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A QUANTITATIVE ANALYSIS OF THE MIGDAL-KADANOFF
RENORMALIZATION SCHEME FOR SU(2) GAUGE THEORY

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A B S T R A C T

By examining the strong and weak coupling limits we select a particular Migdal-Kadanoff renormalization transformation. We compare the results obtained by means of this transformation with the recent Monte Carlo calculations. The prediction for the average plaquette agree within 10% while a less satisfactory agreement is obtained for the function $g_0^2(a)$.

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1. INTRODUCTION

Renormalization group appears as one of the most powerful tools for connecting small with large distances in quantum field theory. An elementary realization of a renormalization group transformation has been proposed by Migdal¹⁾ and applied by various people to different problems²⁾⁻⁴⁾. We recall here the main features and successes of the approximation.

a) It preserves the duality relation for self-dual systems and thus it gives the correct critical point for the two-dimensional Ising system and related four-dimensional discrete gauge theories.

b) It gives a good description of the phase structure for the $x - y$ two-dimensional system.

c) Its application to four-dimensional local gauge theories shows the profound difference between the Abelian and non-Abelian cases. In particular for the Abelian case, i.e., $U(1)$, one gets a confined phase and an unconfined one corresponding to familiar QED³⁾. The transition point for pure $U(1)$ comes out in good agreement with direct Monte Carlo computations⁵⁾.

d) The approximation allows also for the introduction of fermions in gauge theories, giving results in qualitative agreement with the known perturbative calculations for weak coupling^{3),6)}.

While in the intermediate coupling region no direct numerical computation has ever been performed for gauge theories in presence of fermions, recently some detailed Monte Carlo calculations have been performed for pure gauge theories^{5),7)}. This raises the question whether such an approximation can be taken as a good starting point for a quantitative analysis of such systems⁸⁾. The main advantage of the scheme is its renormalization group character and the simplicity of the mathematics involved. Actually there is a variety of such zero order approximations which have been variously interpreted in the literature²⁾. Here we shall perform an analytic comparison of the free energy in the weak and in the strong coupling regions, with the familiar perturbative expansions and this will allow us to choose a particular "zero order" transformation. At the end a comparison is performed between the results obtained in this way and the above quoted Monte Carlo calculations. This extremely simple procedure provides a rather good value for the action per plaquette over the whole range of coupling. A qualitatively correct but quantitatively less satisfactory result is obtained for the dependence of the bare coupling constant on the lattice spacing.

2. THE MIGDAL-KADANOFF APPROXIMATION

The Migdal-Kadanoff (MK) approximation is a particularly simple renormalization transformation in which one allows the maximal generality for the single plaquette functional but avoids the generation of interaction terms corresponding to circuits more complex than the elementary plaquette. One may think of a process of decimation in which one fixes the gauge variables bordering a square of side λa and integrates on all the others. Thus the new Lagrangian depends on the ordered products of the group elements belonging to the side of length λa . These are the new gauge variables.

In the MK approximation such a procedure is realized by adding at every step a particular interaction $\Delta \mathcal{L}$ which allows to perform the above described integrations in an elementary way, without generating interaction terms more complex than the most general plaquette functional.

In the class of additional interactions $\Delta \mathcal{L}$ one can choose those which minimize the errors in the regions of strong and weak couplings, where a direct comparison can be made with perturbative expansions. In order to show how such a general approximation scheme works, let us consider a two-dimensional spin system. A potential moving transformation²⁾ (which respects the isotropy of the system) is shown in Fig. 1a. From the formal point of view such a transformation is obtained acting on the link function, first by superposing two links (which defines the action of the operator S_λ with $\lambda = 2$) followed by the integration on the intermediate spin (which defines the action of the operator I_λ with $\lambda = 2$). If on the other hand one acts first with I_2 and then with S_2 one still obtains a transformation satisfying our general requirements. The corresponding $\mathcal{L} + \Delta \mathcal{L}$ is shown in Fig. 1b and it does not belong to the potential moving category defined by Kadanoff. Completely similar renormalization transformations can be devised for gauge theories in any dimension¹⁾⁻³⁾. We shall bound ourselves for simplicity to the transformations $S_\lambda I_\lambda$ and $I_\lambda S_\lambda$ and we shall look within this class for those which give the best agreement for the free energy both in the strong and in the weak coupling limits.

