THE TRANSITION FROM STRONG COUPLING TO WEAK COUPLING
IN THE SU(2) LATTICE GAUGE THEORY

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A B S T R A C T

The effects of modifications in the action for SU(2) lattice gauge theories are investigated by Monte Carlo computations. We find that Manton's action and a generalization of Villain's U(1) action lead to earlier approaches to scaling and evaluate the relevant parameters. We also find evidence of a higher-order phase transition with Manton's action.

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GENERAL CONSIDERATIONS

The lattice formulation of gauge field theories ¹⁾ is considered by particle physicists as a very convenient way of performing a gauge invariant regularization, which also allows a strong coupling expansion. Eventually, a continuum limit must be taken. From this point of view, the specific form chosen for the lattice action is not so relevant : it is expected that wide classes of actions lead to the same continuum theory when the coupling constant approaches some suitable scaling critical point. Yet much attention has also been paid to the properties of lattice systems away from their critical points : besides being interesting per se, it is believed that these investigations may provide a clue to the understanding of the corresponding continuum theories. It then becomes important to see how a change in the form of the action affects the features of the theory. It is clear that any property of the lattice system, such as the rapid cross-over from a strong coupling to a scaling régime observed with non-Abelian gauge models ²⁾, can be related to properties of the continuum theory ³⁾ only if some degree of universality with respect to a modification of the lattice action exists.

These and other considerations have motivated us to perform a numerical analysis, by the Monte Carlo method, of the SU(2) lattice gauge theory defined with actions different from the one standardly used in these computations.

The dynamical variables of a lattice gauge theory are, we recall, the finite group elements U_{ij} associated with the oriented links of the lattice. The ordered products $U_{\square}$ of the four U_{ij} associated with the links forming the sides of an elementary square constitute the plaquette variables. To form the action, one considers a class function (i.e., a function invariant under similarity transformations $U_{\square} \rightarrow G^{-1} U_{\square} G$, which guarantees gauge invariance) $f(U_{\square})$ and then sums over all plaquettes ^{*)}. The dependence on the coupling constant g , or on the parameter $\beta = 4/g^2$, is contained in f . Most often it is simply multiplicative : $f(U_{\square}, \beta) = \beta \hat{f}(U_{\square})$, but this is not necessary.

If the gauge group is SU(2), the only one which will be considered in this paper, then $U_{\square}$ may be represented in the form :

$$U_{\square} = \cos \theta_{\square} + i \vec{\sigma} \cdot \hat{n} \sin \theta_{\square} \quad (1)$$

^{*)} Effects of change in the form of the action, made to incorporate also variables extending to more than one plaquette, have been considered by Wilson ⁴⁾. This change is of a different nature than the one studied in the present paper.

and f must be an even, periodic function of θ . The so-called Wilson form of the action is obtained choosing

$$f_w = \beta (1 - \frac{1}{2} \text{Tr } U_\square) = \beta (1 - \cos \theta_\square) \quad (2)$$

and, to our knowledge, it is the only one which has been used in Monte Carlo simulations of the $SU(2)$ gauge theory. However, one would expect that any f , with reasonable properties, could be used to define a lattice system with the same continuum limit. The action of a single plaquette should be minimum when $U_\square$ equals the identity ; near the minimum it should have a quadratic behaviour which is convenient to normalize so that, at least for large β ,

$$f(\theta, \beta) \underset{\theta \sim 0}{\sim} \frac{\beta}{2} \theta^2 \quad (3)$$

and it should have no local minima other than the values $\theta = 0 \pmod{2\pi}$.

In particular, interesting candidates which have been proposed and investigated in different contexts are Manton's form ^{5),6)} :

$$f_M = \frac{\beta}{2} \bar{\theta}_\square^2 \quad (4)$$

where by $\bar{\theta}_\square$ we denote the value between $-\pi$ and π congruent to $\theta_\square$ modulus 2π , and a generalization of the form suggested by Villain for the XY model ⁷⁾.

