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HIGHER ORDER INVARIANTS IN EXTENDED SUPERGRAVITY

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A B S T R A C T

On-shell linearized extended supergravity is presented in superspace for all N . The formalism is then used in the construction of higher-order invariants which may serve as counterterm Lagrangians. It is shown that three-loop counterterms exist for $N \leq 3$ and $(N-1)$ loop counterterms for $N \geq 4$. In the full non-linear theory, the presence of a global non-compact symmetry group for $N \geq 4$ does not allow a simple extension of the $(N-1)$ loop term, but N loop counterterms may be constructed.

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1. INTRODUCTION

One of the main hopes entertained for extended supergravity theories has been that they would have better quantum properties than ordinary gravity. Results obtained so far have indicated that this is so for $N = 1$ and 2 , but that the improvement is not sufficient. It has been shown that there is a three-loop counterterm Lagrangian for both $N = 1$ and 2 (these results were first obtained at the linearized level¹⁾ and then generalized to the full case²⁾), and hence the one-loop on-shell finiteness of ordinary gravity becomes improved to the two-loop level for $N = 1, 2$ supergravities. In this paper, we shall adopt a similar philosophy to that employed in the above references in order to investigate the quantum properties of extended supergravity for all N . We shall assume that covariant quantization makes sense, that one can find a regularization scheme which preserves the gauge symmetries and that any divergences should therefore be expressible as invariants constructed out of the field strength tensors of the theory. This leads us naturally to a study of higher-order invariants whose existence would signal the presence of corresponding divergences.

There are two techniques available in the construction of supersymmetric theories which are either unnecessary or not applicable in ordinary field theories. The first of these is the method of dimensional reduction which has had some resounding successes, notably the construction of all supersymmetric Yang-Mills theories in two and four dimensions from simple supersymmetry in higher dimensions³⁾, and the derivation of the full non-linear $N = 8$ supergravity⁴⁾ in four dimensions from eleven-dimensional supergravity^{5)*}). It is tempting to try and apply this technique to the counterterm Lagrangians we are interested in, and indeed some results have been obtained in this way⁷⁾. However, the construction of counterterm Lagrangians for $N = 8$ supergravity starting from an eleven-dimensional formulation seems to be unmanageable, and it would seem that the second technique - superspace - is more suitable for our purposes. Although the superspace formulation of supersymmetric theories is the most natural one from the point of view of manifest covariance, it has generally been the case that theories have been discovered in x space first and then recast in superspace language subsequently, although recently superspace has come more into its own in quantum calculations⁸⁾. For our purposes superspace is almost indispensable, and we review this formulation of extended supergravity theories in the next section. In Section 3 we use this formalism to construct linearized on-shell invariants for all supergravity theories. In particular, our analysis enables us to show that there is an $(N-1)$ loop counterterm

*) The linearized $N = 8$ had been previously constructed (Ref. 6)).

for $N \geq 4$, but we do not prove that the three-loop counterterm which exists for $N \leq 4$ is non-existent for $N > 4$. In Section 4, we extend the results of Section 3 to the full non-linear (on-shell) case, and show that if one demands that all the symmetries of the equations of motion should be implemented, the $(N-1)$ loop counterterm for $N \geq 4$ does not generalize. It is still possible to construct higher loop Lagrangians, however, and the simplest ones occur at the N loop level. Finally, in Section 5, we briefly discuss the off-shell case and make some concluding remarks. For convenience, our notational conventions are given in Appendix A and in Appendix B we formulate the $N = 8$ supergravity in a slightly different way.

2. EXTENDED SUPERGRAVITY IN SUPERSPACE

In this section we shall construct all linearized on-shell extended supergravities in superspace starting from the superspace formulation of the $N = 8$ theory [Ref. 9), hereafter referred to as B.H.]. We shall use the language of differential forms, and for details of this method in superspace we refer the reader to Ref. 10). We recall that superspace has co-ordinates $Z^M = (x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}})$ where the x^m are four commuting co-ordinates and the θ^μ are $2N$ complex anti-commuting "co-ordinates" with complex conjugates $\bar{\theta}^{\dot{\mu}}$. At each point in superspace we can introduce a preferred basis of one-forms E^A which is related to the co-ordinate basis by the vielbein matrix:

$$E^A = dz^M E_M^A \quad (2.1)$$

Any tensor can be expressed in components in either basis; e.g., for a vector X

$$X^A = X^M E_M^A ; \quad X^M = X^A E_A^M \quad (2.2)$$

where E_A^M is the inverse vielbein,

$$E_A^M E_M^B = \delta_A^B$$

$$\delta_A^B = (\delta_\mu^\nu, \delta_\alpha^\beta, -\delta_{\dot{\alpha}}^{\dot{\beta}})$$

(2.3)

