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THE INSTANTON DENSITY IN A QUARK GAS

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ABSTRACT

We consider the fermion determinant in an instanton background for a fixed density of quarks. For heavy quarks, instantons of scale sizes greater than the Fermi momentum (p_F) of the quark gas are strongly suppressed; p_F acts as an infrared cut-off on the instanton density. For light quarks, we present asymptotic estimates of the determinant and suggest that, in the strictly massless case, isolated instantons should be completely suppressed and only instanton - anti-instanton pairs should play a role.

1. INTRODUCTION

At typical nuclear densities, quarks are believed to be confined inside hadrons. However, in superdense matter, it is possible that quarks might become liberated and form a quark plasma. Several authors¹⁾ have considered such a possibility for the interior of neutron stars, where densities might be high enough to allow for a quark matter phase. Perturbative QCD calculations have been carried out for such a phase¹⁾, relying on the fact that asymptotic freedom will guarantee the smallness of the coupling constant, if the Fermi momentum of the quark gas is sufficiently high. The common strategy has been to compare the free energy obtained from these calculations to the ones derived from different nuclear models for the hadron phase, and thus obtain some indication of whether such a scenario can be realized. In order to settle the question, one would need a better understanding of the non-perturbative effects which are responsible for the confinement mechanism. Nevertheless, one can try to improve on the quark matter calculations by including the simplest non-perturbative effects, provided by the instantons of QCD, in a weak coupling approximation to the functional integral for the quark phase. This has been attempted by a number of authors²⁻⁵⁾, by simply using the Fermi momentum of the quark gas as an infra-red cut-off on the instanton density.

The aim of this paper is to try to understand, in detail, the effect of the quark gas on the instanton density of QCD. It is an attempt to calculate quantum fluctuations in the quark plasma around instanton solutions and thereby provide an indication of the changes introduced by the quark plasma in a semi-classical approach. All calculations are carried out at zero temperature and we introduce a chemical potential μ which couples to the quark density. Section 2 discusses the fermion determinant for both heavy (2.1) and light (2.2) quarks. Section 3 is devoted to speculations about the massless quark problem.

2. THE FERMION DETERMINANT

We shall be interested in computing the transition amplitude to go from the $|0\rangle$ to the $|1\rangle$ topological sector of the Yang-Mills vacuum, by carrying out a saddle-point integration around the instanton solution which tunnels between these sectors. The quantity to be calculated can be expressed as a path integral over gauge and fermion (quark) fields:

$$\lim_{\beta \rightarrow \infty} \langle 1 | e^{-\beta H} | 0 \rangle \propto \int [DA_\mu] e^{-S_m(A)} \int [D\bar{\psi}] [D\psi] e^{-\int d^4x [\bar{\psi}(\not{D} - \not{A} + m)\psi]} \quad (1)$$

$S_{YM}(A)$ denotes the pure Yang-Mills action, $D_\nu \equiv \partial_\nu - igA_\nu$ and the term $\xi_\nu \bar{\psi} \gamma_\nu \psi$ imposes the fixed density constraint⁶⁾. In the rest frame of the system of quarks, $\xi_\nu \equiv (\mu, \vec{0})$, μ being the chemical potential of quarks which is diagonal in colour space. For the sake of simplicity we shall work with only one flavour.

By expanding the gauge fields around the instanton and integrating out the fermions we obtain⁷⁾:

$$\lim_{\beta \rightarrow \infty} \langle 1 | e^{-\beta H} | 0 \rangle \simeq C_N \int d^4 x_\pm \int \frac{d\rho}{\rho^5} \left(\frac{8\pi^2}{g^2(M\rho)} \right)^{2N} e^{-\frac{8\pi^2}{g^2(M\rho)}} Z_F(m, \rho, \mu) [1 + O(g^2)] \quad (2)$$

The integer N refers to the gauge group $SU(N)$; C_N is a numerical constant; the integrals are over position and scale sizes of instantons; M is a renormalization point and the calculation has been normalized by the result for $A_\nu = 0$. The quantity Z_F , obtained from the integral over the fermions is defined as:

$$Z_F(m, \rho, \mu) = \frac{\text{Det} [\gamma_4 (\not{D}_{cl} - \not{X} + m)]}{\text{Det} [\gamma_4 (\not{D} - \not{X} + m)]} \quad (3)$$

where the A_ν field in D_{cl} is taken to be the instanton field. We can rewrite (3) in the form:

$$Z_F(m, \rho, \mu) = [Z_F(m, \rho, \mu) Z_F^{-1}(m, \rho, 0)] Z_F(m, \rho, 0) \quad (4)$$

