Ref.TH.2948-CERN

TIME-DEPENDENT PERTURBATION THEORY WITH PERMANENT EFFECTS

F. Englert^{*}

Faculté des Sciences, Université Libre
Bruxelles, Belgique

J.-M. Frère

CERN, Geneva

ABSTRACT

The presence of time-varying external fields invalidates usual perturbation theories as expressed by Feynman diagrams. We derive the correct Feynman graphs, which automatically ensure unitarity and causality. The relevance of these considerations for cosmological problems is pointed out.

* Supported in part by the Belgian State under contract ARC 79/83-12.

1. INTRODUCTION

It is well known in quantum field theory that the perturbation series, as expressed by the conventional Feynman graphs, fails when the Schrödinger vacuum at $t = +\infty$ is not equivalent to the initial vacuum ($t = -\infty$). This failure happens, in general, when particles are created and remain present for all subsequent time. Such an effect arises, for instance, in the presence of time-dependent external forces and is of particular importance in cosmological models where the gravitational field is treated as a classical external background source.

In some particular cases special techniques have been devised to study such particle creation¹⁾. Our purpose is to formulate a generalized perturbation series which includes the above-mentioned permanent effects. This is achieved by adding to the conventional Feynman graphs other diagrams, closely related to "unitarity cut" graphs.

The presentation is as follows. In Section 2 we show, on a trivial example with time-dependent forces, how a naive application of Feynman graphs contradicts both causality and unitarity. The corrections which restore causality are obtained in Section 3 from unitarity requirements and the modified Feynman rules are presented for the general case. Finally, we apply these techniques in Section 4 to compute the expected values of the T^{00} component of the energy-momentum tensor for a scalar field in a conformally flat space. We discuss the constraints to be imposed on the (external) gravitational field in order to make this quantity finite after proper zero point energy subtraction.

2. A SIMPLE EXAMPLE

Consider a scalar field ϕ with a space-time dependent mass $m(x)$. The action is:

$$S = \int \frac{1}{2} [\partial_\mu \phi \partial^\mu \phi - m^2(x) \phi^2(x)] d^4x. \quad (2.1)$$

Such a model arises naturally when massive particles are coupled to a cosmological²⁾ (conformally flat) gravitational metric field $g^{\mu\nu} \equiv g^{0\mu\nu} e^{-\lambda(x)}$ where $g^{0\mu\nu}$ is the Minkowskian metric; the coupling to gravity can then be described by the effective mass $m(x) = m e^{\lambda(x)/2}$ in an underlying Minkowski space.

In this section, however, we shall illustrate the problem posed in the introduction using a simple toy model:

$$m^2(x) = m^2 + A \delta(t) \quad (2.2)$$

To this effect we shall calculate the Green's function

$$G(x', x) = \langle 0 | T \phi(x') \phi(x) | 0 \rangle \quad (2.3)$$

in terms of A and m .

The Heisenberg operator $\phi(x)$ satisfies

$$[\square + m^2 + A \delta(t)] \phi(x) = 0 \quad (2.4)$$

and must coincide with the free field operator for $t < 0$. The solution is thus found to be:

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int d^3p \left[\xi(t) e^{-i\vec{p}\vec{x}} a^\dagger(\vec{p}) + h.c. \right] \\ \xi(t) &= \frac{1}{\sqrt{\omega_p}} \left[e^{i\omega_p t} - \theta(t) \frac{A}{\omega_p} \sin \omega_p t \right] \end{aligned}$$

with

$$\omega_p = \sqrt{p^2 + m^2}$$

and

$$[a(\vec{p}), a^\dagger(\vec{p}')] = \delta^3(\vec{p} - \vec{p}') \quad (2.5)$$

Hence, from Eqs (2.3) and (2.5) we obtain:

$$G(x', x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-i\vec{p}(\vec{x}' - \vec{x})} G_p(t', t) \quad (2.6)$$

$$\begin{aligned}
G_p(t', t) = & \theta(t'-t) e^{-i\omega_p(t'-t)} + \theta(t-t') e^{-i\omega_p(t-t')} \\
& + \theta(t) \theta(t') \frac{A}{2\omega_p i} \left\{ e^{-i\omega_p(t+t')} \left[1 + \frac{A}{2\omega_p i} \right] \right. \\
& \quad \left. + e^{+i\omega_p(t+t')} \left[-1 + \frac{A}{2\omega_p i} \right] \right. \\
& \quad \left. - \left(e^{-i\omega_p(t'-t)} + e^{-i\omega_p(t-t')} \right) \frac{A}{2\omega_p i} \right\} \\
& + \theta(t) \theta(-t') \frac{A}{2\omega_p i} \left[e^{-i\omega_p(t'-t)} - e^{i\omega_p(t+t')} \right] \\
& + \theta(-t) \theta(t') \frac{A}{2\omega_p i} \left[e^{-i\omega_p(t'-t)} - e^{i\omega_p(t+t')} \right]
\end{aligned}
\tag{2.6}$$