The most general plaquette function is given by the general central function of the group, i.e.,

$$\mathcal{Z}(U) = \sum_P d_P e^{k_P} \chi_P(U) \quad (2.1)$$

where $\chi_p(U)$ are the characters of the gauge group, $d_p = \chi_p(1)$ and U is the gauge group element corresponding to the given plaquette. In this paper we shall consider only the gauge group $SU(2)$. Here the irreducible representations are labelled by an integer p going from 1 to ∞ , related to the usual isospin by $p = 2T + 1$. The exponential of the bare Wilson Lagrangian is⁹⁾ $e^{-\beta(1-\frac{1}{2}\chi_2(U))}$ giving

$$e^{k_p} = e^{k_p}[\beta] \equiv \frac{2}{\beta} e^{-\beta} I_p(\beta) \quad (2.2)$$

where $I_p(\beta)$ are the modified Bessel functions and β is connected to the bare coupling constant by $\beta = 4/g_0^2$. U_μ is related to the usual fields of the formal continuum limit by $U_\mu = \exp(\frac{ia}{2} g_0 \underline{A}_\mu \cdot \underline{\sigma})$. The integration operator I_λ acts as a simple power on the weights e^{k_p} , i.e.,

$$I_\lambda \{ e^{k_p} \} = \{ e^{\lambda^2 k_p} \} \quad (2.3)$$

because

$$\int \chi_p(AU^+) \chi_q(UB^+) dU = \delta_{pq} \frac{\chi_p(AB^+)}{d_p} \quad (2.4)$$

On the other hand the superposition operator S_λ in D dimensions is given by

$$S_\lambda(\mathcal{F}(U)) = (\mathcal{F}(U))^{\lambda^{D-2}} \quad (2.5)$$

For integer λ^{D-2} this can be computed in the character representation by repeated use of the relation

$$S(\lambda^{D-2}=2) \{ e^{k_s} \} = \left\{ \sum_{p,t} e^{k_p + k_t} C_{p,t}^s \right\} \quad (2.6)$$

where C_{pt}^s are the Wigner coefficients of $SU(2)$ given by $C_{pt}^s = pt/s$ for $p + t + s = \text{odd}$ and $|p - t| + 1 \leq s \leq p + t - 1$ and zero otherwise.

3. COMPUTATION OF THE PARTITION FUNCTION AND OF THE MEAN ACTION

We are interested in computing the free energy per plaquette given by $f_V = -2\ln Z_V / D(D-1)V$ in the limit of large V . Z_V is obtained by integrating the exponential of the action over all dynamical variables. Such an integration can be explicitly performed using the approximate renormalization scheme described in the previous section, for volumes of the type $a^D (\lambda^D)^N$ for arbitrary large N . The result of such an integration is the effective action computed at the scale $a\lambda^N$, and thus represented by the weight vector

$$[R_\lambda^N(e^k)]_n \equiv e^{k_n(\lambda^N)} \quad (3.1)$$

where R_λ represents the elementary renormalization group transformation, taking from any scale a to λa . The logarithm of $e^{k_1(\lambda^N)}$ represents $-V^f V$ while the rest is a surface term which gives negligible contribution in the large V limit. Thus we obtain the formula

$$f = - \lim_{N \rightarrow \infty} \frac{1}{\lambda^{ND}} k_1(\lambda^N) \quad (3.2)$$

For a practical calculation it is simpler to make use of the normalized weights e^{h_n} defined by $e^{h_n} = e^{k_n} - k_1$. Keeping in mind that

$$R_\lambda(\alpha e^k) = \alpha^{\lambda^D} R_\lambda(e^k) \quad (3.3)$$

we have

$$\begin{aligned} \frac{1}{\lambda^{ND}} k_1(\lambda^N) &= k_1(1) + \frac{1}{\lambda^D} [k_1(\lambda) - \lambda^D k_1(1)] + \\ &+ \frac{1}{\lambda^{2D}} [k_1(\lambda^2) - \lambda^D k_1(\lambda)] + \dots + \frac{1}{\lambda^{ND}} [k_1(\lambda^N) - \lambda^D k_1(\lambda^{N-1})] = \\ &= k_1(1) + \frac{1}{\lambda^D} \ln [R_\lambda(e^{h^{(1)}})]_1 + \frac{1}{\lambda^{2D}} \ln [R_\lambda(e^{h^{(2)}})]_1 + \\ &+ \dots + \frac{1}{\lambda^{ND}} \ln [R_\lambda(e^{h^{(N)}})]_1 \end{aligned} \quad (3.4)$$

and thus¹⁾

$$f = -k_1(1) - \sum_{N=1}^{\infty} \lambda^{-ND} \ln [R_{\lambda}(e^{h(\lambda^{N-1})})]_1 \quad (3.5)$$

