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GENERAL MASSIVE MULTIPLETS IN EXTENDED SUPERSYMMETRY

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ABSTRACT

We consider the extended Poincaré superalgebra in four dimensions with general central charge operators. We find the invariances of this algebra and the particle supermultiplets with their internal symmetry properties. Central charges obtained by dimensional reduction appear as a special case in this classification.

It is well known that in four dimensions the most general supersymmetry algebra, consistent with Poincaré invariance and certain requirements on the S matrix coming from a relativistic quantum field theory, has the following structure¹⁾

$$\begin{aligned} \{Q_\alpha^i, Q_\beta^j\} &= (\gamma^\mu C)_{\alpha\beta} P_\mu \delta^{ij} + C_{\alpha\beta} U^{ij} + (\gamma_5 C)_{\alpha\beta} V^{ij}, \\ [Q_\alpha^i, P_\mu] &= 0, \quad [Q_\alpha^i, M_{\mu\nu}] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta^i \\ [U^{ij}, Q_\alpha^i] &= [V^{ij}, Q_\alpha^i] = [U^{ij}, U^{lm}] = [V^{ij}, V^{lm}] = [V^{ij}, U^{lm}] = 0 \end{aligned} \quad (1)$$

together with additional commutation relations for the Poincaré algebra and the fact that U^{ij} and V^{ij} are Poincaré invariant operators. In Eq. (1), C is the charge conjugation matrix $C = -C^T$, and Q_α^i are Majorana spinors, i.e., $(C\bar{Q})_\alpha^i = Q_\alpha^i$. In the Majorana representation, Q_α^i are Hermitean operators. The U and V are Hermitean operators which belong to the centre of the superalgebra and for that reason are called central charges. From (1) it follows that $U^{ij} = -U^{ji}$, $V^{ij} = -V^{ji}$. There are no further restrictions on U and V from the Jacobi identities. The central charges satisfy a positivity bound which we will exploit later.

When $U^{ij} = V^{ij} = 0$, the general classification of the representations of the algebra (1) acting on one-particle physical states has been given^{2),3)}. This was achieved by using the method of induced representations and by noting that the supersymmetry algebra for fixed timelike or lightlike momentum $P^2 \leq 0$ reduces to a Clifford algebra for $2N$ or N ($P^2 = 0$) creation and destruction operators respectively. It is the aim of the present paper to generalize this classification to the case of the extended supersymmetry algebra (1) with arbitrary central charge operators. In particular we shall classify all possible supermultiplets corresponding to all central charge operators and we shall also exploit the residual symmetries of (1), in the presence of the central charges, which turn out to be subgroups of $U(N)$.

The main motivation of the present analysis is to have a complete classification of massless and massive multiplets and possibly to have a general procedure for a Higgs effect for higher spin states, at least at the level of counting the degrees of freedom. This is in view of the possible relevance of higher spin multiplets of composite fields, which may be used for current particle phenomenology as speculated in the work of Ellis et al.^{4),5)}.

The role of central charges in supersymmetric Lagrangian field theories has been considered by Fayet⁶⁾ and particular realizations of supermultiplets have also been discussed in Refs 7)-9). Central charges avoid the proliferation of states in massive representations by eliminating a discontinuity in the maximum spin between massive and massless multiplets. For this reason, they play a crucial role in super Yang-Mills theories with a spontaneously broken Yang-Mills group but with an unbroken supersymmetry⁶⁾. Supermultiplets with central charges also appear in supergravity models, in which case the central charges are gauge symmetries¹⁰⁾. Of particular interest are models where mass parameters are introduced via dimensional reduction¹¹⁾. They provide unique examples of supermultiplets of elementary fields with non-vanishing central charges and spin $J > 1$ ¹²⁾. The role of central charges in connection with non-trivial topological solutions of supersymmetric Yang-Mills theories has also been investigated¹³⁾. More recently, supersymmetry algebras with (off-shell) central charges have been used in order to obtain a field representation of certain extended supermultiplets¹⁴⁾ ($N = 2$, $N = 4$ Yang-Mills and $N = 8$ supergravity) with the correct number of bosonic and fermionic degrees of freedom which ensure the off-shell closure of the supersymmetry algebra.