The tangent space group acts on the "flat" indices (A,B...) and is taken to be of the form $SL(2,C) \times G$ where G is $U(N)$ or a subgroup thereof. Thus one has

$$\delta X^A = X^B L_B^A \quad (2.4)$$

with

$$\begin{aligned} L_{\mu}^{\alpha} &= -L^{\alpha}_{\mu} \\ L_{\beta}^{\alpha} &= \delta_{\beta}^{\alpha} L_a^b + \delta_a^b L_{\beta}^{\alpha} \\ L_{\dot{\beta}}^{\alpha} &= -\overline{L}_{\dot{\beta}}^{\alpha} = -\overline{(L_{\beta}^{\alpha})} \\ L_{\beta\alpha} &= \frac{1}{2} (\sigma^{\mu\nu})_{\beta\alpha} L_{\mu\nu} \\ \overline{L}_b^a &= -L^a_b \end{aligned} \quad (2.5)$$

where there will be more restrictions on L^a_b if G is a subgroup of $U(N)$. All other parts of L , e.g., L_a^b , are zero. Since the transformations (2.4) are local, we introduce a set of connection one-forms Ω_A^B belonging to the same Lie-algebra as L , but transforming inhomogeneously:

$$\delta \Omega_A^B = -dL_A^B + \Omega_A^C L_C^B - L_A^C \Omega_C^B \quad (2.6)$$

Using Ω one can then construct a covariant exterior derivative D , and one has as an example

$$DX^A = dX^A + X^B \Omega_B^A \quad (2.7)$$

The geometric field strengths are the torsion and the curvature:

$$T^A = DE^A = \frac{1}{2} E^C_{\alpha} E^B_{\beta} T_{BC}^A \quad (2.8)$$

$$R_A^B = d\Omega_A^B + \Omega_A^C \wedge \Omega_C^B = \frac{1}{2} E^D \wedge E^C R_{CD A}^B \quad (2.9)$$

where the latter has the same symmetry properties as L (2.5) on the indices A, B . The Bianchi identities are

$$D T^A = E^B \wedge R_B^A \quad (2.10)$$

$$D R_A^B = 0 \quad (2.11)$$

The structure of the tangent space group (2.5) allows one to show that all the components of the curvature tensor may be expressed in terms of the torsion from (2.10) and further, that the identity (2.11) is satisfied if (2.10) is¹¹⁾.

For $N > 1$, we observe that the supergravity cannot be solely described in terms of the geometric variables that we have discussed above. This is because there are $\binom{N}{2}$ plus $\binom{N}{6}$ [$\binom{N}{2}$ for $N = 8$] spin-one fields, and the invariances of the geometry do not contain the appropriate gauge transformations for these fields. If we insist on G being local, it will follow that the $U(1)$ potentials cannot transform under G but must be representations of some global group G' . Thus we have potentials ϕ with

$$\delta \phi_{m, m_2} = d \Lambda_{m, m_2} \quad (2.12)$$

$$\delta \phi_{m, m_2 \dots m_6} = d \Lambda_{m, m_2 \dots m_6} \quad (2.13)$$

where the ϕ 's and Λ 's are totally antisymmetric on the global G' indices (m 's) and have corresponding field strengths F

$$F_{\dots} = d\phi_{\dots} \quad (2.14)$$

satisfying Bianchi identities

$$dF_{\dots} = 0 \quad (2.15)$$

Evidently, since $N = 8$ is self-conjugate and the six-index potential is zero for $N \leq 5$, it follows that this latter field is only present in the $N = 6$ theory.

We now turn to a brief review of the $N = 8$ formalism of BH, where G is taken to be $SU(8)$ ($L^a_a = 0$). We recall that the spin-one identities were found to be expressible in the $SU(8)$ covariant form

$$DF_{ab} = P_{abcd} \wedge \bar{F}^{cd} \quad (2.16)$$

where the one-form P is self-dual on the $SU(8)$ indices:

$$P_{abcd} = \frac{1}{4!} \epsilon_{abcdefgh} \bar{P}^{efgh} \quad (2.17)$$

In addition there are two further identities

$$D P_{abcd} = 0 \quad (2.18)$$

$$R^a{}_{cd} = -2 \delta^a_{[c} R^{b]}_{d]} = \bar{P}^{abef} \wedge P_{efcd} \quad (2.19)$$

which may be rewritten in the form

$$d\hat{\Omega} - \hat{\Omega} \wedge \hat{\Omega} = 0 \quad (2.20)$$

$$\hat{\Omega} = \begin{pmatrix} Q & \bar{P} \\ P & \bar{Q} \end{pmatrix} \quad (2.21)$$

$$Q^{ab}_{cd} = -2 \delta^{[a}_{[c} \Omega^{b]}_{d]}$$

(2.20) shows that $\hat{\Omega}$ is a pure gauge field, the relevant group being E_7 . Hence

$$\hat{\Omega} = v^{-1} dv \quad (2.22)$$

where v is an element of E_7 ;

$$v = \begin{pmatrix} U^{mn}_{ab} & \bar{V}^{mnab} \\ V_{mnab} & \bar{U}_{mn}^{ab} \end{pmatrix} \quad (2.23)$$