From Eq. (3.2) one derives a relation between the free energy computed by means of the transformation $I_{\lambda} S_{\lambda}$ and by means of $S_{\lambda} I_{\lambda}$.

$$f_{IS} = - \lim_{N \rightarrow \infty} \frac{1}{\lambda^{ND}} \ln [I_{\lambda} (S_{\lambda} I_{\lambda})^{N-1} S_{\lambda} (e^k[\beta])]_1 \quad (3.6)$$

Keeping in mind that from Eqs (2.2) and (2.5)

$$S_{\lambda} e^k[\beta] = e^k[\lambda^{D-2} \beta] \quad (3.7)$$

and Eq. (2.3), we get the relation

$$\begin{aligned} f_{IS}(\beta) &= - \lim_{N \rightarrow \infty} \frac{1}{\lambda^{ND-2}} \ln [(S_{\lambda} I_{\lambda})^{N-1} e^k[\lambda^{D-2} \beta]]_1 = \\ &= \frac{1}{\lambda^{D-2}} f_{SI}(\lambda^{D-2} \beta). \end{aligned} \quad (3.8)$$

We shall now examine how good is this family of renormalization transformations in giving the free energy in the strong and in the weak coupling regions.

The strong coupling ($\beta \ll 1$) is the simplest region to examine as $\Delta \ell$ is of the order of β . The straightforward perturbative strong coupling computation gives as a result for the free energy

$$f = \beta - \ln \left[\frac{2}{\beta} I_1(\beta) + O(\beta^6) \right] \quad (3.9)$$

As

$$I_1(\beta) = \frac{\beta}{2} \left[1 + \frac{1}{2} \left(\frac{\beta}{2} \right)^2 + O(\beta^4) \right]$$

we have

$$f = \beta - \frac{\beta^2}{8} + O(\beta^4) \quad (3.10)$$

giving for the mean value of a plaquette

$$E = \frac{\partial f}{\partial \beta} = 1 - \frac{\beta}{4} + O(\beta^3). \quad (3.11)$$

Consider now the same computation with the simplest renormalization transformation, i.e., $I_\lambda S_\lambda$ with $\lambda = \sqrt{2}$. According to Eq. (3.5) we have

$$f = -k_1(1) - \frac{1}{4} \ln [I S(e^{h^{(1)}})]_1 + \dots \quad (3.12)$$

where

$$k_1(1) = \ln \left[\frac{2}{\beta} e^{-\beta} I_1(\beta) \right] \quad (3.13)$$

and

$$\left[S_\lambda e^{h^{(1)}} \right]_1 = 1 + 2\lambda^2(\lambda^2 - 1) \left(\frac{\beta}{4} \right)^2 + O(\beta^4) \quad (3.14)$$

Equation (3.14) in this region ($\beta \ll 1$) can be derived either by using Wigner coefficients or more simply by the SU(2) harmonic analysis of Eq. (2.5) with

$$\gamma(U) = 1 + \frac{\beta}{2} \chi_2(U) + O(\beta^2)$$

Moreover

$$\left[I_\lambda S_\lambda e^{h^{(1)}} \right]_1 = 1 + 2 \lambda^4 (\lambda^2 - 1) \left(\frac{\beta}{4} \right)^2 + O(\beta^4) \quad (3.15)$$

$$\left[I_\lambda S_\lambda e^{h^{(1)}} \right]_2 = \left(\frac{\lambda^2 \beta}{4} \right)^{\lambda^2} \left[1 + O(\beta^2) \right] \quad (3.16)$$

as $e^{h_2^{(1)}} = \beta/4 + O(\beta^3)$. For $\lambda^2 = 2$ we have

$$f_{IS} = \beta - \frac{\beta^2}{4} + O(\beta^4) \quad (3.17)$$

Thus we see that the IS recursion goes wrong already at the order β^2 , while still preserving the correct term in β as assured by Kadanoff's analysis. The other renormalization transformation $S_\lambda I_\lambda$ gives for $\lambda^2 \leq 2$

$$\begin{aligned} f &= -k_1(1) - \frac{1}{\lambda^D} \ln \left[S_\lambda I_\lambda e^{h^{(1)}} \right]_1 + \dots = \\ &= \beta - \frac{\beta^2}{8} + O(\beta^4) - \frac{2 \lambda^2 (\lambda^2 - 1)}{\lambda^D} \left(\frac{\beta}{4} \right)^{2 \lambda^2} + O(\beta^{2 \lambda^4}) \end{aligned} \quad (3.18)$$

which for $\lambda^2 = 2$ gives

$$f = \beta - \frac{\beta^2}{8} + O(\beta^4) \quad (3.19)$$

while for $\lambda^2 = \sqrt{2}$, which we shall use later,

$$f = \beta - \frac{\beta^2}{8} - (2 - \sqrt{2}) \left(\frac{\beta}{4} \right)^{2 \sqrt{2}} \quad (3.20)$$

Thus we see that in this case one gets also the correct second term in the strong coupling expansion.