contd

Clearly, $G(x|x)$ in Eq. (2.6) reduces to the free propagator for t and $t' < 0$, as required by causality. For both $t, t' \gg m$, the relevant part of $G_p(t', t)$ reduces to (we neglect rapidly oscillating terms):

$$\begin{aligned}
G_p(t', t) \longrightarrow & \theta(t'-t) e^{-i\omega_p(t'-t)} + \theta(t-t') e^{-i\omega_p(t-t')} \\
& + \frac{A^2}{2\omega_p^2} \left[e^{-i\omega_p(t'-t)} + e^{-i\omega_p(t-t')} \right]
\end{aligned}
\tag{2.7}$$

This describes propagation in a state $\vec{p}$ with particle density $n(\vec{p})$ given by:

$$\begin{aligned}
n(\vec{p}) = & \lim_{\epsilon \rightarrow 0^+} \left[i \frac{d}{d\epsilon} + \omega_p \right] \left[-i \frac{d}{d\epsilon} + \omega_p \right] \frac{G_p(t', t)}{4\omega_p^2} \\
= & \frac{A^2}{4\omega_p^2}
\end{aligned}
\tag{2.8}$$

The total particle density $n = 1/(2\pi)^3 \cdot \int n(\vec{p}) d^3p$ diverges. This divergence results from the singular nature of the potential (2.2) and can be cured by an appropriate smoothing of the δ function; it is therefore of no concern. The interesting point is that a constant density of particles per mode has been generated out of the vacuum by the time-dependent potential, thereby providing the type of permanent effect which will break the validity of the conventional perturbation series.

We now sum the usual Feynman graphs in order to check this failure explicitly. The sum, depicted in Fig. 1, yields, instead of (2.6), the (incorrect) Green's function $G'_p(t', t)$:

$$G'_p(t', t) = \theta(t'-t) e^{-i\omega_p(t'-t)} + \theta(t-t') e^{-i\omega_p(t-t')} + \frac{A/2\omega_p i}{1 - A/2\omega_p i} \begin{cases} \theta(t) \theta(t') e^{-i\omega_p(t'+t)} \\ \theta(t) \theta(-t') e^{-i\omega_p(t-t')} \\ \theta(-t) \theta(t') e^{-i\omega_p(t'-t)} \\ \theta(-t) \theta(-t') e^{+i\omega_p(t'+t)} \end{cases} \quad (2.9)$$

The result, (2.9), clearly violates causality since the effect of the external potential is "felt", even for $t' < 0$, $t < 0$, i.e., before it is applied.

We conclude this section by noting that both Eq. (2.6) and Eq. (2.9) satisfy the inhomogeneous equation

$$[\square + m^2 + A \delta(t)] G(x', x) = -i \delta^4(x' - x)$$

but differ by a solution of the homogeneous equation. We shall now see that the perturbative expansion for the correct solution, (2.6), can be deduced straightforwardly from the Feynman graphs yielding (2.9) by performing unitarity cuts.

3. GENERALIZED PERTURBATION THEORY

We first consider the action (2.1), and now leave $m(x)$ arbitrary. We want to evaluate the Green's functions; in the Heisenberg representation these are given by:

$$G(x_n \dots x_1) = \langle 0 | T \phi(x_n) \dots \phi(x_1) | 0 \rangle \quad (3.1)$$

Going to the interaction representation we obtain in the familiar way, assuming that the perturbation is switched on adiabatically from $t = -\infty$

$$G(x_n \dots x_1) = \langle 0 | S^\dagger(t, -\infty) [T S(t, -\infty) \phi_{int}(x_n) \dots \phi_{int}(x_1)] | 0 \rangle \quad (3.2)$$

where t is any time greater than all the x_i^0 . Inserting a complete set of states we have:

$$G(x_n \dots x_1) = \sum_i \langle 0 | S^+(t, -\infty) | i \rangle \times \quad (3.3)$$

$$\times \langle i | T S(t, -\infty) \phi_{int}(x_n) \dots \phi_{int}(x_1) | 0 \rangle$$