We consider the anticommutation in (1) in the Weyl basis

$$\gamma^\mu = i \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma_\mu & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (2)$$

$\gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$, $\sigma_0 = -\sigma^0 = I$ and σ_i 's are the Pauli matrices. Correspondingly, we introduce Weyl spinors:

$$\begin{aligned} Q_\alpha^i &= (Q_L^i)_\alpha & (Q_R^i)^{\dot{\alpha}} &= \epsilon^{\dot{\alpha}\dot{\beta}} Q_{\dot{\beta}}^{*i} \\ (\alpha, \beta = 1, 2; \dot{\alpha}, \dot{\beta} = 1, 2) & & &= (i\sigma_2 Q_L^{*i})^{\dot{\alpha}} \\ \epsilon^{\alpha\beta} &= \epsilon^{\dot{\alpha}\dot{\beta}} = (i\sigma_2)_{\alpha\beta} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}} \end{aligned} \quad (3)$$

The four-component spinor Q_α ($\alpha = 1, \dots, 4$) has lower and upper components given by $(Q_L)_\alpha$ and $\epsilon^{\dot{\alpha}\beta} Q_{L\beta}^*$ respectively. In the two-component notation, the anticommutation in (1) becomes

$$\{Q_{\alpha}^i, Q_{\beta}^{*j}\} = (\sigma_{\mu})_{\alpha\beta} P^{\mu} \delta^{ij} \quad (4)$$

$$\{Q_{\alpha}^i, Q_{\beta}^j\} = \epsilon_{\alpha\beta} Z^{ij}$$

$$\{Q_{\alpha}^{*i}, Q_{\beta}^j\} = \epsilon_{\alpha\beta} Z^{ij} \quad (5)$$

$$Z^{ij} = -V^{ij} + iU^{ij}$$

For massive states at rest, $P^{\mu} = (M, \vec{0})$, we define $2N$ component spinors¹⁵⁾

$$Q_{\alpha}^a = \begin{pmatrix} Q_{\alpha}^i \\ Q^{*i\alpha} \end{pmatrix} \quad (6)$$

i.e., $Q_{\alpha}^a = Q_{\alpha}^i, a=i=1,2,\dots,N$
 $= Q^{*i\alpha} = \epsilon^{\alpha\beta} Q_{\beta}^{*i}, a=N+i=N+1,\dots,2N$

Q_{α}^a satisfy the reality condition

$$Q_{\alpha}^{*a} = \epsilon^{\alpha\beta} \Omega_{ab} Q_{\beta}^b$$

with

$$\Omega^{ab} = -\Omega_{ab} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}_{ab}$$

With these notations, (4) and (5) become

$$\{Q_{\alpha}^a, Q_{\beta}^b\} = \epsilon_{\alpha\beta} \begin{pmatrix} Z & MI \\ -MI & Z^{*} \end{pmatrix}_{ab} = \epsilon_{\alpha\beta} \Lambda^{ab} \quad (7)$$

where the $(2N \times 2N)$ matrix Λ^{ab} has been written in terms of the $N \times N$ numerical matrices Z^{ij} and $M\delta^{ij}$. The numbers Z^{ij} are the eigenvalues of the central charge operators for a given irreducible supermultiplet.

Before considering the supersymmetry algebra with central charges, let us recall the classification of massive one-particle states in the absence of central charges. When $Z^{ij} = 0$, (7) becomes¹⁵⁾

$$\{Q_\alpha^a, Q_\beta^b\} = \epsilon_{\alpha\beta} \Omega^{ab} M \quad (8)$$

These anticommutators define the Clifford algebra with an invariance group $SO(4N)$. In the form (8), the algebra has a manifest $SU(2) \otimes USp(2N) \subset SO(4N)$ invariance. The spinor charges Q_α^i , which behave like a vector under $SO(4N)$, remain irreducible with respect to $SU(2) \otimes USp(2N)$: $4N \rightarrow (2, 2N)$. The fundamental irreducible massive multiplet, i.e., the lowest dimensional supermultiplet, is obtained from the Clifford vacuum Ω , defined by the condition $Q_\alpha^i \Omega = 0$ ($\forall \alpha, i$), by repeated application of the creation operators Q_α^{*i} :

$$\Omega; Q_\alpha^{*i} \Omega; Q_\alpha^{*i} Q_\beta^{*j} \Omega; \dots; Q_{\alpha_1}^{*i_1} Q_{\alpha_2}^{*i_2} \dots Q_{\alpha_k}^{*i_k} \Omega; \dots$$