The global group $G'(=E_7)$ acts on the m,n indices to the left by^{*}

$$\begin{aligned} \delta v &= -\Lambda v \\ \Lambda &= \begin{pmatrix} 2 \delta^{[m}_{[r} \Lambda^{n]}_{s]} & \bar{\Sigma}^{mnrs} \\ \bar{\Sigma}_{mnrs} & 2 \delta^{[r}_{[m} \bar{\Lambda}^{s]}_{n]} \end{pmatrix} \\ \Lambda^m_n &= -\bar{\Lambda}_n^m ; \Lambda^m_m = 0 \end{aligned} \quad (2.24)$$

$$\bar{\Sigma}_{mnrs} = \frac{1}{4!} \epsilon_{mnrspqtu} \bar{\Sigma}^{pqtu}$$

whilst the local $SU(8)$ acts on the a,b indices to the right. With

$$(F_{mn}, \bar{F}^{mn}) = (F_{ab}, \bar{F}^{ab}) v^{-1} \quad (2.25)$$

^{*}) Details of E_7 can be found in Ref. 4).

the new F 's on the left-hand side satisfy ordinary spin-one Bianchi identities (2.15), and form a 56-dimensional representation of E_7 . The apparent doubling of the number of spin-one degrees of freedom is avoided by a space-time self-duality condition and is perhaps more easily seen at the linearized level (see below). The components of T^A , R_A^B , F_{ab} and P_{abcd} satisfying the identities (2.10), (2.16), (2.18) and (2.19), were tabulated by BH, and we shall not reproduce them here, but merely draw attention to one or two salient points. At dimension zero, the non-vanishing field strengths are

$$T_{\alpha\beta}{}^\gamma = -i \delta_b^a (\sigma^a)_{\alpha\beta} \quad (2.26)$$

$$F_{\alpha\beta, cd} = -2i \epsilon_{\alpha\beta} \delta_{[cd]}^{ab} \quad (2.27)$$

These remain in the global limit and correspond to the structure constants of the translational part of the rigid supersymmetry algebra, which has 28 complex "central charges" giving rise to (2.27). These charges do not commute with the entire algebra [they transform as a 28 under $SU(8)$], and hence they are not central charges in the sense of Haag et al¹²⁾. No physical field carries such a charge, and the generators correspond to the additional Abelian gauge invariances of the theory.

At dimension one-half (mass), the only non-vanishing component of the torsion was found to be ^{*)}

$$T_{\alpha\beta}{}^{\dot{\gamma}} = 2 \epsilon_{\alpha\beta} \bar{\chi}^{abc \dot{\gamma}} \quad (2.28)$$

The spinor superfield χ is totally antisymmetric on the $SU(8)$ indices and its leading component corresponds to the 56 spin one-half fields of the theory. BH then showed that all the components of the torsion and curvature tensors were expressible in terms of the superfield χ whose independent (θ expansion) components are in one-to-one correspondence with the $N=8$ spectrum (one spin 2, eight spin 3/2, 28 spin J, 56 spin 1/2, and 70 spin 0). Furthermore, since

$$P_{\alpha, bcde} = \delta_{[b}^a \chi_{cde] \alpha} \quad (2.29)$$

*) 2χ was called W in BH.

and

$$\begin{pmatrix} 0 & \bar{P} \\ P & 0 \end{pmatrix} = U^{-1} D U \quad (2.30)$$

we see that χ itself is derivable from the superfield U . This superfield has 133 components, but since we have 63 gauge parameters for $SU(8)$, it follows that (at $\theta = 0$) it describes the 70 spin-0 fields of the theory. To summarize, the $N = 8$ theory in superspace is described by a scalar superfield U which is an element of the coset space $E_7/SU(8)$, and from which torsions, curvatures and spin-one field strengths may be constructed satisfying the required Bianchi identities. We mention that this theory can be formulated in a completely geometrical way, but since this is outside the main arguments of the paper, we relegate the details to Appendix B.

Since we have a solution which satisfies the Bianchi identities and have shown that all fields are components of the scalar superfield U , we may straightforwardly linearize the results tabulated in BH. We set

$$U = \begin{pmatrix} \delta_{[ab]}^{mn} & 0 \\ 0 & \delta_{[mn]}^{ab} \end{pmatrix} + \begin{pmatrix} 0 & \delta_{[cd]}^{mn} \bar{V}^{abcd} \\ \delta_{[mn]}^{cd} V_{abcd} & 0 \end{pmatrix} \quad (2.31)$$

where V is totally antisymmetric and self-dual, and where we have used the linearized local $SU(8)$ to remove the additional non-physical degrees of freedom. (We have also absorbed a factor of κ into V so that it has zero dimension.) The invariances of the linearized theory are thus global supersymmetry (with internal $SU(8)$) together with linearized general (superspace) co-ordinate transformations and linearized $SL(2, C)$. These latter two symmetries are local and one can use the $SL(2, C)$ to gauge away unwanted fields; however, the linearized $SL(2, C)$ curvature does not vanish, whilst from (2.19) we see that the $SU(8)$ part does.