The usual procedure at this stage is to assume that for $t \rightarrow \infty$ $S(t, -\infty)$ connects the vacuum only to itself (this is the case, as we will check, if a "constant" perturbation is also switched off adiabatically). Then:

$$1 = \langle 0 | S^+(t, -\infty) S(t, -\infty) | 0 \rangle \quad (3.4)$$

$$= \lim_{t \rightarrow \infty} \langle 0 | S^+(t, -\infty) | 0 \rangle \langle 0 | S(t, \infty) | 0 \rangle$$

and we have the well-known form:

$$G(x_n \dots x_1) = \frac{\langle 0 | T S(+\infty, -\infty) \phi_{int}(x_n) \dots \phi_{int}(x_1) | 0 \rangle}{\langle 0 | S(+\infty, -\infty) | 0 \rangle} \quad (3.5)$$

This is, in general, incorrect when the interaction is time-dependent, as exemplified by the simple model of Section 2. Clearly, permanent effects require keeping intermediate states in Eq. (3.3). For the action (2.1) this amounts to:

$$G(x_n \dots x_1) = \sum_{p=0}^{\infty} \frac{1}{p!} \int d^3k_1 \dots d^3k_n \langle 0 | S(-\infty, +\infty) a_{\vec{k}_p}^+ \dots a_{\vec{k}_1}^+ | 0 \rangle \times \quad (3.6)$$

$$\times \langle 0 | a_{\vec{k}_p} \dots a_{\vec{k}_1} T S(+\infty, -\infty) \phi_{int}(x_n) \dots \phi_{int}(x_1) | 0 \rangle$$

Each term of this sum thus appears as the product of two Feynman graphs, where the Wick contractions are performed between ϕ operators and also between ϕ and a_k . The Feynman diagrams corresponding to the left bracket in (3.6) obey rules which are the complex conjugate of the ordinary ones:

$$\langle 0 | S(-\infty, +\infty) a_{\vec{k}_p}^+ \dots a_{\vec{k}_1}^+ | 0 \rangle = \langle 0 | a_{\vec{k}_1} \dots a_{\vec{k}_p} S(+\infty, -\infty) | 0 \rangle^+$$

We make the convention of representing them by dotted lines. Examples of first order graphs are depicted in Fig. 1 for $n = 2$ and $p = 0, 2$.

We now observe that the products of contractions $\langle 0 | a_k \phi(x_i) | 0 \rangle$ and $\langle 0 | a_k \phi(x_j) | 0 \rangle^*$ are equivalent when summed on k , to an effective propagator

$$D'_{ji} \equiv \langle 0 | \phi(x_j) \phi(x_i) | 0 \rangle \quad (3.7)$$

$$= \frac{1}{(2\pi)^3} \int d^3k \frac{e^{i\omega_k(t_i - t_j)} e^{-i\vec{k}(\vec{x}_i - \vec{x}_j)}}{2\omega_k}$$

Each term of (3.6) can thus be rewritten as a single Feynman graph by introducing two types of vertices, corresponding to $\pm i m(x)$, and three types of propagators, corresponding to plain and dashed lines, or to the contraction (3.7). For the sake of completeness we list them in the Table.

In fact, each propagator appearing in the Table is fully determined by its end-point vertices ("circled" or not) and we may thus represent all of them by the same symbol, say a solid line. In this way we have recovered the notation introduced by 't Hooft and Veltman³⁾ in the context of unitarity cuts: each term in (3.6) is now represented by a generalized Feynman graph constructed from two types of vertices instead of one. The rules for the generalized graphs follow from the Table.

The summation in Eq. (3.6) is over all generalized Feynman graphs, connected or disconnected, where the external vertices $(x_1 \dots x_n)$ are never circled. In the usual case, disconnected graphs are removed by the denominator in Eq. (3.5). In the present generalization, disconnected Feynman graphs again drop out of the expansion because they sum up to:

$$\sum_n \langle 0 | S^+ | n \rangle \langle n | S | 0 \rangle = 1. \quad (3.8)$$

The unitarity relation (3.8) can be checked diagrammatically³⁾ as illustrated in Fig. 2. The final result for $G(x_n \dots x_1)$ is thus simply given by the sum of all connected generalized Feynman graphs. Clearly, the validity of our construction does not rely upon the special form of the action (2.1) and we can generalize it trivially to other interactions.

Do the above rules cure the diseases of conventional perturbation theory, as exemplified in Section 2? It is an easy matter to prove that, according to the rules of the Table, the value of the potential in the future does not influence the vacuum expected of fields in the present or the past. This proposition is in fact a simple rephrasing of the "largest time equation" of Ref. 3).