In this generalization one takes for $\exp(-f)$ the heat kernel over the group manifold ^{8),9)} which is given by

$$\begin{aligned} \exp(-f_v) &= \frac{\sum_{n=-\infty}^{\infty} (\sin \theta)^{-1} (\theta + 2n\pi) \exp \left[-\frac{\beta}{2} (\theta + 2n\pi)^2 \right]}{\sum_{n=-\infty}^{\infty} (1 - 4\beta n^2 \pi^2) \exp \left[-2\beta n^2 \pi^2 \right]} \\ &= \frac{\sum_{\ell=0}^{\infty} (\ell+1) (\sin \theta)^{-1} \sin[(\ell+1)\theta] \exp \left[-\frac{1}{2\beta} \ell(\ell+2) \right]}{\sum_{\ell=0}^{\infty} (\ell+1)^2 \exp \left[-\frac{1}{2\beta} \ell(\ell+2) \right]} \quad (5) \end{aligned}$$

We have performed Monte Carlo simulations of the $SU(2)$ lattice gauge system with these two actions, measuring the average value of the internal energy, its derivative, related to the specific heat of the system, and the string tension. We have found that using Manton's form of the action one obtains a smoother behaviour of the internal energy versus β than with Wilson's action ^{2),10)-12)} ; in particular the inflexion point, which with Wilson's action leads to a pronounced peak ¹³⁾ in the specific heat, is absent.

Measurements of the string tension, done both with Manton's action and with the generalized Villain's action, exhibit very clearly the scaling behaviour characteristic of the approach to the continuum limit and already observed with Wilson's form of the action ^{2),10),11)}. The transition from the strong coupling régime to the scaling régime occurs earlier in β , at $\beta \approx 1.5$ with Manton's action and at $\beta \approx 2.0$ with the generalized Villain's form. This gives larger values for the Λ_0 parameters than in Wilson's case, $\Lambda_0^{(M)} = (0.0616 \pm 0.0020)\sqrt{\sigma}$ and $\Lambda_0^{(V)} = (0.0206 \pm 0.0011)\sqrt{\sigma}$ [$\Lambda_0^{(W)} = (0.012 \pm 0.002)\sqrt{\sigma}$ ^{10),11)}], which agree with theoretical expectations (see later).

Apart from this difference in the scales, the transition appears still rather abrupt and similar to what one observes with Wilson's action. But a most surprising result came up when we evaluated the derivative of the internal energy with Manton's action. The very good statistics of our simulation allowed us to determine this function very accurately, both from a direct evaluation of the derivative and from a computation of the energy fluctuations. The curve we obtained reveals a very marked, discontinuous change in slope at $\beta \approx 1.6$. Although a numerical computation can never distinguish between a real singularity and regular, but rapidly varying behaviour, the discontinuity in the second derivative of the internal energy, if indeed present, would signal a third-order phase transition.

In our computation we have approximated the full $SU(2)$ gauge group with its 120 element icosahedral subgroup. That gauge system with finite subgroups of $SU(2)$ may give very good approximations to the systems with finite subgroups of $SU(2)$ has been shown already in the literature ^{12),14),11)}. The technical advantage of working with a finite group is enormous : the possibility of using a multiplication table, stored in the computer memory, to implement the group operation and to perform other logical manipulations allows a very marked reduction in computer time. On the CDC 7600 at CERN it takes our algorithm ≈ 0.8 seconds to perform a full Monte Carlo iteration with an 8^4 lattice, i.e., to go through the upgrading procedure of all 16384 dynamical variables. The Monte Carlo method has been by now presented many times in the literature and we shall not dwell on it. We mention only that we have used a generalization of the algorithm of Metropolis et al. ¹⁵⁾, in which the new trial value for a U_{ij} variable is obtained by multiplying U_{ij} by one of the 12 neighbours of the identity (i.e., the group elements of the form $\cos\theta + i\vec{\sigma} \cdot \hat{n} \sin\theta$ with $|\theta| = 2\pi/10$, the lowest non-vanishing value) chosen at random ; this upgrading procedure is then repeated six times before going to another U_{ij} . We have done all computations on an 8^4 lattice, except for the analysis of the energy fluctuations, which has been done with long runs of 10000 iterations on a 4^4 lattice.