The basic field V fulfils the following conditions:

$$V_{abcd} = \frac{1}{4!} \epsilon_{abcdefgh} \bar{V}^{efgh} \quad (2.32)$$

$$\begin{aligned}
D_\alpha^a V_{bcde} &= 4 \delta_{[b}^a \chi_{cde]\alpha} \\
D_\alpha^a \chi_{bcd\beta} &= 3 \delta_{[b}^a M_{cd]\alpha\beta} \\
D_\alpha^a M_{bc\beta\gamma} &= 2 \delta_{[b}^a \psi_{c]\alpha\beta\gamma} \\
D_\alpha^a \psi_{b\beta\gamma\delta} &= \delta_b^a C_{\alpha\beta\gamma\delta}
\end{aligned} \tag{2.33}$$

where each of the superfields V, χ, M, ψ, C is totally antisymmetric on its internal indices and totally symmetric on its spinor indices. The leading ($\theta = 0$) components are thus 70 scalar fields, 56 spin- $\frac{1}{2}$ fields, and 28 spin-one field strengths (observe that there is no $\hat{\alpha}\hat{\beta}$ field in the same representation of $SU(8)$, so that we have space-time self-duality), eight on-shell gravitino field strengths and the spin-two Weyl spinor respectively. In addition to the chain (2.33) obtained by differentiating with D 's, we can obtain another chain by differentiating with $\bar{D}$'s:

$$\begin{aligned}
\bar{D}_{\dot{\alpha}a} V_{bcde} &= \bar{\lambda}_{\dot{\alpha}abcde} \\
\bar{D}_{\dot{\alpha}a} \bar{\lambda}_{\dot{\beta}bcdef} &= \bar{N}_{\dot{\alpha}\dot{\beta}abcdef} \\
\bar{D}_{\dot{\alpha}a} \bar{N}_{\dot{\beta}\dot{\gamma}bcdefg} &= \bar{\varphi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}abcdefgh} \\
\bar{D}_{\dot{\alpha}a} \bar{\varphi}_{\dot{\beta}\dot{\gamma}\dot{\delta}bcdefgh} &= \epsilon_{abcdefgh} \bar{C}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}
\end{aligned} \tag{2.34}$$

where the new fields λ, N, φ, C are again antisymmetric on the internal indices and symmetric on the spinor indices and, because of (2.32), are related to χ, M and ψ . For example,

$$\bar{\lambda}_{\dot{\alpha}abcde} = \frac{1}{3!} \epsilon_{abcdefgh} \bar{\chi}_{\dot{\alpha}}^{fgh} \tag{2.35}$$

It is now straightforward to truncate the theory to lower N by merely restricting the internal indices in (2.33), (2.34) to run from 1 to N . For $N < 8$, (2.32) is no longer valid, and the chains (2.33) and (2.34) are independent. For $N < 4$, V itself is zero and we only have the chain (2.33) with the leading field being

χ, M, ψ for $N = 3, 2, 1$ respectively^{*)}. In all these cases, (as well as $N = 4$), the leading field in the chain is chiral. In restricting the invariance group itself, we notice that the traceless condition in $SU(8)$ disappears, so that we have a $U(N)$ invariance for $N \neq 8$. (For $N = 1$, this global $U(1)$ invariance arises as a special case of superconformal transformations¹⁴⁾ and corresponds to chiral transformations of the gravitino field¹⁵⁾.) To summarize, extended supergravity in superspace is described by a generating superfield, W , which is a scalar for $N \geq 4$, has spin $2-N/2$ for $N < 4$, and is totally antisymmetric on N internal indices. Its components are the scalar fields, spin-half fields, and field strengths for spins ≥ 1 which are represented by symmetric spinors with $2s$ (for spin s) spinor indices, and from (2.33), (2.34) we see that the spectrum falls into the familiar binomial coefficient pattern. For $N \leq 4$, W is a chiral superfield.

To conclude this section, we shall explicitly show that the theory is on-shell^{**)}. We consider first the case $N \geq 4$ where W is a scalar superfield. One has

$$\epsilon^{\dot{\beta}\dot{\gamma}} \bar{D}_{\dot{\beta}b} \bar{D}_{\dot{\gamma}c} \bar{V}^{def} = 0 \quad (2.36)$$

Differentiating with D_{α}^a yields

$$i(N+1) \partial_{\alpha}^{\dot{\beta}} \bar{D}_{\dot{\beta}c} \bar{V}^{def} = \epsilon^{\dot{\beta}\dot{\delta}} \bar{D}_{\dot{\beta}b} \bar{D}_{\dot{\delta}c} D_{\alpha}^b \bar{V}^{def} \quad (2.37)$$

after summing over a, b and c, d . Since $D_{\alpha}^b \bar{V}^{cefg}$ is totally skew, the right-hand side of (2.37) is zero and one has

$$\partial_{\alpha}^{\dot{\beta}} \bar{\chi}_{\dot{\beta}}^{abc} = 0 \quad (2.38)$$

Successively differentiating this equation with D 's, one finds

^{*)} In $N = 1$, ψ is just the Wess-Zumino superfield W ¹³⁾.