In Figs. 3 and 4 we present the situation for the first and second order contributions to $G(x_2, x_1)$. Graph 3a is directly associated to graph 3b, which differs in sign (due to the vertex) but not in analytic form, as long as $x_1^0 > x_2^0$ and $x_1^0 > x_1^0$ since the propagators D_{1t}, D_{2t} coincide with D'_{1t}, D'_{2t} in these conditions. In a similar way, the graphs 4c and 4d cancel the graphs 4a and 4b when $t_j > t_i > t_2 > t_1$. This result generalizes directly to all orders.

According to the Table, the propagator D'_{ij} is non-vanishing only if energy flows from an uncircled to a circled vertex. Graphs containing circled vertices would thus vanish if energy would be conserved at each internal vertex (remember that external vertices are never circled). This, however, is not the case for the action (2.1) since the interaction $m(t)$, being time-dependent, is able to supply or absorb energy at those vertices. Note that in the case of a constant external field $m(t)$ all graphs containing circled vertices vanish identically, hence in these conventional cases the usual Feynman rules are indeed recovered. Their validity extends, in fact, in all cases where the energy fed by the time-dependence remains below threshold for particle production; in particular, the adiabatic switching on and off of an otherwise time-independent potential does not, in general, invalidate the usual Feynman expansion.

Going back to the simplest example considered in Section 2, we observe that for any value of x_1^0 and x_2^0 the generalized series for $G(x_1 x_2)$ terminates after the second order. Summing up the terms presented in Fig. 5 then reproduces identically the exact solution (2.6) instead of the incorrect (2.9).

4. APPLICATION TO THE ENERGY MOMENTUM TENSOR

The expectation value in the vacuum of the trace of the energy momentum tensor, as well as of its T^{00} component, are of great importance in cosmological models. We will consider as an example the value of

$$T^{00}(t) = \int d^3x \langle 0 | T^{00}(x, t) | 0 \rangle \quad (4.1)$$

Setting, for convenience, $m^2(x) \equiv m^2 + \mu^2(t)$ we obtain from (2.1) the standard expression for T^{00} (in flat space)*)

*) In order to respect the notation of Ref. 2), all quantities appearing in this section should be affected by an upper index 0 ($T \rightarrow T^0$) since they refer to "rescaled" fields in flat space (see Ref. 2)). Note also that the quantum field was denoted ψ in Ref. 2) instead of ϕ as here.

$$T^{\circ\circ}_{(x,t)} = \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \partial_i \phi \partial^i \phi + \frac{1}{2} \mu^2(t) \phi^2 \quad (4.2)$$

The first contributions to $T^{\circ\circ}(t)$ are represented in Fig. 6 for "ordinary" graphs and in Fig. 7 for "cut" graphs.

The graphs of Fig. 6 are, in general u.v. divergent. We know already, however, that a subtraction is needed in this model. In Ref. 2) we have shown that the divergences of the trace of the energy momentum tensor could be absorbed in a purely "gravitational" counterterm, i.e.- a rescaling of the cosmological constant in n dimensions. Rephrased in Minkowskian language, this amounts to subtracting from any graph evaluated at time t , a similar graph where all vertices $\mu(t_i)$ are replaced by $\mu(t)$ ²⁾. According to this subtraction scheme we thus note that no counterterm is available for cut graphs since they would vanish identically, as shown in Section 3.

We now proceed to isolate the potential divergences in the expression $T^{\circ\circ}$. According to the subtraction rule introduced, we find that the contribution of Fig. 6.1 (corresponding to the zero point energy) vanishes identically, while Fig. 6.2 gives

$$\begin{aligned} I_2 &= \int \frac{d^3 k}{(2\pi)^3} I_2 k \\ \bar{I}_2 k &= (-i) \frac{1}{2} \frac{\mu^2(t)}{(2\omega)^2} \int_{-\infty}^t (\mu^2(t') - \mu^2(t)) e^{2i\omega(t'-t)} dt' \\ &\quad + \int_t^{+\infty} (\mu^2(t') - \mu^2(t)) e^{-2i\omega(t'-t)} dt' \end{aligned} \quad (4.3)$$

It is an easy matter to check that Figs 6.3a and 6.3b will only provide convergent contributions after integration over k ; as in Fig. 6.2 the time derivatives yield a term $-\omega^2$, which cancels against the term $p^2 + m^2$ due to space derivatives and leaves us (before integration over k) with a factor $\mu^2(t)/(2\omega)^3$ in front of an integral itself bounded by $1/2\omega$. The graph 6.3c, however, does not present this cancellation and we have instead a contribution:

$$I_{3k} \text{ div.} = \left(\frac{-2\omega^2}{8\omega^3} \right) \int_t^\infty dt_2 \int_{-\infty}^t dt_1 [\mu^2(t_1) \mu^2(t_2) - \mu^4(t)] e^{-2i\omega(t_2-t_1)} \quad (4.4)$$