Because the chemical potential does not affect the short-distance behaviour of the theory, by dividing out the $\mu = 0$ contribution we render the term inside the brackets ultra-violet finite. Taking the logarithm of (4):

$$\Omega_F(m, \rho, \mu) = \Delta \Omega_F(m, \rho, \mu) + \Omega_F(m, \rho, 0) \quad (5)$$

The expression for $\Delta \Omega_F$ can be formally expanded in the gauge fields to yield a series of one-loop graphs:

$$\Delta \Omega_F = \text{Tr} \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} i^n \left[\text{Diagram 1} - \text{Diagram 2} \right] \quad (6)$$

The wavy lines represent the instanton background field, the broken lines are free propagators in the quark gas, $(\not{\partial} - \not{\zeta} + m)^{-1}$, and the continuous lines are just the usual free fermion propagators, $(\not{\partial} + m)^{-1}$. The sum of all these terms yields a gauge invariant expression. Each term, however, is not individually gauge invariant. A gauge which is particularly suited for our calculations is the singular gauge⁸⁾, where the instanton field is more "localized", falling off at infinity as $|x|^{-3}$. In such a gauge every term in the series is finite.

2.1 Heavy Quarks ($m_Q \gg 1$)

If we consider quarks whose rest masses are large compared to the typical energy of the instanton field, i.e., $m_p \gg 1$ where ρ is the instanton scale size, the expansion depicted in (6) will yield a series in inverse powers of m_p (apart from logarithms). The dominant contribution will come from the $n = 2$ term (the $n = 1$ term is traceless in colour) and the others will add corrections which are higher order in $(m_p)^{-1}$:

$$\Delta \Omega_F = \frac{1}{2} \left[\text{dashed circle} - \text{solid circle} \right] + \text{higher orders in } (m_F)^{-1} \quad (7)$$

The lowest order term $(\Delta_F^{(2)})$, shown in brackets, is given by:

$$\Delta\Omega_F^{(2)} = \int \frac{d^4q}{(2\pi)^4} \text{tr} [A_\tau(q) A_\nu(-q)] \Delta\pi_{\tau\nu}(q^2, q^2) \quad (7.a)$$

$$\Delta \Pi_{\sigma\nu} \equiv -\frac{1}{2} \left\{ \int \frac{d^4 p}{(2\pi)^4} \text{Sp} \left[\gamma_\sigma \frac{1}{\not{p} + i\mu \gamma_4 - im} \gamma_\nu \frac{1}{(\not{p} - \not{q}) + i\mu \gamma_4 - im} \right] \right\} - \{(\mu=0)\} \quad (7.b)$$

In these expressions tr denotes colour trace, while Sp denotes spin trace. In (7.b) the second curly bracket indicates that we are to subtract the $\mu = 0$ result from the integral in the first bracket.

Because ξ_v introduces a preferred direction in Euclidean space, $\Delta\Pi_{\sigma v}$ will only possess three-dimensional rotational invariance. This, together with current

conservation ($q_\sigma \Delta \Pi_{\sigma\nu} = \Delta \Pi_{\sigma\nu} q_\nu = 0$), allows us to write a most general form for $\Delta \Pi_{\sigma\nu}$:

$$\Delta \Pi_{\sigma\nu} = \left(\delta_{\sigma\nu} - \frac{q_\sigma q_\nu}{q^2} \right) \Delta \Pi_A + \delta_{\sigma m} \delta_{\nu n} \left(\delta_{mn} - \frac{q_m q_n}{q^2} \right) \Delta \Pi_B \quad (8.a)$$

$$\Delta \Pi_A \equiv \frac{q^2}{q^2} \Delta \Pi_{44} ; \Delta \Pi_B \equiv \frac{1}{2} \left(\Delta \Pi - \frac{3q^2}{q^2} \Delta \Pi_{44} \right) ; \Delta \Pi \equiv \sum_\nu \Delta \Pi_{\nu\nu} \quad (8.b)$$

