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Addendum

SU(2) STRING TENSION, GLUEBALL MASS
AND INTERQUARK POTENTIAL BY MONTE CARLO COMPUTATIONS

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ADDED NOTE

After completion of the manuscript we have learnt of a work by B. Berg (DESY preprint, to appear in Physics Letters B) where results on the glueball mass in agreement with ours are presented.

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A B S T R A C T

Monte Carlo simulations of the $SU(2)$ gauge system on a large (16^4) lattice and with high statistics are performed to determine several quantities of physical interest. Previous evaluations of the ratio between string tension, σ , and scale constant, Λ_0 , are confirmed. The mass of the glueball is found to be approximately $3\sqrt{\sigma}$ and the potential between static charges at very small separation is measured.

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1. INTRODUCTION

Monte Carlo simulations have provided us with extremely important results about lattice gauge theories¹⁻⁴⁾. Yet the fact of working in four dimensions imposes severe constraints on the computation: the size of the lattice is necessarily small, one runs into boundary effects and the statistics are often poor. We have tried to refine the analyses done so far and to obtain new results by performing Monte Carlo simulations on a fairly large lattice (lattice size 16^4 , with periodic boundary conditions; number of gauge variables $4 \times 16^4 = 262\,144$), for a non-Abelian model defined by the 120-element icosahedral subgroup $\tilde{Y}$ of $SU(2)$. In this article, we present the results of our investigation.

It has been shown already that models using a finite subgroup of $SU(2)$ as gauge group can give a very good approximation to the $SU(2)$ theory itself throughout a range of values of the parameter $\beta = 4/g^2$ ^{3,4)}. In particular, the $\tilde{Y}$ system undergoes the phase transition to the broken symmetry phase, characteristic of the finite nature of the gauge group, at $\beta_c \approx 5.99$. This value far exceeds the value $\beta \sim 2$, where the transition from the strong coupling to the weak coupling behaviour takes place, and one expects the $\tilde{Y}$ -model to reproduce quite well the properties of the $SU(2)$ gauge theory throughout the interesting physical region. The possibility of using a multiplication table for the group operation and of storing many group elements (spins) in the same word of the computer permits a substantial reduction in processing time and memory requirements⁵⁾. This has allowed us to perform Monte Carlo simulations on a 16^4 lattice, still keeping the time needed for one sweep of the entire lattice (on a CDC 7600) to the acceptable value of ~ 15 seconds.

Working with a large lattice presents some definite advantages. Finite size effects are less likely to occur. For the determination of the string tension, for instance, we have measured the expectation value of Wilson loops of size up to six by six. Since the loops are embedded in a lattice extending for 16 sites in each direction, the distortion of their expectation values due to boundary effects is almost certainly negligible. A more subtle advantage has to do with the possible occurrence of a deconfining phase transition in finite systems. A study of temperature effects reveals that on a lattice, in principle infinite in its spatial extent, but periodic over d lattice sites in the time direction, the system undergoes a transition to a Debye-screened Coulomb phase as β increases beyond some definite $\beta_c(d)$ ⁶⁾. d acts like an inverse temperature ($da = 1/T$, where a is the lattice spacing), and the larger d is, the greater is the value

of β_c . The point is that a signal of this transition is also observed with lattices of finite spatial extent (even of the order of d), so, to play it safe, one should not rely on Monte Carlo computations at a value of β which exceeds β_c to derive properties of the confining phase. With a larger system, higher values of β are allowed.