Turning now to the cut diagrams we conclude by the same arguments as above that only two divergent contributions may arise, namely, from graph 7.1 for all t_1 , and graph 7.2 for $t_2 < t, \forall t_1$. We obtain, respectively:

$$7.1 \rightarrow I_{4k} = \frac{i}{2} \frac{\mu^2(t)}{4\omega^2} \int_{-\infty}^{+\infty} \mu^2(t') e^{2i\omega(t'-t)} dt' \quad (4.5)$$

$$7.2 \rightarrow I_{5k} = \frac{2\omega^2}{8\omega^3} I_6 \cdot I_7$$

$$I_6 = \int_{-\infty}^{+\infty} dt_1 e^{-2i\omega t_1} \mu^2(t_1) \quad (4.6)$$

$$I_7 = \int_{-\infty}^t dt_2 e^{2i\omega t_2} \mu^2(t_2)$$

In Ref. 2) we paid special attention to a sample case where $\mu^2(t) = \theta(t)$. In this case, it is straightforward to check that the possible divergences arising from Eq. (4.3) and (4.4) actually cancel each other. This is not true for the contributions arising for (4.5) and (4.6) and a divergence persists in the expression of T^{00} . This result is expected since we encountered this situation already in Ref. 2). In fact, it is easy to track the origin of the divergence to the discontinuity in $\mu^2(t)$. This leads us to ask what are the regularity conditions requested from $\mu^2(t)$ to ensure the finiteness of T^{00} .

A sufficient condition to make the integral over k converge would be to bring a factor ω^{-2} in front of (4.5) and a factor ω^{-3} in front of (4.6). This can easily be done if we are allowed to integrate I_4 twice by parts. A sufficient condition for this is

$$\int_{-\infty}^{+\infty} \left| \frac{d^2}{dt^2} \mu^2(t) \right| dt' < \infty.$$

It is an easy matter to check that this condition also ensures the finiteness of the contribution of uncut graphs to T^{00} .

The above example shows how expectation values of Heisenberg operators, such as $\langle 0 | T^i_i | 0 \rangle$, $\langle 0 | T^{00} | 0 \rangle$ can be handled perturbatively in a varying potential equivalent to a cosmological background. The method may be useful to compute various cosmological effects, such as particle production induced by trace anomalies in conformally flat spaces, a potentially important effect in the very early universe²⁾.

ACKNOWLEDGEMENTS

One of us (F.E.) wishes to thank CERN for its warm hospitality while part of this work was completed.

Table 1

$-i m(t)$

$+i m(t)$

$\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}_1 - \vec{x}_2)}}{2\omega_k}$		
$\underset{1}{\text{---}} \underset{2}{D_{21}}$	$\times [\theta(t_1 - t_2) e^{-i\omega_k(t_1 - t_2)} + \theta(t_2 - t_1) e^{i\omega_k(t_1 - t_2)}]$
$\underset{1}{\text{---}} \underset{2}{D'_{21}}$	$\times e^{-i\omega_k(t_2 - t_1)}$
$\underset{1}{\text{---}} \underset{2}{D''_{21}}$	$\times [\theta(t_1 - t_2) e^{i\omega_k(t_1 - t_2)} + \theta(t_2 - t_1) e^{-i\omega_k(t_1 - t_2)}]$

REFERENCES

- 1) J. Schwinger, Phys. Rev. 82 (1951) 664;
E. Brezin and C. Itzykson, Phys. Rev. D2 (1979) 1191;
H. Neuberger, Finite time corrections to the chromoelectric flux model,
Tel Aviv preprint TAUP 721/79;
An approach closer to ours is found in: S. Dimopoulos and L. Susskind, Phys.
Rev. D18 (1978) 4500.
- 2) R. Brout, F. Englert, J.M. Frère, E. Gunzig, P. Nardone, C. Truffin and
Ph. Spindel, Cosmogenesis and the origin of the fundamental length scale,
Brussels University preprint, 1979, to appear in Nucl. Phys. B.
- 3) G. 't Hooft and M. Veltman, "Diagrammar", CERN 73-9.

FIG.2

fig.5

$$(\text{wavy line} + \text{wavy line} = 0.)$$

fig. 6.1

6.2

6.3a

6.3b

6.3c

7.1

7.2

7.3

fig.