If we use the explicit form of the instanton field in singular gauge⁸⁾:

$$A_\nu(x) = \frac{2}{g} \bar{\eta}_{a\nu\sigma} \frac{f^2(x-x_I)_\sigma}{(x-x_I)^2 [(x-x_I)^2 + f^2]} \cdot \frac{\zeta_a}{2} \quad (9.a)$$

$$A_\nu(q) = \frac{16\pi^2 i}{g} \bar{\eta}_{a\nu\sigma} \frac{q_\sigma}{q^4} \left[1 - \frac{1}{2} (qf)^2 K_2(qf) \right] \cdot \frac{\zeta_a}{2} \quad (9.b)$$

expression (7.a) turns out to reduce to:

$$\Delta \Omega_F^{(2)} = 8 \int \frac{d^4 q}{q^6} f(qf) \Delta \Pi(q^2, q, \zeta) \quad (10)$$

where $\Delta \Pi$ is defined in (8.b). The fact that only the trace of $\Delta \Pi_{\nu\sigma}$ plays a role is associated to the spherical symmetry of the instanton solution. The function $f(x)$ is given by:

$$f(x) = \left[1 - \frac{1}{2} x^2 K_2(x) \right]^2 \quad (11)$$

The calculation of $\Delta\Pi$ can be performed by first integrating over dp_4 in the complex p_4 -plane. It is essentially the same calculation as for QED⁹⁾ and the result is:

$$\Delta\Pi = -\frac{1}{\pi^2} \int_0^{\sqrt{\mu^2 - m^2}} \frac{dp_4 p_4^2}{\epsilon_p} \left[1 + \frac{2m^2 - q^2}{8p|\vec{q}|} \ln \frac{(q^2 + 2p|\vec{q}|)^2 + 4\epsilon_p^2 q_4^2}{(q^2 - 2p|\vec{q}|)^2 + 4\epsilon_p^2 q_4^2} \right] \quad (12)$$

where $\epsilon_p = \sqrt{p^2 + m^2}$. Using $q_4 = qx$ and $|\vec{q}| = q\sqrt{1-x^2}$ and introducing dimensionless variables $y = qp$, $z = pp$ and $z_F = p_F p = \sqrt{\mu^2 - m^2} p$ we get:

$$\Delta\Omega_F^{(2)} = -\frac{32}{\pi} \int_0^\infty \frac{dy}{y^3} f(y) \int_0^{z_F} \frac{dz z^2}{\sqrt{z^2 + (mp)^2}} \int_{-1}^1 dx \sqrt{1-x^2} [1 + \Phi(x, y, z)] \quad (13.a)$$

$$\Phi(x, y, z) \equiv \frac{2(mp)^2 - y^2}{8zy\sqrt{1-x^2}} \ln \frac{(y^2 + 2zy\sqrt{1-x^2})^2 + 4x^2 y^2 \sqrt{z^2 + (mp)^2}}{(y^2 - 2zy\sqrt{1-x^2})^2 + 4x^2 y^2 \sqrt{z^2 + (mp)^2}} \quad (13.b)$$

The integrations over dx and dz can be performed analytically (see Appendix), however, due to the complicated form of the instanton field in momentum space, the integration over dy was performed numerically. The final result can be written as:

$$\Delta\Omega_F^{(2)} = -F(p_F/m)(mp)^2 [A(1-\alpha) + B] \quad (14)$$

where $A = A(mp)$, $\alpha = \alpha(mp, p_F/m)$, $B = 1$ and $F(x) = x\sqrt{1+x^2} - \ln(x + \sqrt{1+x^2})$. Plots of A and α are shown in Figures 1 and 2. In Figure 3 we show a plot of $\ln |\Delta\Omega_F^{(2)}|$ versus mp for $p_F/m = 10^{-1}$. The curve reaches zero for $mp \sim 13-14$, i.e., $p_F p \sim O(1)$. This means that in the determinant we obtain a suppression by a factor $e^{\Delta\Omega_F^{(2)}} \sim e^{-1}$. Clearly, fluctuations with $p \gg p_F^{-1}$ will be strongly suppressed. To give an idea of how strong the suppression is we have plotted values for $|\Delta\Omega_F^{(2)}|$ versus mp in Figure 4. We remind the reader that the calculations can only be trusted for larger values of mp , so that higher order corrections may

be neglected. Nevertheless, one can safely conclude that instanton fluctuations with $\rho \gg p_F^{-1}$ will be completely screened by the heavy quark plasma.

