory, and for $N=4$ we need to impose an extra constraint which is connected with the global $SU(1,1)$ invariance [found in Ref. 4)]. Explicitly, we choose

$$\epsilon^{\alpha\beta} D_\alpha^a \chi_{\beta abc} = 0 \quad (21)$$

so that in $SU(4)$ notation we have a $\underline{10}$ of dimension one scalars. It is now straightforward, albeit somewhat tedious to verify that at higher dimensions we obtain the correct fields after we have used up all available W gauges. In terms of linearized superfields^{2),4),11),12)} the answer is as expected, i.e., conformal supergravity is described by a chiral superfield $V_{\alpha_1, \dots, \alpha_{4-N}}$ which is totally symmetric on its spinor indices, has dimension $(2 - N/2)$ and which satisfies an additional constraint, namely that the dimension two scalars are real. Furthermore, it has been shown that at the linearized level, this superfield may be used to construct an action^{4),11),12)}.

We now turn to a discussion of the $SU(1,1)$ invariance in $N = 4$, where, as we have seen, it was necessary to impose the additional constraint (21). We introduce the field

$$\overline{\chi}_{\dot{\alpha}a} = -\frac{1}{3!} \epsilon_{abcd} \overline{\chi}_{\dot{\alpha}}^{bcd} \quad (22)$$

which transforms as a $\underline{4}$ under $SU(4)$ and which also transforms under the remaining $U(1)$ by

$$\delta \overline{\chi}_{\dot{\alpha}a} = -\frac{3}{4} L^b{}_b \overline{\chi}_{\dot{\alpha}a} \quad (23)$$

We can construct the following 1 form

$$P = E^a P_a - E^{\dot{\alpha}} P_{\dot{\alpha}}$$

with

$$P_{\alpha\dot{\alpha}} = \frac{i}{4} D_{\alpha}^a \overline{\chi}_{\dot{\alpha}a} \quad (24)$$

$$P_{\dot{\alpha}} = \overline{\chi}_{\dot{\alpha}a}$$

Then one finds that the constraints (7), (8) (13) and (21) imply^{*)}

$$DP = 0 \quad (25)$$

and

$$\frac{1}{2} R^a{}_a = \overline{P} \wedge P \quad (26)$$

These two equations may be rewritten as

$$d\hat{\Omega} - \hat{\Omega} \wedge \hat{\Omega} = 0 \quad (27)$$

where

$$\hat{\Omega} = \begin{pmatrix} \frac{1}{2} \Omega^a{}_a & \overline{P} \\ P & \frac{1}{2} \overline{\Omega}_a{}^a \end{pmatrix} \quad (28)$$

Hence $\hat{\Omega}$ is a Lie-algebra valued one-form, the relevant group being $SU(1,1)$.

From (27) it is pure gauge and thus may be written

$$\hat{\Omega} = v^{-1} dv \quad (29)$$

*) The curvature 2 form is $R_C^D = 1/2 E^B E^A R_{ABC}^D$, See, for example Ref. 6). In the linearized limit (25) has the solution $P = dv$ where from (24) V is chiral.

where $\mathcal{V}$ is an element of $SU(1, 1)$. At $\theta = 0$, $\mathcal{V}$ therefore has three components, but only two are physical since (27) implies that the $U(1)$ gauge field is composite and does not propagate, so that we have an additional $U(1)$ which may be used to remove the non-physical component of $\mathcal{V}$ (and its supersymmetric counterparts).

To summarize, we have shown that it is possible to formulate conformal supergravity for $N \leq 4$ in superspace by explicitly gauging only $SL(2, C) \times U(N)$. The method requires the introduction of auxiliary superfields whose unwanted components correspond precisely to the additional parameters in the scale superfield W . Alternatively, one may say that the x space formulations^{1),2),4)} have been given in a Wess-Zumino gauge with respect to W transformations. It is natural to ask whether the formalism we have outlined above may be generalized to higher N (although one does not expect to obtain any actions, but merely a representation⁴⁾). As far as we have checked, the constraints (7) and (8) do not imply any equations of motion, but there are some problems; in particular, there are no dimension zero scalars appearing naturally, and the only way we could introduce them in $N = 4$ was by imposing the constraint (21) to obtain a coset representation $SU(1,1)/U(1)$. If dimension zero scalars appear à la Cremmer and Julia¹³⁾ for higher N , then this has the drawback (from the conformal point of view) that all the $U(N)$ gauge fields would have to be composite.

Finally, we make some observations about the Poincaré theories. For $N = 1$ and 2 it has been shown in x space^{2),14)} how to obtain off-shell Poincaré theories starting from the corresponding conformal theories. In superspace, one expects to be able to reproduce these results by suitably fixing the $U(N)$ gauges, but for $N \geq 3$ the procedure is not so obvious. This is because the Poincaré theories require a Yang-Mills sector and the only known off-shell version of $N = 4$ Yang Mills involves a central charge¹⁵⁾ and its attendant difficulties¹⁶⁾. However, the above formalism does yield a (reducible) off-shell representation with high dimension fields and the problem is how to obtain a submultiplet which may be used to form an action. There are therefore many interesting problems outstanding, and it may be that the conformal approach in superspace could be of some help in their resolution.

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