In the weak coupling region ($\beta \gg 1$) one obtains an elementary estimate of the free energy by using the spin wave approximation. The well-known result for SU(2) in four dimensions is

$$f = \frac{3}{4} \ln \beta \quad (3.21)$$

A renormalization group treatment using directly Eq. (3.2) for the free energy would involve also the intermediate coupling region; an analytical computation, however, can be performed recalling that the result of the transformation IS or SI in the weak coupling gives¹⁾⁻³⁾

$$\beta(\lambda) = \beta(1) + \text{const} + O\left(\frac{1}{\beta}\right) \quad (3.22)$$

For small coupling one can write setting $\beta' \equiv \beta(\lambda)$

$$R_\lambda e^k[\beta] = e^{x(\beta)} e^k[\beta'] \quad (3.23)$$

where $x(\beta)$, for $R_\lambda = I_\lambda S_\lambda$ can easily be computed to be

$$x(\beta) = \frac{\lambda^2 - 1}{2} \ln \frac{2}{\pi} - \frac{3}{2} (\lambda^2 - 1) \ln \beta - \frac{3}{2} \lambda^2 \ln \lambda^2 + O\left(\frac{1}{\beta}\right) \quad (3.24)$$

Thus

$$\begin{aligned} f(\beta) &= - \lim_{N \rightarrow \infty} \frac{1}{\lambda^{ND}} \ln [R_\lambda^{N-1} e^{x(\beta)} e^k[\beta']]_1 = \\ &= - \frac{x(\beta)}{\lambda^D} - \frac{1}{\lambda^D} \lim_{N \rightarrow \infty} \frac{1}{\lambda^{(N-1)D}} [R_\lambda^{N-1} e^k[\beta']]_1 = \\ &= - \frac{x(\beta)}{\lambda^D} + \frac{f(\beta')}{\lambda^D} \end{aligned} \quad (3.25)$$

from which

$$\frac{f(\beta') - f(\beta)}{\beta' - \beta} = \text{const} \left[(\lambda^D - 1) f(\beta) - \frac{3}{2} (\lambda^2 - 1) \ln \beta - \frac{3}{2} \lambda^2 \ln \lambda^2 + \frac{\lambda^2 - 1}{2} \ln \frac{2}{\pi} + O\left(\frac{1}{\beta}\right) \right] \approx \frac{\partial f}{\partial \beta} \quad (3.26)$$

The consistency of this equation implies for $D = 4$

$$(\lambda^4 - 1) f(\beta) - \frac{3}{2} (\lambda^2 - 1) \ln \beta = O(1) \quad (3.27)$$

giving

$$f_{IS}(\beta) = \frac{3}{2} \frac{1}{\lambda^2 + 1} \ln \beta + \text{const} \quad (3.28)$$

for $\beta \rightarrow \infty$. From the general relation (3.8) we have

$$f_{SI}(\beta) = \frac{3}{2} \frac{\lambda^2}{\lambda^2 + 1} \ln \beta + \text{const} \quad (3.29)$$

Comparing Eqs (3.28) and (3.29) with Eq. (3.21) we see that both formulas agree with the spin wave result for $\lambda \rightarrow 1$. However, for general λ MK renormalization group approach assures only the correct logarithmic behaviour for $\beta \rightarrow \infty$ but not the correct coefficient in front of it. The average of the two formulas for IS and SI always gives the correct result; thus the absolute error one performs by keeping $\lambda \neq 1$ is the same for both procedures.