The presence of a chemical potential μ introduces a screening length in the polarization tensor $\Delta\Pi_{\nu\sigma}$, $\lambda^2 \sim (g^2 p_F^2)^{-1}$, which acts as an infra-red cut-off. Because our background field has a $1/g$ dependence and is characterized by a scale ρ , it is no surprise that the combination $p_F \rho$ should dictate the suppression [one could also accomplish such a suppression by using a condensate of bosons. 't Hooft⁷⁾ has shown that, in that case, the expectation value of the scalar field $\langle\phi\rangle$ provides the cut-off, i.e., $e^{-\langle\phi\rangle^2 \rho^2}$].

2.2 Light Quarks ($m \ll \mu$)

We shall now consider light quarks, a situation which is physically meaningful since it is believed that quark matter should be made up mostly of up and down quarks, whose masses are much smaller than typical estimates for the chemical potential μ for neutron stars (300-500 MeV). In that which follows we shall set the masses to zero whenever possible.

(i) The limit $\mu\rho \gg 1$

Let us go back to the diagrammatic expansion (6). One can easily write down a general expression for the $\Delta\Omega_F^{(n)}$:

$$\Delta\Omega_F^{(n)} = (-1)^{2n+1} \prod_{j=1}^{n-1} \int \frac{d^4 q_j}{(2\pi)^4} \text{tr} [A_{\sigma_1}(q_1) \dots A_{\sigma_n}(-\sum_{l=1}^{n-1} q_l)] \Delta\Pi_{\sigma_1 \dots \sigma_n}^{(n)}(q_1, \dots, q_{n-1}) \quad (15.a)$$

$$\Delta\Pi_{\sigma_1 \dots \sigma_n}^{(n)} = \int \frac{d^4 p}{(2\pi)^4} \text{Sp} \left\{ \gamma_{\sigma_1} \frac{1}{\tilde{p} + im} \prod_{k=2}^{n-1} \gamma_{\sigma_k} \frac{1}{(\tilde{p} + q_k) + im} \gamma_{\sigma_n} \frac{1}{\tilde{p} - q_1 + im} \right\} - \{\mu=0\} \quad (15.b)$$

where $\tilde{p} \equiv p + i\mu\gamma_4$ and in (15.b) we subtract the $\mu = 0$ contribution. If we neglect the masses (which can be done without running into infra-red problems), the expression for $\Delta\Pi^{(n)}$ will depend on μ and on the $(n-1)$ independent external momenta. The momentum space dependence of the instanton field is such that most of the contribution to $\Delta\Pi^{(n)}$ will come from $|q_j| \lesssim 1/\rho$. For higher momenta the field can be approximated by a power law, q^{-3} . If we consider $\mu\rho \gg 1$, the contribution from the region where the field is stronger will correspond to $|q_j/\mu| \lesssim 1/\mu\rho \ll 1$. Rescaling variables in $\Delta\Pi^{(n)}$:

$$\Delta \Pi^{(n)}(\mu, q_j) = \mu^{4-n} \Delta \tilde{\Pi}^{(n)}(q_j/\mu) \quad (16)$$

We can now expand $\Delta \tilde{\Pi}^{(n)}$ asymptotically for small $x_j = q_j/\mu$. Upon integrating over the external momenta dq_j^4 we shall obtain (apart from logarithms) terms of the form $(\mu\rho)^{4-n-i}$, with $i \in (0, \infty)$. On the other hand, the contribution from higher momenta will always be $O(1)$, as can easily be seen by rescaling the whole integral for $\Delta \Omega_F^{(n)}$ to obtain $\mu^{4(n-1)}$ from the integration variables, μ^{-3n} from A_j 's and μ^{4-n} from the $\Delta \Pi^{(n)}$. From these arguments one can conclude that the leading contribution to $\Delta \Omega_F$ behaves like $\mu^2 \rho^2$ and comes solely from the $n = 2$ graph, and that all terms $\Delta \Omega_F^{(n)}$ contribute to $O(1)$.