^{**)} The arguments we use here are similar to those applicable in the Yang-Mills case, see Ref. 16).

$$\partial_\alpha{}^i \bar{M}_{\dot{\alpha}\dot{\beta}}{}^{ab} = \partial_\alpha{}^i \bar{\Psi}_{\dot{\alpha}\dot{\beta}}{}^a = \partial_\alpha{}^i \bar{C}_{\dot{\alpha}\dot{\beta}}{}^i = 0 \quad (2.39)$$

The easily derivable relation

$$D_\alpha{}^a \bar{D}_{\dot{\beta}b} \bar{V}^{cdef} + D_\alpha{}^c \bar{D}_{\dot{\beta}b} \bar{V}^{adef} = i \delta_b^a \partial_{\alpha\dot{\beta}} \bar{V}^{cdef} + i \delta_b^c \partial_{\alpha\dot{\beta}} \bar{V}^{adef} \quad (2.40)$$

gives rise to the identities

$$D_\alpha{}^a \bar{D}_{\dot{\beta}a} \bar{V}^{cdef} = 4i \partial_{\alpha\dot{\beta}} \bar{V}^{cdef} \quad (2.41)$$

and

$$D_\alpha{}^c \bar{D}_{\dot{\beta}b} \bar{V}^{bdef} = i(N-3) \partial_{\alpha\dot{\beta}} \bar{V}^{cdef} \quad (2.42)$$

Differentiating (2.37) with a D and using (2.41), one obtains the spin zero equation of motion

$$\square V_{abcd} = 0 \quad (2.43)$$

A similar identity to (2.36) is

$$\epsilon^{\beta\gamma} D_\rho{}^b D_\gamma{}^c \bar{V}^{defg} = 0 \quad (2.44)$$

Differentiation of this equation with $\bar{D}_{\dot{\alpha}a}$ and summation over a, b gives

$$i(N+1) \partial_{\dot{\alpha}}^{\gamma} \bar{D}_{\gamma}^c \bar{V}^{defg} = \epsilon^{\beta\gamma} \bar{D}_{\beta}^b \bar{D}_{\gamma}^c \bar{D}_{\dot{\alpha}b} \bar{V}^{defg} \quad (2.45)$$

and the right-hand side of this equation can be shown to vanish by employing (2.42). Thus one finds

$$\partial_{\dot{\alpha}}^{\alpha} \lambda_{\alpha}^{abcde} = 0 \quad (2.46)$$

from which

$$\partial_{\dot{\alpha}}^{\alpha} N_{\alpha\beta}^{abcdef} = \partial_{\dot{\alpha}}^{\alpha} \mathcal{F}_{\alpha\beta\gamma}^{abcdefg} = 0 \quad (2.47)$$

Equations (2.38), (2.39), (2.43), (2.46) and (2.47) are the equations of motion and Bianchi identities for the fields of spins zero to two which occur in the theory. For $N < 4$, one starts with

$$\epsilon^{\dot{\beta}\gamma} \bar{D}_{\dot{\beta}b} \bar{D}_{\gamma c} \bar{W}_{\dot{\delta}_1 \dots \dot{\delta}_k}^{d_1 \dots d_N} = 0, \quad k = 4 - N \quad (2.48)$$

and differentiates with D 's to obtain the required equations.

3. COUNTERTERM LAGRANGIANS IN SUPERSPACE

The types of n loop counterterm Lagrangians that we will be interested in and that have been previously discussed have the linearized form

$$\mathcal{L}^{(n)}(x) = K^{2(n-1)} \left\{ R^{n+1} + \text{Supersymmetric completion} \right\} \quad (3.1)$$

where R^{n+1} is shorthand for an n fold product of the Riemann tensor with suitably contracted indices. (We can also have terms where two derivatives replace

one factor of R , etc.) On-shell, R is just the Weyl tensor and for the no-derivative case there are just two algebraically independent invariants¹⁷⁾,

$$\Delta_1 = C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \quad (3.2)$$

$$\Delta_2 = C_{\alpha\beta\gamma\delta} C^{\alpha\beta\epsilon\varphi} C^{\gamma\delta}_{\epsilon\varphi} \quad (3.3)$$

all others being the products of these two and their complex conjugates. $\Delta_1 + \bar{\Delta}_1$ is the one-loop counterterm for ordinary gravity and is a total divergence, whilst $\Delta_2 + \bar{\Delta}_2$ is the two-loop invariant which has no supersymmetric counterpart (at least for $N = 1, 2$). The basic problem is therefore to find the supersymmetric completion of R^{n+1} , and as N and hence the number of fields present in the theory increases, this task becomes progressively more complicated. To attack the problem we therefore switch to superspace, where the n loop counterterm action is of the form

$$S^{(n)} = \int d^4x d^{4N}\theta \mathcal{L}^{(n)}(z) \quad (3.4)$$

where

$$\dim. \left[\frac{\mathcal{L}^{(n)}(z)}{\kappa^{2(n-1)}} \right] = -2(N-n-1) \text{ mass} \quad (3.5)$$

From (3.5) we see that we encounter a new problem, namely that for large enough N and low enough n the right-hand side of (3.5) becomes negative and there are no covariant fields of negative dimension in the theory. This does not mean that the corresponding counterterms do not exist, however, since one can integrate suitably constrained superfields over subspaces of the whole superspace and obtain a supersymmetric result. Examples of this type are known for superfields other than chiral superfields, e.g., six-dimensional supersymmetric Yang-Mills¹⁸⁾, but as yet the general technology has not been fully developed, and we shall therefore restrict ourselves to integrating over the whole of superspace so that supersymmetry is manifest. (For $N \leq 4$, we shall also consider chiral superfields.)