These 2^{2N} states span the spinorial representation of $SO(4N)$ which splits into two irreducible ones, each of dimension 2^{2N-1} , corresponding to bosons and fermions respectively. The physical states are obtained by decomposing the spinor representation with respect to $SU(2) \otimes USp(2N)$ which classify the states of given intrinsic $SU(2)$ J-spin, as follows¹⁵⁾

$$\underline{2^{2N}} = (N+1, 1) + (N, 2N) + \dots + (N+1-k, [2N]_k) + \dots + (1, [2N]_N) \quad (9)$$

The first label is the dimension of the $SU(2)$ multiplet ($J = (N-k)/2$) and $[2N]_k$ denotes the totally antisymmetric traceless representation of $USp(2N)$. All the other massive supermultiplets are obtained by taking a vacuum which is not a singlet under $SU(2) \times U(N)$. One can further decompose the massive supermultiplet given by (9) into "massless" ones. This means considering a suitable subgroup $U(1) \otimes SU(N) \subset SU(2) \otimes USp(2N)$ when $U(1)$ is generated by the $SU(2)$

J-spin projection, λ , and the $2N$ representation of $USp(2N)$ decomposes under $U(N)$ as $2N \rightarrow N + \bar{N}$. Then the massive multiplet splits into 2^N "massless" multiplets, each of dimension 2^N , as follows¹⁵⁾

$$\underline{2^{2N}} = \left\{ \frac{N}{2}, 1 \right\} + \left\{ \frac{N-1}{2}, N \right\} + \dots + \left\{ \frac{N-k}{2}, [N]_k \right\} + \dots + \{ 0, 1 \} \quad (10)$$

where the curly bracket denotes a "massless" supermultiplet specified by a state of maximal spin projection, λ_{MAX} , belonging to the antisymmetric representation $[N]_k$ of $SU(N)$, namely

$$\begin{aligned} \{ \lambda_{MAX}, [N]_k \} &= (\lambda_{MAX}, [N]_k) + (\lambda_{MAX} - \frac{1}{2}, [N]_k \otimes \bar{N}) + \\ &\dots + (\lambda_{MAX} - \frac{k}{2}, [N]_k \otimes [\bar{N}]_k) + \dots + (\lambda_{MAX} - \frac{N}{2}, [N]_k). \end{aligned} \quad (11)$$

We now consider the modification of this analysis when the central charges Z^{ij} do not vanish. If we perform a $U(N)$ transformation on the two component (Weyl) charges $Q_\alpha^i, Q_\alpha^i \rightarrow U^{ij} Q_\alpha^j = \tilde{Q}_\alpha^i, Q_\alpha^{*i} \rightarrow U^{*ij} Q_\alpha^{*j} = \tilde{Q}_\alpha^{*i}$, Eq. (4) remains unchanged. In the rest frame it reduces to

$$\{ \tilde{Q}_\alpha^i, \tilde{Q}_\beta^{*j} \} = \delta_{\alpha\beta} \delta^{ij} M \quad (12)$$

In Eq. (5) the Z^{ij} matrix transforms as

$$Z^{ij} \rightarrow (U Z U^T)^{ij} = \tilde{Z}^{ij} \quad (13)$$

i.e., the central charges transform like the complex antisymmetric $[N]_2$ representation of $U(N)$. We can now use a well-known theorem¹⁶⁾ which states that for any complex antisymmetric matrix $Z = -Z^T$, one can find a unitary transformation such that $\tilde{Z}$ has normal form, namely for even N ,

$$\tilde{Z} = i\sigma_2 \otimes \hat{Z}_{N/2} ; \quad \hat{Z}_{N/2} = \text{diag} (z_1, \dots, z_{N/2}) \quad (14)$$

and for odd N

$$\tilde{Z} = \left(\begin{array}{c|c} i\sigma_2 \otimes \hat{Z}_{\frac{N-1}{2}} & 0 \\ \hline 0 & 0 \end{array} \right) \quad (15)$$