The calculation of $\Delta \Omega_F^{(2)}$ corresponds to repeating the analysis of the previous section for $m = 0$. The result is:

$$\Delta \Omega_F^{(2)} = -8 \int_0^{2\mu\rho} \frac{dy}{y^3} f(y) \left[\mu^2 \rho^2 - \frac{y^2}{4} - \frac{y^2}{4} \ln \frac{2\mu\rho}{y} \right] \quad (17)$$

To estimate the coefficient of $\mu^2 \rho^2$ we change variables to $t = y/2\mu\rho$ and keep the leading term for small t :

$$\Delta \Omega_F^{(2)} \xrightarrow{\mu\rho \gg 1} -8 \int_0^{2\mu\rho} \frac{dy}{y^3} f(y) \mu^2 \rho^2 \simeq \left[-8 \int_0^\infty \frac{dy}{y^3} f(y) \right] \mu^2 \rho^2 \quad (18)$$

The integral can be estimated numerically so that:

$$\Delta \Omega_F^{(2)} \xrightarrow{\mu\rho \gg 1} -\mu^2 \rho^2 \quad (19)$$

One can check this result by doing the whole integral (17) numerically. Figure 5 shows a plot of $|\Delta \Omega_F^{(2)}|/\mu^2 \rho^2$ where one can see that, for large $\mu\rho$, the curve levels off toward unity. The result for $\Delta \Omega_F$ in the limit $\mu\rho \gg 1$ is then:

$$\Delta \Omega_F^{(2)} = -\mu^2 \rho^2 \left[1 + O(m\rho, \frac{1}{\mu\rho}) \right] \quad (20)$$

Because every graph contributes in $O(1)$, when we exponentiate the result (20) to obtain the expression multiplying the $\mu = 0$ determinant in (4), our answer

will not produce an over-all factor coming from the $O(1)$ term. All one can say is that there will be an $e^{-\mu^2 \rho^2}$ suppression.

(ii) The limit $\mu\rho \ll 1$

In order to estimate what happens for $\mu\rho \ll 1$ let us go back to formula (4) for $Z_F(m, \rho, \mu)$. We can rearrange the term in brackets so that $Z_F(m, \rho, \mu)$ can be rewritten:

$$Z_F(m, \rho, \mu) = \left\{ \frac{\text{Det}[\gamma_4(\not{D}_{cl} - \not{\zeta} + m)]}{\text{Det}[\gamma_4(\not{D}_{cl} + m)]} \cdot \frac{\text{Det}[\gamma_4(\not{\zeta} + m)]}{\text{Det}[\gamma_4(\not{\zeta} - \not{\zeta} + m)]} \right\} Z_F(m, \rho, 0) \quad (21)$$

Taking the logarithm and differentiating with respect to the chemical potential:

$$\frac{\partial \Omega_F}{\partial \mu} = \frac{\partial \Delta \Omega_F}{\partial \mu} = \text{Tr} [\gamma_4 S(A, \mu) - \gamma_4 S(0, \mu)] \quad (22)$$

where $S(A, \mu)$ denotes the fermion propagator in a quark gas in an instanton background $(\not{D}_{cl} - \not{\zeta} + m)^{-1}$; $S(0, \mu)$ is just the free fermion propagator in a quark gas $(\not{\zeta} - \not{\zeta} + m)^{-1}$ and Tr denotes a total trace over space, spin and colour. Since we assume $\mu\rho \ll 1$ we shall be interested in computing (22) for $\mu = 0$. We can thus use $S(A, 0)$, the usual fermion propagator in an instanton background, which for small masses can be written¹⁰⁾:

$$\langle x | S(A, 0) | y \rangle = \sum_E \frac{\psi_E(x) \psi_E^+(y)}{E - m} = -\frac{\psi_0(x) \psi_0^+(y)}{m} + S_A(x, y) + m \int d^4 z S_A(x, z) S_A(z, y) + O(m^2) \quad (23)$$

where $\psi_0(x)$ denotes the normalizable zero eigenmode of the Dirac operator in an instanton background⁷⁾, $S_A(x, y)$ corresponds to the massless fermion propagator in the subspace orthogonal to the zero modes¹¹⁾ and in the calculations we only keep terms which are non-vanishing as $m \rightarrow 0$. By carefully defining¹²⁾ the right-hand side of (22) for $\mu = 0$, one can show that the result vanishes, i.e., no linear contribution in $\mu\rho$ will appear. However, taking a second derivative:

$$\left. \frac{\partial^2 \Omega_F}{\partial \mu^2} \right|_{\mu=0} = \text{Tr} [\gamma_4 S(A, 0) \gamma_4 S(A, 0)] - \text{Tr} [\gamma_4 S(0, 0) \gamma_4 S(0, 0)] \quad (24)$$