We begin by considering the one-loop counterterm for $N \leq 4$. We recall that W is chiral ($\bar{D}W = 0$) and has the form

$$W = W_{(\alpha_1 \dots \alpha_{4-N})} [a_1, a_N] \quad (3.6)$$

The dimension of W is $(2-N/2)$ so that we can construct the following Lagrangian

$$\mathcal{L}^{(1)}(z) = \bar{\theta}^{2N} W^2 + \text{h.c.} \quad (3.7)$$

which has the correct dimension for all $N \leq 4$, and which is supersymmetric because of chirality. From (2.33) one has

$$W \sim \theta^N C \quad (3.8)$$

so that upon integrating out the θ 's, one obtains

$$\mathcal{L}^{(1)}(x) = \Delta_1 + \bar{\Delta}_1 \quad (3.9)$$

in all cases. We observe that the supersymmetric completion is identically zero and $\mathcal{L}^{(1)}(x)$ is thus a total divergence whose integral is a topological invariant (Euler number).

Our next result is the three-loop Lagrangian for $N \leq 4$, which has already been constructed for $N = 1, 2$ as we have mentioned previously^{1,2)}. For $N = 1$, the on-shell linearized form has also been given in superspace¹⁹⁾ and generalizes readily to the general ($N \leq 4$) case. One has

$$\mathcal{L}^{(3)}(\tau) = K^4 W^2 \bar{W}^2 \quad (3.10)$$

We observe that the increase in the dimension of the measure is precisely balanced by the decrease in dimension of W . The pure spin-two contribution to the x space Lagrangian is easily obtained using (3.8), and we find

$$\mathcal{L}^{(3)}(x) = \kappa^4 \left\{ \Delta, \overline{\Delta}, + \text{s.s.c.} \right\} \quad (3.11)$$

where the manifest supersymmetry of (3.10) guarantees that the supersymmetric completion (s.s.c) exists. We therefore conclude that for $N \leq 4$ there is a three-loop counterterm whose spin-two part is the square of the Bel-Robinson tensor (at least at the linearized on-shell level).

We now turn to the case $N \geq 5$ where W is a scalar field and is no longer chiral. This latter fact has the consequence that the simplest Lagrangians we can construct occur at the $(N-1)$ loop level and have the form

$$\mathcal{L}_p^{(N-1)}(z) = \kappa^{2(N-2)} (V_{abcd} \overline{V}^{abcd})^p \quad (3.12)$$

where $p \geq 2$. From (2.33) we have

$$V \sim \Theta^4 C \left(+ \overline{\Theta}^4 \overline{C} \text{ for } N=8 \right) \quad (3.13)$$

and, as examples, one obtains x space Lagrangians of the following type:

$$\begin{aligned} \mathcal{L}^4(x) &= \kappa^6 \left\{ \Delta, \square \overline{\Delta}, + \text{s.s.c} \right\} \quad N=5, p=2 \\ \mathcal{L}^5(x) &= \kappa^8 \left\{ \Delta_2 \overline{\Delta}_2 + \text{s.s.c} \right\} \quad N=6, p=3 \\ \mathcal{L}^7(x) &= \kappa^{12} \left\{ \Delta_1^2 (\overline{\Delta}_1)^2 + \text{s.s.c} \right\} \quad \begin{matrix} N=8, \\ p=4 \end{matrix} \quad (3.14) \end{aligned}$$

There are also $(N-1)$ loop Lagrangians which have no pure spin-two part, for example $N = 8, p = 5$.

4. THE NON-LINEAR CASE

We have seen that at the linearized on-shell level, there are three-loop counterterms for $N \leq 4$ and $(N-1)$ loop counterterms for $N \geq 5$, and it is natural to ask whether or not these results generalize to the full non-linear case. For $N \leq 3$, this generalization may be accomplished by the insertion of a factor of the superdeterminant of the vielbein E so that (3.10) becomes

$$\mathcal{L}^{(3)}(\tau) = \kappa^4 E W^2 \bar{W}^2 ; N \leq 3 \quad (4.1)$$

Here W is of the same type as in the linearized case (i.e., for $N = 3$, W is a spinor, etc.) but the equations (2.33) are altered by non-linear contributions. For $N \geq 4$, however, the scalars are represented in the full theory as elements of the coset spaces G'/G , where $G' = SU(4) \times SU(1,1)$, $SU(5,1)$, $SO^*(12)$ and E_7 for $N = 4, 5, 6, 8$ respectively⁴⁾. One can use local G invariance to eliminate the non-physical degrees of freedom so that, for example in $N = 8$, one may write⁴⁾