THE RESULTS

With Wilson's and Manton's actions we define the average plaquette contribution to the action of a certain lattice configuration with the factor of β removed :

$$S_P^W \equiv \frac{1}{N} \sum_{\square} (1 - \cos \theta_{\square}) \quad (6)$$

$$S_P^M \equiv \frac{1}{N} \sum_{\square} \frac{1}{2} \bar{\theta}_{\square}^2 \quad (7)$$

where N is the number of plaquettes ; we call the expectation value $\langle S_P \rangle$ the "internal energy" E . In the generalized Villain's model the more complicated dependence of the action on β does not lend itself to a simple definition of internal energy and we have not considered this quantity.

In Fig. 1a, taken from Ref. 11), we reproduce the results of a Monte Carlo evaluation of the function $E_W(\beta)$. In Fig. 1b we show the results of our determination of $E_M(\beta)$. The measurement has been done with a very large number of Monte Carlo iterations on an 8^4 lattice with periodic boundary conditions. The number of iterations performed to evaluate E varied from 120, for the values of β where less accuracy was sought ($0.8 \div 1$), to 240 or 300 for $\beta = 1.1 \div 1.45, 1.55, 2.4$ and 2.5, and to over 800 for the other points in the range $1.5 \div 2$ where we wanted to have a quite precise determination of E . The efficiency of our algorithm allowed us to perform all these iterations while still keeping the cost of the computing moderate ; the errors inherent in the statistical fluctuations are thus sensibly reduced. A straightforward evaluation of these errors, according to the formula $\Delta E = (1/\sqrt{n-1}) \langle S_P^2 - \langle S_P \rangle^2 \rangle^{\frac{1}{2}}$ (n being the number of iterations), would indeed give $\Delta E \lesssim 1.3 \times 10^{-4}$ for the points between $\beta = 1.5$ and $\beta = 2$, but this number underestimates the possible effects of fluctuations with a range comparable with or longer than n , and therefore the true errors may be larger. Another detail that should be mentioned about the computational procedure is that we took as initial configurations for the runs used to evaluate E (and the string tension) those encountered along a very slow cooling process : starting from a completely disordered lattice several Monte Carlo iterations were performed to bring β to 0.8 and then 100 (or 200 for larger β) further iterations were done at each value of β before proceeding to the next one. This, of course, reduces any errors due to possible deviations from statistical equilibrium.

In Fig. 1b we also reproduce the strong coupling expansion for E [carried up to $O(\beta^{14})$]. In deriving it one utilizes the character expansion for Manton's action

$$\exp[-f_M(\theta, \beta)] = a(\beta) \left[1 + \sum_{r>0} \nu_r c_r(\beta) \chi_r(U) \right] \quad (8)$$

where the sum is over all non-trivial representations with dimensions ν_r . It should be noticed that, in distinction to Wilson's case, all $c_r(\beta)$ are of order β and in principle all representations have to be taken into account at every order. In practice, the contributions from higher representations become rapidly negligible. The strong coupling expansion for E is then obtained by standard methods [see, for instance, Ref. 16)]. [Incidentally, up to the order considered, insertion of the characters for the $SU(2)$ group or its icosahedral subgroup makes very little difference in the curve for $E(\beta)$, thus supporting the validity of the discrete group approximation.] We notice in Fig. 1b a rather marked difference in the behaviour of the function $E(\beta)$ with respect to Wilson's case. In particular, the inflection point, which leads to a pronounced peak in the specific heat, is absent. However, as with Wilson's action, the strong coupling expansion (to the order considered) appears to reproduce quite well the data up to some value of β , after which it departs from the numerical results rather abruptly.