This can be calculated by using the known result^{10),13)} for $\Delta\pi_{\nu\sigma}$ defined as:

$$\Delta\pi_{\nu\sigma}(x,y) = \text{tr} S_P \langle x | \gamma_\nu S(A,0) \gamma_\sigma S(A,0) - \gamma_\nu S(0,0) \gamma_\sigma S(0,0) | y \rangle \quad (25.a)$$

$$\begin{aligned} \Delta\pi_{\nu\sigma} = & \frac{3}{4\pi^4} (h_x h_y)^2 p^2 \left\{ - \frac{\Sigma^2 \Delta_\nu \Delta_\sigma}{\Delta^4} + \frac{\Sigma \cdot \Delta (\Sigma_\nu \Delta_\sigma + \Delta_\nu \Sigma_\sigma - \Sigma \cdot \Delta \delta_{\nu\sigma})}{\Delta^4} \right. \\ & \left. + \frac{\Delta^2 \delta_{\nu\sigma} - \Delta_\nu \Delta_\sigma + \Delta_\nu \Sigma_\sigma - \Delta_\sigma \Sigma_\nu}{\Delta^2} \right\} \end{aligned} \quad (25.b)$$

where $h_x \equiv 1/(x^2 + p^2)$, $\Sigma \equiv x + y$ and $\Delta \equiv x - y$. Because of the spherical symmetry of the $\mu = 0$ theory, $\Delta\pi_{44} = 1/4 \Delta\pi$ where $\Delta\pi \equiv \sum_\nu \Delta\pi_{\nu\nu}$. Therefore:

$$\left. \frac{\partial^2 \Omega_F}{\partial \mu^2} \right|_{\mu=0} = \int d^4x d^4y \Delta\pi_{44}(x,y) = \frac{1}{4} \int d^4x d^4y \Delta\pi(x,y) \quad (26)$$

Using (25.b) and integrating we obtain:

$$\left. \frac{\partial^2 \Omega_F}{\partial \mu^2} \right|_{\mu=0} = - \frac{9}{64} p^2 \simeq - 0.14 p^2 \quad (27)$$

Clearly, apart from corrections $O(m_p)$ (we are assuming $m_p \ll \mu p \ll 1$) we shall have:

$$\begin{aligned} \Omega_F(p, \mu) = & \Omega_F(p, 0) + \frac{1}{2} \left. \frac{\partial^2 \Omega_F}{\partial \mu^2} \right|_{\mu=0} \mu^2 + O(\mu^4) = \Omega_F(p, 0) - \\ & - 0.07 \mu^2 p^2 + O(\mu^4 p^4) \end{aligned} \quad (28)$$

If we put together the information obtained from both limits, we see that the fermion determinant for the light quark gas behaves like:

$$Z_F(m, p, \mu) = e^{-G(\mu_F, m_F)} Z_F(m, p, 0) \quad (29)$$

$$G_0(\mu_f) \equiv \lim_{m \rightarrow 0} G(\mu_f, m_f) = \begin{cases} \mu_f^2 p^2 + \dots & \mu_f \gg 1 \\ 0.07 \mu_f^2 p^2 + \dots & \mu_f \ll 1 \end{cases} \quad (30)$$

In the expression for G there will be corrections due to the mass:

$$G(\mu_f, m_f) = G_0(\mu_f) + G_1(\mu_f) m_f + G_2(\mu_f) m_f \ln m_f + \dots \quad (31)$$

We have, however, confined our discussion to the first term.

3. MASSLESS QUARKS

The results of the previous section seem to suggest that if one were to take the $m \rightarrow 0$ limit of (29), then:

$$Z_F(m, \mu, \mu) \stackrel{m \rightarrow 0}{\sim} e^{-G_0(\mu_f)} m_f \quad (32)$$

where we have used the fact that $Z_F(m, \mu, 0)$, for small m , becomes proportional to $m\mu$. This is due to the existence of normalizable zero eigenmodes of the Dirac equation for massless fermions; for small m , one obtains the $m\mu$ dependence by simply using ordinary perturbation theory in the mass. What equation (32) implies is that, just as in the $\mu = 0$ case, the determinant for the quark gas should also vanish in the strictly massless limit.