The intermediate coupling region can be dealt with only numerically and results in a very simple computation. The character representation is used throughout. In this representation the operation I_λ is trivial while S_λ for $\lambda^2 \neq \text{integer}$ can be computed with sufficient accuracy by quadratic interpolation on neighbouring results. The analytic study reported above induces us to prefer the $S_\lambda I_\lambda$ renormalization transformation with $\lambda - 1$ near zero, to have a good weak coupling behaviour but not too small as to save the strong coupling expansion in the limit of small β . $\lambda = 2^{\frac{1}{4}}$ appears to be a good compromise and possesses the advantage

of allowing a direct check on the interpolation of S_λ in λ : in fact one must have $S_{2\frac{1}{4}} S_{2\frac{1}{4}} = S_{2\frac{1}{2}}$ and the last one is directly calculable. We notice how our procedure with $S_\lambda I_\lambda^2$ with $\lambda = 2\frac{1}{4}$ gives for $\partial f/\partial\beta$ a value which stays below the strong coupling for small β and above the weak coupling expansion for large β . In Fig. 2 we show the value for the average plaquette superposed with the points of the Monte Carlo computation by Creutz⁷⁾ and by Rebbi⁷⁾. The maximum discrepancy is of the order of 10%. The curve given by the MK renormalization scheme appears smoother than the one given by the Monte Carlo calculation. This tendency appears as a general feature of the approximation which essentially interpolates between the weak and the strong coupling regions and is probably due to the absence of interactions more complicated than the single plaquette. Thus we expect a worse result for the $U(1)$ case where a sharp transition is observed⁵⁾.

4. BEHAVIOUR OF THE BARE COUPLING CONSTANT

From the analysis given elsewhere¹⁾⁻³⁾ we know that in this approximate renormalization group treatment the $SU(2)$ theory is simultaneously confined and asymptotically free. For $U(1)$ it gives a beta function $-(\partial g^2(L))/(\partial \ln L)$ which essentially departs from zero for $g^2 \gtrsim 1$ and has all the coefficients in the power expansion in g^2 equal to zero^{1),3)}. For the non-Abelian case in absence of fermions $-(\partial g^2(L))/(\partial \ln L) = \beta_4 g^4 + \beta_6 g^6 + O(g^8)$, has β_4 and β_6 both negative and of the correct relative size but these coefficients are too large by about a factor 2. Thus it appears that the beta function is a more delicate quantity to compute than the free energy. We have computed directly, using the above discussed renormalization group, the dependence of the bare coupling constant g_0^2 , on the lattice spacing, a , keeping fixed the asymptotic value of the string tension. The calculation of such a dependence is performed by fixing the strong coupling theory so as to produce a well-defined area constant κ , and to count, chosen a g_0^2 at will, how many steps in the renormalization procedure are necessary to reach the given strong coupling theory. The number of such steps equals $\log_{2\frac{1}{4}}(1/\sqrt{\kappa}a)$. This provides the functional relation $g_0^2(a)$ at a given fixed κ . The fact that for $a \rightarrow 0$, $g_0^2 \rightarrow 0$ is related to the coexistence of the asymptotic freedom and confinement in the theory. Due to the above mentioned discrepancy between the true and the approximate beta functions, we cannot expect a good agreement for small g_0^2 . In Fig. 3 we show our curve, in which the asymptotic string tension is kept fixed, and compare it with the points obtained by Creutz⁷⁾ using loops of side up to six lattice spacings, for all values of g_0^2 .

Even though the general characteristics are correct, we see only a very shallow variation of the slope. Again this may be due to the strong restrictions on the complexity of the effective interactions at all scales, of the MK approximation.

5. CONCLUSIONS

In the class of MK renormalization transformations we have looked for those which in the strong and in the weak coupling regions depart as little as possible from the known perturbation expansions. We have compared the results for the average plaquette and for the dependence of g_0^2 on the lattice spacing a with the recent Monte Carlo calculations. As for the average plaquette the obtained results depart from the computer simulation with a maximum discrepancy of about 10%. The general trend of the MK approximation is to give a smoother behaviour than the "experimental" values. This smoothing effect may be due to the absence in the Lagrangian of interaction more complex than the most general single plaquette. At least for the computation of the average plaquette MK renormalization group appears to be a good starting point for a systematic improvement⁸⁾. The prediction for $g_0^2(a)$ are less satisfactory. In addition to the smoothing effect, as we have mentioned the β_4 coefficient is off by about a factor 2. On the whole, however, this is the simplest renormalization scheme which applied to the whole range of coupling possesses asymptotic freedom and confinement and thus encompasses correctly at "zero order" what appear to be the general features of the theory.

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FIGURE CAPTIONS

- Fig. 1 : Effective Lagrangian $\mathcal{L} + \Delta\mathcal{L}$ (a) for a potential moving type.
(b) for a more general transformation.
- Fig. 2 : Action per plaquette computed with the transformation $S_\lambda I_\lambda$ with
 $\lambda = 2^{\frac{1}{4}}$ (full line) compared with Creutz' points (dots).
- Fig. 3 : $g_0^2(a)$ for fixed asymptotic string tension (full line) compared
with Creutz' points (dots).

Fig. 1

(a)

(b)