$$\mathcal{V} = \exp \left\{ \begin{array}{cc} 0 & \bar{V}^{abcd} \\ V_{abcd} & 0 \end{array} \right\} \quad (4.2)$$

with V totally skew and self-dual. Under an E_7 transformation, V transforms non-linearly and there must also be a compensating V dependent local $SU(8)$ transformation, so that the form of the gauge choice (4.2) is preserved⁴⁾. Under the $SU(8)$ subgroup of E_7 however, V transforms linearly and one may therefore generalize (3.44) to the non-linear case by inserting a factor of E . This procedure may be carried through for all $N \geq 4$ and leads to $(N-1)$ loop counterterms which do not, however, have the full G' invariance of the equations of motion.

In order to construct non-linear counterterms possessing the full $G' - G$ invariances, it is not possible to use the scalar field matrix $\mathcal{V}$, since it behaves in a similar fashion to the vierbein, e_m^a , from which one can only form the scalar density $\det e_m^a$ and $\det \mathcal{V} = 1$. The simplest way to construct invariants possessing all the symmetries of the equations of motion is therefore to use superfields which occur directly in the torsions. The lowest of these in terms

of mass dimensions is the spinor superfield $\chi_{\alpha,abc}$ and using this field, one can construct the following N loop Lagrangian:

$$\mathcal{L}^{(N)}(\tau) = \kappa^{2(N-1)} E \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \chi_{\alpha abc} \chi_{\beta def} \times \bar{\chi}_{\dot{\alpha}}^{abc} \bar{\chi}_{\dot{\beta}}^{def} \quad (4.3)$$

This is manifestly locally G and globally G' invariant, since χ is a scalar under the latter group. At the linearized level χ may be expressed in terms of V , and upon integrating out the θ 's, one finds an x space Lagrangian whose spin-two part has the schematic form

$$\mathcal{L}^{(N)}(x) (\text{spin } 2) = \kappa^{2(N-1)} \Delta_1 \square^{(N-3)} \bar{\Delta}_1, \quad (4.4)$$

This is easy to see for $N = 4$ where the linearized version of (4.3) may be re-written (up to total divergences) in the form

$$\mathcal{L}^{(4)}(\tau) = \kappa^6 V^2 \square \bar{V}^2 \quad (4.5)$$

and therefore corresponds to (3.10) with a box insertion so that (4.4) is obtained immediately.

5. CONCLUSIONS

We have shown that there are on-shell counterterm Lagrangians for all extended supergravities which begin at the three-loop level for $N \leq 3$ and at the N loop level for $N \geq 4$ in the full non-linear case, and that in the linearized version there are $(N-1)$ loop Lagrangians for $N \geq 4$, although these may only be generalized to the full case at the cost of G' invariance. We have not shown that a three-loop counterterm is not present in $N > 4$, although it is tempting to conjecture that this may be the case. Another partially optimistic result is that all the counterterms we have discussed are of the type allowed by Christensen et al.¹⁵⁾ and vanish for half-flat spaces.

At the off-shell level the Lagrangians we have constructed will still be valid in form (although (2.33) and (2.34) will no longer hold, since there will be auxiliary field contributions), provided that there is no change in the field representations and that there are no central changes²⁰⁾. If either of these two latter cases should arise, then the structure of the Bianchi identities would be radically altered and the analysis would have to be redone anew. One would still expect that there would be off-shell counterterms that give rise to the ones we have exhibited upon imposing the equations of motion, however. We remark in passing that the linearized one-loop off-shell Lagrangians (3.7) are the Lagrangians for extended superconformal gravity which exists up to $N = 4$. This has been pointed out elsewhere²¹⁾ and the fact that chirality is valid off-shell has been explicitly shown in component form²²⁾.

In conclusion it seems to be difficult to argue away the existence of counterterms in supergravity theories in any obvious way, but it may be that we have more to learn about quantum supersymmetry and that our analysis has been too naive. In this context we may adduce the results obtained for $N = 4$ Yang-Mills in four dimensions^{8,23)} and $N = 4$ non-linear sigma models in two dimensions²⁴⁾ where three-loop finiteness has been shown, although as yet, the reasons for this behaviour are not fully understood.

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NOTE: After this paper had been written, we received a preprint by R.E. Kallosh in which similar results were obtained. In addition, the author also proposed a three-loop counterterm for $N = 8$ (at the linearized on-shell level).