$\hat{Z}_{N/2}$ and $\hat{Z}_{(N-1)/2} = \hat{Z}_{[N/2]}$ are $N/2 \times N/2$ and $(N-1)/2 \times (N-1)/2$ diagonal matrices respectively, with non-negative eigenvalues $z_1, z_2, \dots, z_{[N/2]}$. We first consider N even and introduce labels $i = (a, m)$, $a = 1, 2$, $m = 1, \dots, N/2$. Then, after the unitary transformation U with the property (14), Eqs (5) and (12) become

$$\{\tilde{Q}_\alpha^{am}, \tilde{Q}_\beta^{bn}\} = \epsilon_{\alpha\beta} \epsilon^{ab} \delta^{mn} z_n \quad (16)$$

$$\{\tilde{Q}_\alpha^{am}, \tilde{Q}_\beta^{*bn}\} = \epsilon_{\alpha\beta} \epsilon^{ab} \delta^{mn} M \quad (17)$$

where

$$\tilde{Q}_a^{*am} = \epsilon_{ab} \epsilon^{\alpha\beta} \tilde{Q}_\beta^{*bm}$$

and

$$\epsilon^{ab} = -\epsilon_{ab} = (i\sigma_2)_{ab}$$

If we now define the new spinors

$$S_{\alpha(1)}^{am} = \frac{\tilde{Q}_\alpha^{am} + \tilde{Q}_\alpha^{*am}}{\sqrt{2}} ; S_{\alpha(2)}^{am} = \frac{\tilde{Q}_\alpha^{am} - \tilde{Q}_\alpha^{*am}}{\sqrt{2}} \cdot (-1)^{a+1} \quad (18)$$

equations (16) and (17) take the form

$$\begin{aligned} \{S_{\alpha(1)}^{am}, S_{\beta(1)}^{bn}\} &= \epsilon_{ab} \epsilon^{\alpha\beta} \delta^{mn} (M + z_n) \\ \{S_{\alpha(2)}^{am}, S_{\beta(2)}^{bn}\} &= \epsilon_{ab} \epsilon^{\alpha\beta} \delta^{mn} (M - z_n) \\ \{S_{\alpha(1)}^{am}, S_{\beta(2)}^{bn}\} &= 0 \end{aligned} \quad (19)$$

where the $S_{\alpha(i)}^{am}$ satisfy the reality conditions

$$S_{\alpha(i)}^{am} = \epsilon^{ab} \epsilon_{\alpha\beta} S_{\beta(i)}^{*bm} \quad (20)$$

and we used the property $\epsilon^{ac}(-)^{c+1} = \epsilon^{ac}(-)^a$. Because of (20), $a = b = 2$ give redundant relations, so we can rewrite (19) only in terms of $S_{\alpha(i)}^m = S_{\alpha(i)}^{1m}$ and their complex conjugates:

$$\begin{aligned} \{S_{\alpha(i)}^m, S_{\beta(j)}^{*n}\} &= \delta_{\alpha\beta} \delta^{mn} \delta_{ij} (M - (-)^j z_n) \\ \{S_{\alpha(i)}^m, S_{\beta(j)}^n\} &= \{S_{\alpha(i)}^{*m}, S_{\beta(j)}^{*n}\} = 0 \end{aligned} \quad (21)$$

$$(\alpha, \beta = 1, 2 ; i, j = 1, 2 ; m, n = 1, \dots, N/2)$$

For N odd we can repeat the same procedure for the first $2(N-1)$ spinor charges and the remaining Q_α^N , which corresponds to the zero eigenvalue in (15), satisfies

$$\{Q_{\alpha}^N, Q_{\beta}^{*N}\} = \delta_{\alpha\beta} M ; \{Q_{\alpha}^N, Q_{\beta}^N\} = 0 \quad (22)$$

$$\{Q_{\alpha}^N, S_{\beta(i)}^{*m}\} = \{Q_{\alpha}^N, S_{\beta(i)}^m\} = 0 \quad (m=1, \dots, \frac{N-1}{2})$$