Figures 2a and 2b illustrate the results of our measurements of the string tension. If U_γ is the product of the U_{ij} variables along a closed path γ , the Wilson loop factor is defined as

$$W_\gamma = \left\langle \frac{1}{2} \text{Tr } U_\gamma \right\rangle \quad (9)$$

The coefficient κ of an area term in the exponential fall-off

$$W_\gamma \sim \exp(-\kappa A_\gamma) \quad (10)$$

may be evaluated comparing loop factors for square and rectangular loops of the same perimeter and different areas^{11),12)} or, anyway, of suitably related sizes¹⁰⁾. If γ is a rectangular path extending for m and n plaquettes and we denote W_γ by $W(m,n)$, some of the possible determinations of κ are given by

$$\kappa^{(A)}(\beta, m) = -\ln \frac{W(m, m) W(m-1, m-1)}{W(m, m-1)^2} \quad (11)$$

$$\kappa^{(B)}(\beta, m) = -\ln \frac{W(m, m)}{W(m+1, m-1)} \quad (12)$$

and

$$K^{(C)}(\beta, m) = -\frac{1}{2} \ln \frac{W(m, m) W(m-2, m-2)}{W(m-1, m-1)^2} \quad (13)$$

In any case, one evaluates a size dependent function $\kappa(\beta, m)$ which would tend to the correct $\kappa(\beta)$ only in the limit $m \rightarrow \infty$. Since in the actual computation one is necessarily limited to finite and rather small m , the function $\kappa(\beta)$ is seen as an envelope of the curves $\kappa(\beta, m)$. Following Ref. 10), we have used $\kappa^{(A)}(\beta, m)$, with m up to 4, to measure the string tension ; with the help of the other two determinations $\kappa^{(B)}$ and $\kappa^{(C)}$ we have estimated the systematic errors (due to the presence of terms other than the area and perimeter ones in the exponential fall-off of W), which, for higher values of β , are typically larger than the statistical ones. In the figures we plot $\kappa^{(A)} \pm \frac{1}{2} (|\kappa^{(A)} - \kappa^{(B)}| + |\kappa^{(A)} - \kappa^{(C)}|)$.

Basically, the same number of Monte Carlo iterations on the 8^4 lattice has been used for the measurement of the loop factors as for the evaluation of $E_M(\beta)$. However, W was calculated on fewer configurations, chosen at random along the sequence of Monte Carlo iterations to minimize correlations ; on average the measurements were made every fourth iteration.

The upper parts of Figs. 2a and 2b show the values found for $\kappa(\beta, m)$ with Manton's and the generalized Villain's action, respectively. In Fig. 1a we also show the result of a strong coupling expansion (correct up to fifth order in β) for Manton's action

$$K(\beta) = -\ln c_{1/2} - 4 c_{1/2}^4 - 4 \sum_{r \geq 1/2} (2r+1)(r+1) (c_r^5 c_{r+1/2} + c_{r+1/2}^5 c_r) c_{1/2}^{-1} + O(\beta^6) \quad (14)$$

where the $c_r(\beta)$ have been defined above. The almost straight solid lines represent the best fits of the expected scaling behaviour as predicted by the renormalization group

$$K(\beta) \sim \frac{\sigma}{\Lambda_0^2} \left(\frac{6\pi^2}{11} \beta \right)^{-\frac{102}{121}} \exp \left(-\frac{6\pi^2}{11} \beta \right) \quad (15)$$

(σ being the physical string tension and Λ_0^2 the lattice regularization parameter) through the points in the envelope. The lower parts of Figs. 2a and 2b display the products

$$K_{\text{measured}} \left(\frac{6\pi^2}{11} \beta \right)^{-\frac{102}{121}} \exp \left(\frac{6\pi^2}{11} \beta \right) \quad (16)$$

which should tend asymptotically to σ/Λ_0^2 .