APPENDIX A: NOTATION AND CONVENTIONS

Curved (tangent space) indices are taken from the middle (beginning) of the alphabet, Latin indices are internal, Gothic vectorial, Greek spinorial, and Greek underlined run over both spinorial and internal. Capital indices run over all types. A tangent space vector is thus

$$X^A = (X^\mu, X^\alpha, X^{\dot{\alpha}}) = (X^\mu, X^\alpha_a, X^{\dot{\alpha}a}) \quad (\text{A.1})$$

and a covector

$$Y_A = (Y_\mu, Y_\alpha, Y_{\dot{\alpha}}) = (Y_\mu, Y^a_\alpha, Y_{\dot{a}a}) \quad (\text{A.2})$$

The summation convention is

$$X^A Y_A = X^\mu Y_\mu + X^\alpha Y_\alpha - X^{\dot{\alpha}} Y_{\dot{\alpha}} \quad (\text{A.3})$$

The metric is time-favoured:

$$\eta_{ab} = \text{diag. } (+1, -1, -1, -1) \quad (\text{A.4})$$

and the two-component spinor metric is

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{\alpha\beta} \quad (\text{A.5})$$

and the same for dotted indices. For spinors, one has

$$X_{\alpha a} = X^\beta_a \epsilon_{\beta\alpha}; \quad X^\alpha_a = \epsilon^{\alpha\beta} X_{\beta a} \quad (\text{A.6})$$

Complex conjugation changes a dotted index to an undotted (and vice versa) and raises or lowers internal U(N) indices, e.g.

$$\chi_a^{\dot{\alpha}} \rightarrow (\overline{\chi}_a^{\dot{\alpha}}) = \overline{\chi}^{\dot{\alpha} \alpha} \quad (\text{A.7})$$

The σ matrix is

$$(\sigma^{\mu})_{\alpha\dot{\alpha}} = (1, \vec{\sigma}) \quad (\text{A.8})$$

and may be used to convert vector indices to a pair of spinor indices, thus

$$\chi_{\alpha\dot{\alpha}} = (\sigma^{\mu})_{\alpha\dot{\alpha}} \chi_{\mu} \quad (\text{A.9})$$

The matrix $(\sigma^{\hat{A}\hat{B}})_{\alpha\beta}$ is defined by

$$(\sigma^{\hat{A}\hat{B}})_{\alpha\beta} = \frac{1}{4} \left[(\sigma^{\mu})_{\alpha\dot{\beta}} (\sigma^{\nu})^{\dot{\beta}\beta} - \mu \leftrightarrow \nu \right] \quad (\text{A.10})$$

and is symmetric on α, β .

APPENDIX B - ALTERNATIVE FORMULATION OF N = 8 SUPERGRAVITY

We consider a superspace with co-ordinates $Z^{\underline{m}} = (Z^{\underline{M}}, y^{\underline{m}})$ where $Z^{\underline{M}}$ are the co-ordinates of ordinary superspace and $y^{\underline{m}} = (y_{m_1 m_2}, \bar{y}^{\bar{m}_1 \bar{m}_2})$ are 56 additional bosonic co-ordinates ($y_{m_1 m_2} = -y_{m_2 m_1}$; $m_1, m_2 = 1, \dots, 8$). A tangent space vector has components

$$\chi^{\underline{A}} = (\chi^{\underline{A}}, \chi^{\underline{a}}) ; \quad \chi^{\underline{a}} = (\chi_{a, a_2}, \bar{\chi}^{a, a_2}) \quad (\text{B.1})$$

and the tangent space group remains $SL(2, \mathbb{C}) \times SU(8)$ with $SU(8)$ acting on the additional components in the natural way (as 28's and $\bar{28}$'s). The inverse "sehrvielbein" has the form

$$\varepsilon_{\underline{A}}^{\underline{m}} = \begin{pmatrix} E_{\underline{A}}^{\underline{M}} & \varphi_{\underline{A}}^{\underline{m}} \\ 0 & (\mathcal{V}^{-1})_{\underline{a}}^{\underline{m}} \end{pmatrix} \quad (\text{B.2})$$

where $\varphi_{\underline{A}}^{\underline{m}}$ is the set of 28 complex Abelian gauge fields and $\mathcal{V}$ the scalar superfield of Section 2. The Bianchi identities (2.10), (2.16), (2.18) and (2.19) now appear as components of the Bianchi identity

$$\mathcal{D} T^{\underline{A}} = \varepsilon^{\underline{B}}_{\underline{A}} R_{\underline{B}}^{\underline{A}} \quad (\text{B.3})$$

where $T^{\underline{A}}$ and $R_{\underline{A}}^{\underline{B}}$ are built in exactly the same way as for ordinary superspace and where we make the following identifications:

- i) $T_{AB}^{\underline{C}}, R_{AB}^{\underline{D}}, \underline{C}$ are those of ordinary superspace;
- ii) All curvatures which do not have AB as the first two indices are zero;
- iii) $T_{A, b_1 b_2, c_1 c_2} = P_{A, b_1 b_2, c_1 c_2}$
- iv) $T_{AB, c_1 c_2} = F_{AB, c_1 c_2}$

- v) All other components of the torsion apart from those cited above and their complex conjugates are zero;
- vi) $\partial_{\underline{m}}$ (any field) = 0.

Those parts of the Identities (B.3) which do not correspond to the $N = 8$ identities are then trivially satisfied.

The theory cast in this form may be truncated to lower N straightforwardly and this approach is similar to that of Ref. 21).

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