From (21), one gets the positivity bound

$$z_n \leq M, \quad n=1, \dots, \frac{N}{2} \left(\frac{N-1}{2} \text{ for odd } N \right) \quad (23)$$

The above inequality also shows that all z_n (and therefore z^{ij}) vanish when $M = 0$. We note that the transformations (13) and (18) are the following symplectic rotation on the $Q_{\alpha}^a = (Q_{\alpha}^1, Q_{\alpha}^{*1})$ spinorial charges

$$Q_{\alpha}^a \rightarrow T^a_b Q_{\alpha}^b = S_{\alpha}^a$$

$$T = \left[\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \otimes I_{N/2} \right] \cdot \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \quad (24)$$

with

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

Under the symplectic transformation (24) the matrix Λ^{ab} given by (7) goes into

$$\tilde{\Lambda}^{ab} = \begin{pmatrix} 0 & M + \hat{z}_{N/2} \otimes \sigma_3 \\ -M - \hat{z}_{N/2} \otimes \sigma_3 & 0 \end{pmatrix} = \Omega^{ab} \rho_b$$

with

$$\rho_{2i-1} = \rho_{N+2i-1} = M + z_i ; \quad \rho_{2i} = \rho_{N+2i} = M - z_i \quad (i=1, \dots, N/2)$$

For odd N obvious modifications occur. The matrix $\tilde{\Lambda}^{ab}$ becomes the symplectic metric after rescaling

$$S_{\alpha}^a \rightarrow \left(\frac{M}{\rho_a} \right)^{1/2} S_{\alpha}^a$$

It is clear that the transformation T_b^a is not unique. In particular, the unitary transformation U is defined as a transformation of the invariance group of the central charge matrix Z^{ij} which will be discussed later.

We now discuss the structure of the fundamental multiplets which arise from the anticommutation relations given by (21). If no z_i 's saturate the positivity bound (23), after the rescaling

$$S_{\alpha(i)}^m \rightarrow \left(\frac{M}{M - (-)^i z_m} \right)^{1/2} S_{\alpha(i)}^m$$

one again obtains the Clifford algebra (8) with

$$Q_{\alpha}^a = \begin{pmatrix} S_{\alpha(i)}^m \\ S_{(i)}^{*\alpha m} \end{pmatrix} \quad (25)$$

For odd N one adds Q_{α}^N to the $S_{\alpha(i)}^m$ in (25). The Clifford vacuum Ω defined by the condition $S_{\alpha(i)}^m \Omega = 0$ has to be doubled, since under a PCT transformation $Q_{\alpha}^j \rightarrow i Q_{\alpha}^{*j}$, the central charges transform as follows

$$Z^{ij} \rightarrow -Z^{*ij}, \quad \text{i.e., } z_n \rightarrow -z_n \quad (26)$$

This has the consequence that the massive multiplet with central charges has twice the (real) degrees of freedom of the spinorial representation associated with the Clifford algebra (8). The invariance group of (8) is $SO(4N)$ and the 2^{2N} complex states are classified by $SU(2) \otimes USp(2N)$. If $z_1 = z_2 = \dots = z_q = M$, then the $2q$ operators $S_{\alpha(2)}^m$ will vanish and the remaining Clifford algebra will have an invariance $SO(4N - 4q)$ with the states classified by $SU(2) \otimes USp(2N - 2q)$. The one particle states are then those of a $(N-q)$ extended supersymmetry without central charges, repeated twice. The $SU(2)$ spin will run, inside fundamental multiplet from $J = 0$ up to $J = (N-q)/2$. If all z_i saturate