Motivated by this, we decided to search for zero eigenmodes of the Dirac equation in an instanton background with the chemical potential:

$$(\mathcal{D}_{cl} - \not{\beta}) \psi = 0 \quad (33)$$

Solutions to this equation can be related to the usual Dirac equation by simply noting that, if we write $\psi = e^{\xi x} \psi_0$, ψ_0 will then satisfy $\mathcal{D}_{cl} \psi_0 = 0$. It is obvious that, if we were to take the usual zero eigenmode⁷⁾ for ψ_0 , the expression for ψ would blow exponentially. In the hope of obtaining a normalizable ψ , we looked for solutions ψ_0 which, in the rest frame, behaved asymptotically like $e^{-\mu x_4}$, so as to cancel the exponential behaviour in ψ . We employed the techniques of Refs. 14), 15), working in singular gauge with a representation of the γ matrices given by:

$$\gamma_4 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad \gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix} \quad (34)$$

σ_j being the Pauli matrices. We split ψ_0 into upper and lower components and write:

$$[\tilde{\psi}_0]_{u,L} = [\psi_0^T]_{u,L} \sigma_2 = M_\alpha(x) \Gamma_\alpha \quad (35)$$

where T stands for transposed and Γ_α are 2×2 matrices which mix colour and spin indices (numerically, the fourth component is i times the identity whereas the spatial components are the Pauli matrices). Looking at one of the chiral subspaces we can arrive at the following equations^{14),15)}:

$$M_\alpha(x) = \Lambda^{\frac{1}{2}}(x) N_\alpha(x) = \Lambda^{\frac{1}{2}}(x) \left[\partial_\alpha \left(\frac{\varphi}{\Lambda} \right) + b_\alpha \right] \quad (36.a)$$

$$\square \varphi = -2b_\alpha \partial_\alpha \Lambda \quad ; \quad \Lambda(x) = 1 + \frac{\rho^2}{x^2} \quad (36.b)$$

$$b_\mu = \partial_\nu \chi_{\mu\nu} \quad ; \quad \square \chi_{\mu\nu} = 0 \quad (36.c)$$

$\chi_{\mu\nu}$ is antisymmetric and antiselfdual. Just as in the $\mu = 0$ case, we set $b_\alpha = 0$ since (36.a) would otherwise lead to singular solutions. This reduces (36.b) to $\square \varphi = 0$ and since we want $e^{-\mu x_4}$ behaviour one can verify that the solution is:

$$\varphi(x) = \frac{e^{-\mu x_4}}{x^2} \left\{ \cos \mu |\vec{x}| + \frac{x_4}{|\vec{x}|} \sin \mu |\vec{x}| \right\} \quad (37)$$

where $x^2 = x_4^2 + \vec{x}^2$. This reduces to the known solution ($\phi = 1/x^2$) for $\mu \rightarrow 0$. Unfortunately, when one computes the norm of ψ obtained from (37) the answer is:

$$\langle \psi | \psi \rangle = \mu^2 \ln \left(\frac{V^{\frac{1}{3}}}{\rho} \right) + \text{finite terms} \quad (38)$$

where V is the spatial three-volume of Euclidean space. Despite our efforts we were unable to find a normalizable zero eigenmode and, although short of proof, we believe that Eq. (37) expresses the change in the previously normalizable mode, as we introduce a chemical potential.

Nevertheless, the fact that our mode is only logarithmically divergent may still be compatible with the speculation that the determinant for massless fermions vanishes. In the $\mu = 0$ case such a phenomenon is essentially due to the axial anomaly, which relates the divergence of the axial current to the topological winding number. Instantons would tend to flip the chirality of the fermionic states which is forbidden by the chiral invariance of the massless theory. Since the addition of a chemical potential does not change the short-distance behaviour of the theory, similar arguments should still hold. In fact, in two-dimensional models this is indeed what happens⁶⁾ and there the zero modes are also logarithmically divergent¹⁶⁾.

Our findings disagree with the claim of Ref. 17) that instantons should have no effect on the massless quark gas. That analysis tries to relate the $\mu \neq 0$ and $\mu = 0$ exact fermion propagators. However, the whole treatment is restricted to subspaces orthogonal to zero modes and it is assumed that projections onto these zero modes are trivially related (by factors of $e^{\xi x}$), which we believe is incorrect.