The numerical values we find for $\kappa(\beta)$ agree extremely well with the prediction of the strong coupling expansion for small β and with the expected scaling behaviour, when β exceeds some critical value. The Λ_0 parameters are determined to be

$$\Lambda_0^{(M)} = (0.0616 \pm 0.0020) \sqrt{6} \quad (17)$$

and

$$\Lambda_0^{(V)} = (0.0206 \pm 0.0011) \sqrt{6} \quad (18)$$

in the two cases. Both values are larger than the one determined with Wilson's form^{10),11)}

$$\Lambda_0^{(W)} = (0.012 \pm 0.002) \sqrt{6} \quad (19)$$

as may be expected because of the absence or smallness of the term $O(\bar{\theta}^4)$ in the action^{*)}. The passage from the strong coupling behaviour to the scaling behaviour may appear slightly smoother in the graphs of Figs. 2a and 2b than in the corresponding diagrams for the model with Wilson's action. However, in the present computation, the system had the possibility of relaxing toward statistical equilibrium for more iterations than in previous Monte Carlo work and, given the relative slowness of the approach to equilibrium in the transition region, this might justify slightly smoother curves. On the whole, the transition between the two régimes is still abrupt and this gives further support to the idea that some relevant physical effect may be occurring there^{3),18)}.

The high accuracy in our determination of E_M on the 8^4 lattice has allowed us to obtain the derivative $-(\partial E_M / \partial \beta)$ just by forming finite differences. In the infinite volume limit this quantity is related to the specific heat of the (four-dimensional) system

$$C_M(\beta) = -\beta^2 \frac{\partial E_M}{\partial \beta} \quad (20)$$

*) This may be seen by examining the relevance of the various terms in the perturbative computation of $\Lambda_0^{(W)} / \Lambda_{\text{mom}}$ in Ref. 17). Adapting that computation to the derivation of $\Lambda_0^{(M)} / \Lambda_{\text{mom}}$ and $\Lambda_0^{(V)} / \Lambda_{\text{mom}}$ we find values for the ratios of scales which agree with $\Lambda_0^{(M)} > \Lambda_0^{(V)} > \Lambda_0^{(W)}$ but deviate somewhat from the numerical results. A more detailed investigation of perturbation theory for the ratios Λ_M / Λ_W and Λ_V / Λ_M is in progress.

The results are represented by the crosses in Fig. 3. One notices immediately that while $-(\partial E_M/\partial\beta)$ is continuous, there is an apparent break in the slope of the curve at $\beta = 1.6$. The point at $\beta = 1.625$ seems actually to lie outside any reasonable extrapolation of the curve. Assuming that the errors in E_M are entirely due to normally distributed statistical fluctuations, then the errors in $-(\partial E_M/\partial\beta)$ should be of size smaller than the crosses in the Figure. However, as we have discussed already, long-range fluctuations may be responsible for larger errors in E , especially near a critical point, and this could justify the rather odd position of the point at $\beta = 1.625$.

The slope of the internal energy may also be evaluated from an analysis of the fluctuations in E itself, according to the formula

$$-\frac{\partial E}{\partial\beta} = N [\langle S_P^2 \rangle - \langle S_P \rangle^2] \quad (21)$$

where the bracket denotes averages over the values measured after each Monte Carlo iteration and N is the number of plaquettes in the lattice. An accurate determination of $-(\partial E/\partial\beta)$ by this method requires averages over very large numbers of Monte Carlo iterations, precisely to allow possible long-range fluctuations to take place. We have therefore done the measurement on a 4^4 lattice with Monte Carlo runs of 10000 iterations (required computing time ≈ 500 seconds on the CDC 7600). The results are given by the open circles in Fig. 3. The agreement with the determination of $-(\partial E/\partial\beta)$ by finite differences is very good and a break in the slope of the curve is again quite noticeable.