the bound, $z_1 = \dots z_{[N/2]} = M$, then the Clifford algebra has $SO(2N)$ invariance for even N and $SO(2N+2)$ invariance for odd N , the states being classified by $SU(2) \otimes USp(N)$ and $SU(2) \otimes USp(N+1)$ respectively. Notice that for even N , $SO(2N)$ has the same Clifford algebra of the massless case, $M = 0$, in N extended supersymmetry. In the latter case the massless states are classified by $U(1) \times SU(N) \subset SO(2N)$. Then, by decomposing these $SU(N)$ representations according to $USp(N) \subset SU(N)$ one can read out how massless multiplets may be recombined into a massive one with central charges, i.e., how the Higgs effect may operate for extended supermultiplets without increasing their maximum spin^(6),11). We now classify the invariances of the relativistic supersymmetry algebra (1) when $Z^{ij} \neq 0$. In a given irreducible supermultiplet, the effect of the Z^{ij} 's is to reduce the $U(N)$ invariance of this algebra for $Z^{ij} = 0$ to the subgroups of $U(N)$ which leave the complex skew symmetric numerical $N \times N$ matrix Z^{ij} invariant. One, therefore, has to classify the orbits of the twofold antisymmetric representations $[N]_2$ of $U(N)$. The generic orbit has little group $(USp(2))^{N/2}$ for even N or $(USp(2))^{(N-1)/2} \otimes U(1)$ for odd N . This can be seen, e.g., for even N by decomposing $[N]_2 + [\bar{N}]_2$ with respect to the subgroup $(USp(2))^{N/2}$

$$[N]_2 + [\bar{N}]_2 = N(1, \dots, 1) + \text{perm}(2, 2, 1, \dots, 1), \quad (27)$$

where $\text{perm}(2, 2, 1, \dots, 1)$ denotes the sum of all the $N/2$ $((N/2)-1)$ permutations of the $USp(2)$ subgroups in $(USp(2))^{N/2}$. From (27) one must subtract the representation content of the coset space

$$\begin{aligned} U(N)/[USp(2)]^{N/2} &= \text{Adj } U(N) - \text{Adj } [USp(2)]^{N/2} \\ &= \frac{N}{2}(1, \dots, 1) + \text{perm}(2, 2, 1, \dots, 1) \end{aligned} \quad (28)$$

obtaining

$$[N]_2 + [\bar{N}]_2 - [\text{Adj } U(N) - \text{Adj } (USp(2))^{N/2}] = \frac{N}{2}(1, \dots, 1) \quad (29)$$

An analogous analysis can be done for odd N . We conclude that, in general, the supersymmetry algebra (1) is invariant under $(USp(2))^{N/2}$ (N even) or $(USp(2))^{(N-1)/2} \otimes U(1)$ (N odd). These are indeed the symmetries of $\tilde{Z}^{ij}$ given by (14) and (15) in the general case $z_1 > z_2 > \dots > z_{[N/2]}$, i.e., $N/2$ real parameters.

The critical orbits have little groups

$$USp(2q) \otimes U(N-2q) \quad (q=1 \dots [N/2])$$

corresponding to the invariances of $\tilde{z}^{ij}$ in the special one parameter cases $z_1 = z_2 = \dots = z_q > 0, z_{q+1} = \dots = z_{[N/2]} = 0$. This can be checked again by decomposing $[N]_2 + [\bar{N}]_2$ with respect to the little groups:

$$[N]_2 + [\bar{N}]_2 = (1, [N-2q]_2) + ([q]_2, 1) + (1, 1) + (2q, N-2q) + h.c. \quad (30)$$

The coset space decompositions are as follows

$$U(N)/USp(2q) \otimes U(N-2q) = Adj U(N) - Adj USp(2q) - Adj U(N-2q) = ([2q]_2, 1) + (1, 1) + (2q, N-2q) + (2q, \overline{N-2q}) \quad (31)$$