4. CONCLUSIONS

The analysis of instanton contributions to the study of quark matter seems to be plagued with technical difficulties introduced by the reduction of the symmetry of the problem due to the existence of a preferred frame. One can safely conclude that instanton fluctuations will be suppressed, the Fermi momentum acting as an infra-red cut-off for large scale fluctuations. The strictly massless case seems to point to the complete suppression of one-instanton fluctuations. Presumably, only instanton-anti-instanton pairs should be allowed and a correct treatment would require a better understanding of the role of zero modes, which we lack at the moment.

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APPENDIX

Expression (13.a) can be written as a sum of two terms $I_1 + I_2$, of which only the latter depends on Φ . The first term is trivially reduced to:

$$I_1 = - \left[8 \int_0^\infty \frac{dy}{y^3} f(y) \right] F(p_F/m) m^2 \rho^2 \quad (A.1)$$

where $F(x) = x \sqrt{1+x^2} \ln(x + \sqrt{1+x^2})$. The Φ dependent part can be rewritten as:

$$I_2 = - \frac{64}{\pi} \int_0^\infty \frac{dy}{y^3} f(y) \frac{2(mp)^2 - y^2}{8y} \int_0^{z_F} \frac{dz z}{\sqrt{z^2 + (mp)^2}} \int_0^1 \frac{d\eta \eta}{\sqrt{1-\eta^2}} \ln W(y, z, \eta) \quad (A.2)$$

where $\eta = \sqrt{1-x^2}$ and W is the argument of the log in (13.b). The η integral can be performed by expressing the $\ln W$ as a sum of logs of monomials. Then:

$$\begin{aligned} I_2 = & - \frac{64}{\pi} \left\{ \int_0^{z_F} \frac{dy}{y^3} f(y) \frac{2(mp)^2 - y^2}{8y} \left[\int_0^{y/2} \frac{dz z}{\sqrt{z^2 + (mp)^2}} \frac{\pi z}{(mp)^2} \left(\sqrt{y^2 + 4(mp)^2} - y \right) + \right. \right. \\ & \left. \left. + \int_{y/2}^{z_F} \frac{dz z}{\sqrt{z^2 + (mp)^2}} \frac{\pi y}{(mp)^2} \left(\sqrt{z^2 + (mp)^2} - z \right) \right] \right\} + \int_{2z_F}^\infty \frac{dy}{y^3} f(y) \frac{2(mp)^2 - y^2}{8y} \int_0^{z_F} \frac{dz z}{\sqrt{z^2 + (mp)^2}} \\ & \cdot \frac{\pi z}{(mp)^2} \left[\sqrt{y^2 + 4(mp)^2} - y \right] \end{aligned} \quad (A.3)$$

Adding and subtracting the expression:

$$- \frac{\theta}{(mp)^2} \int_0^{z_F} \frac{dy}{y^3} f(y) \frac{2(mp)^2 - y^2}{y} \left[\sqrt{y^2 + 4(mp)^2} - y \right] \int_{y/2}^{z_F} \frac{dz z^2}{\sqrt{z^2 + (mp)^2}} \quad (A.4)$$

and performing the dz integrals, we get:

$$I_2 = - 8(mp)^2 F(p_F/m) \int_0^\infty \frac{dy}{y^3} f(y) \left[1 - \frac{y^2}{2(mp)^2} \right] \left[\frac{\sqrt{y^2 + 4(mp)^2}}{y} - 1 \right] + \quad (A.5)$$

$$+ 8(m_p)^2 \int_0^{2z_F} \frac{dy}{y^3} f(y) \left[1 - \frac{y^2}{2(m_p)^2} \right] \left\{ \frac{\sqrt{y^2 + 4(m_p)^2}}{y} \left[F(R/m) - F(y/2m_p) \right] - \frac{z_F^2 - y^2/4}{(m_p)^2} \right\}$$

The first term of I_2 , when computed numerically, yields the A term in (14).
The second term is related to α whereas I_1 is proportional to B in (14).

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Fig. 1

Fig. 2

Fig. 3

Fig.4

$$|\Delta\Omega_F^{(2)}|/\mu^2\rho^2$$

Fig.5