A real discontinuity in the second derivative of the internal energy would be a quite interesting feature of the model. It would correspond to a non-deconfining phase transition of a novel and intriguing nature (presumably third order). We notice that the results of Lautrup and Nauenberg¹³⁾, who studied the specific heat of the $SU(2)$ system with Wilson's action, are not incompatible with a discontinuity in $-(\partial^2 E/\partial\beta^2)$. The non-scaling peak which they find could actually correspond to a sudden change of slope, which appears as a peak or a cusp because it occurs at the inflection point of $E(\beta)$ and thus is superimposed on a local maximum. It is also interesting that a discontinuity in the derivative of the specific heat is present in the $N \rightarrow \infty$ limit of the two-dimensional $SU(N)$ model with Wilson's action¹⁹⁾, although the discontinuity disappears if Manton's action is used instead⁶⁾.

Unfortunately, from a numerical computation one can obtain only strong evidence for a discontinuity and the possibility that the variation be continuous, albeit very abrupt, is still open. In any case, we believe this unexpected

behaviour of the specific heat to constitute a very interesting result and hope that further investigations will clarify its origin.

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REFERENCES

- 1) K.G. Wilson - Phys.Rev. D10 (1974) 2445.
- 2) M. Creutz - Phys.Rev.Letters 43 (1979) 553, and Phys.Rev. D21 (1980) 2308.
- 3) C.G. Callan, R.F. Dashen and D.J. Gross - Phys.Rev.Letters 44 (1980) 435.
- 4) K.G. Wilson - Cargèse Lecture Notes, Vol. 59 (1979) (Plenum Press Publishing Company, New York, 1980).
- 5) N.S. Manton - Phys.Letters 96B (1980) 328.
- 6) C.B. Lang, P. Salomonson and B.S. Skagerstam - CERN Preprints TH 2993 (1980) ; TH 2995 (1980) to be published in Phys.Letters.
- 7) J. Villain - J.Phys. (Paris) 36 (1975) 581.
- 8) J.M. Drouffe - Phys.Rev. D18 (1978) 1174.
- 9) P. Menotti and E. Onofri - Private communication.
- 10) M. Creutz - Phys.Rev.Letters 45 (1980) 313.
- 11) G. Bhanot and C. Rebbi - CERN Preprint TH. 2979 (1980), to be published in Nuclear Phys. (FS1).
- 12) C. Rebbi - Phys.Rev. D21 (1980) 3350.
- 13) B. Lautrup and M. Nauenberg - Phys.Rev.Letters 45 (1980) 755.
- 14) D. Petcher and D.H. Weingarten - Phys.Rev. D22 (1980) 2465.
- 15) V. Metropolis, A.W. Rosenbluth, A.H. Teller and E. Teller - J.Chem.Phys. 21 (1953) 1087.
- 16) J.M. Drouffe - Nuclear Phys. B170 (1980) 91 (FS1).
- 17) A. Hasenfratz and P. Hasenfratz - Phys.Letters 93B (1980) 165 ;
R. Dashen and D.J. Gross - Princeton University Preprint (1980).
- 18) G. Mack and V.B. Petkova - Ann.Phys. 123 (1979) 442.
- 19) D.J. Gross and E. Witten - Phys.Rev. D21 (1980) 446.

FIGURE CAPTIONS

- Figure 1 The average plaquette contribution to the internal energy with Wilson's [upper part, from Ref. 11)] and Manton's form of the action ; the curve is the strong coupling expansion as described in the text.
- Figure 2a In the upper part we give the results for Wilson's loop averages W_γ (loop sizes up to 4×4) using Manton's action. In the lower part we display the points on the envelope again, but now the expected scaling behaviour has been divided out and the values should tend asymptotically to σ/Λ_0^2 . The solid lines represent the strong coupling expansion and the fit to the scaling behaviour.
- Figure 2b Like Fig. 2a, but using the generalization of Villain's form of the action.
- Figure 3 The crosses denote $-\partial E/\partial \beta$ as obtained by taking the slopes between any two neighbour points of the internal energy (Fig. 1) obtained on an 8^4 lattice with Manton's action. The open circles denote the same quantity, but derived from the fluctuations of the internal energy on a 4^4 lattice over 10000 Monte Carlo iterations. The curve represents the strong coupling expansion as given in the text.

Fig. 1

Fig. 2 a

Fig. 2 b

Fig. 3