Comparing (31) and (30) one sees that

$$[N]_2 + [\bar{N}]_2 - U(N)/USp(2q) \otimes U(N-2q)$$

contains only one singlet of the corresponding little groups, i.e., the orbits are critical. In general, all the orbits of the $[N]_2$ representation of $U(N)$ have little groups

$$G = USp(2q_1) \otimes \dots \otimes USp(2q_n) \otimes U(N-2q), \quad (q = \sum_i q_i) \quad (32)$$

if the eigenvalues of the corresponding matrix $\hat{z}_{[N/2]}$ in (14) or (15) are such that $z_1 = z_2 = \dots = z_{q_1} > z_{q_1}, \dots, z_{q_1+q_2} > \dots > z_{q_1+q_2} = \dots = z_q > 0, z_{q+1} = \dots = z_{[N/2]} = 0$. The matrix $\tilde{z}^{ij}$ depends in these cases on n parameters. It is evident that (32) classifies all possible non-trivial internal symmetries of the relativistic extended supersymmetry algebra given by (1). Observe that these symmetries of the algebra are also symmetries of the Lagrangian but only in some cases. In the above, we have always considered a single irreducible supermultiplet. If, in a theory, there is more than one, it will not be possible in general to simultaneously bring to normal form the central charge matrices of each supermultiplet. This means that the symmetries G are the largest possible invariances in a given theory.

In view of this analysis, we expect that in supergravity theories, if supermultiplets with central charges emerge, the possible gauge symmetry $U(N)$ will be broken and the unbroken gauge symmetry will be at most one of the subgroups G given by (32). This result could be relevant in the application of supergravity models to particle phenomenology and in particular in the mass generation for unwanted states in superunification schemes^{4),5)}.

Finally, we give the relation between the algebra with central charges obtained by dimensional reduction and the general classification of this paper. We consider a spacetime dimension $D > 4$ when a Majorana representation exists^{17),18)}, i.e., $D = 10, 11, 12 \bmod 8$. The anticommutators of the $N = 1$ supersymmetry algebra in D dimension are

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= -(\gamma^\mu C)_{\alpha\beta} P_\mu - (\gamma^A C)_{\alpha\beta} P_A \\ (\alpha, \beta &= 1, \dots, 2^{[D/2]}; \mu = 0, \dots, 3; A = 4, \dots, D-1) \quad (33) \\ \gamma_0^2 &= -I, \quad \gamma_i^2 = \gamma_A^2 = +I \end{aligned}$$

C is the charge conjugation matrix in D dimension. We can always choose a basis in D dimension such that

$$\begin{aligned} P_\mu (\gamma^\mu C)_{\alpha\beta, ij} &= P_\mu (\gamma^\mu C_4)_{\alpha\beta} \delta_{ij} \\ P_A (\gamma^A C)_{\alpha\beta, ij} &= (C_4)_{\alpha\beta} U_{ij} + (\gamma_5 C_4)_{\alpha\beta} V_{ij} \quad (34) \\ (\alpha, \beta &= 1, \dots, 4; i, j = 1, \dots, N = 2^{[D/2]-2}) \end{aligned}$$

($N = 2^{(D/2-3)}$ for $D = 10 \bmod 8$ when the Weyl condition can also be imposed.)¹⁸⁾

In Eq. (34), C_4 is the charge conjugation matrix in four dimension and U^{ij} , V^{ij} are antisymmetric commuting matrices. In the Weyl basis in four dimension one has

$$(\gamma_A C)_{\alpha\beta ij} P^A = \begin{pmatrix} \epsilon_{\alpha\beta} Z^{ij} & 0 \\ 0 & \epsilon_{\alpha\beta} Z^{*ij} \end{pmatrix} \quad (35)$$

Since

$$(P^A \gamma_A C)(P^A \gamma_A C)^\dagger = P^A P_A I$$

the corresponding $\tilde{Z}^{ij}$ defined by (13) has the form

$$\tilde{Z} = i\sigma_2 \otimes \hat{Z}_{N/2}, \text{ with } \hat{Z}_{N/2} = \sqrt{P_A P^A} I_{N/2} \quad (36)$$

This corresponds to the special critical orbit with an $USp(N)$ invariance. If the theory was massless in D dimension then

$$\sqrt{P_A P^A} = \sqrt{-P_\mu P^\mu} = M$$

and the positivity bounds (23) are saturated. In this case the massive four dimensional multiplets are classified by $SU(2) \otimes USp(N)$ and the lowest dimensional multiplet has maximum spin $J = N/4$.

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