A large lattice size implies a very large number of spin variables: so, even averaging over a single lattice configuration in thermodynamical equilibrium, the statistics are quite good. We have made definite use of this improvement in statistics to evaluate quantities which have previously escaped measurement. Of course, there is really no true gain from working with a larger lattice. The statistics are improved if averages are taken over a fixed number of Monte Carlo iterations, but, with a smaller lattice, one could do more iterations during the same processing time. Smaller systems (as long as one does not run into finite size effects) may be more convenient if large-scale fluctuations are present: the relaxation time becomes longer when the size of the system increases and more steps become necessary to obtain reliable results. This typically happens near a phase transition. It has been recently argued⁷⁾ that a phase transition (of the roughening type) may occur in a variety of lattice gauge theories, including the one under consideration. This transition, however, would not destroy confinement or otherwise alter the basic properties of the theory, and, as we have not observed long-range fluctuations in the quantities we have measured, we believe that the possible presence of a roughening transition would not spoil the accuracy of our results. Finally, a lattice measuring 16^4 may be large enough to permit two or three iterations of the renormalization transformation in real space⁸⁾. However, this interesting application is left to future investigations.

2. OUTLINE OF THE COMPUTATION

We have simulated the dynamics of the $\tilde{Y}$ model using the so-called improved Metropolis method⁹⁾. Given a definite group element U_{ij} on the link between the sites i and j , a new, trial value, $\hat{U}_{ij}$, is obtained by multiplying U_{ij} by one of the 12 neighbours G_i of the identity chosen at random. By "neighbours of the identity" we mean the following. Writing any element of $\tilde{Y}$ as $U = \cos\theta/2 + i\vec{\sigma} \cdot \hat{n} \sin\theta/2$, the identity is, of course, obtained with $\theta = 0$, and the G_i 's are those elements which correspond to the lowest non-zero value of $|\theta|$ (namely $|\theta| = 2\pi/5$). They correspond to the minimal non-trivial rotations which leave the icosahedron invariant in the factor group $Y = \tilde{Y}/Z_2$, and any two of them, (provided they are not the inverses of each other), generate the whole group.

$\hat{U}_{ij}$ is accepted to replace U_{ij} if the change lowers the action S ; otherwise, if S is increased by ΔS , it is accepted with conditional probability $e^{-\beta\Delta S}$. The action is given by

$$S = \sum_{\square} (1 - W_{\square}), \quad (2.1)$$

where the sum is extended over all elementary squares of the lattice (plaquettes) and the Wilson factor for any closed loop $\gamma(i_1 i_2 i_3 \dots i_N i_1)$ is defined as

$$W_{\gamma} = \frac{1}{2} \text{tr} (U_{i_1 i_2} U_{i_2 i_3} \dots U_{i_N i_1}). \quad (2.2)$$

To optimize the rate of convergence to statistical equilibrium, six upgrading steps are performed on any definite link, keeping the values of all the other U_{ij} 's fixed. Then a new spin is upgraded, and so on, until all the U_{ij} 's have been analyzed, thus completing a Monte Carlo iteration. The sequence of MC iterations defines a Markovian chain, where eventually the probability of encountering any configuration Σ of the gauge variables becomes proportional to the Boltzmann factor $e^{-\beta S(\Sigma)}$. Quantum mechanical expectation values of observables $A(\Sigma)$ may then be approximated by

$$\langle A \rangle \approx \frac{1}{N} \sum_{i=i_0+1}^{i_0+N} A(\Sigma_i), \quad (2.3)$$

where the Σ_i 's are the configurations occurring in the sequence.

We have started from a completely random system at $\beta = 0$ and have gradually increased β in steps of $\Delta\beta = 0.01$ up to $\beta = 4.5$, performing, in general, one iteration at each value of β . The procedure corresponds to a slow cooling of the whole system, which should maintain it reasonably close to statistical equilibrium at each step. Measurements of the internal energy, $E \equiv \langle S_{\square} \rangle$, produced results in agreement with independent determinations of $E(\beta)$ from longer MC runs or MC simulations with different initial conditions, thus giving assurance that any departure from equilibrium cannot be too drastic.

27 values of β have been singled out ($\beta = 0.25, 0.5, \dots, 1.5, 1.6, 1.7, \dots, 3, 3.25, 3.5, \dots, 4.5$). At these values of β , two additional MC iterations have been performed and then the whole set of gauge variables was recorded on magnetic tape. Thus we obtained 27 configurations of the system in approximate statistical equilibrium at the corresponding values of β . Some of the measurements have then been made averaging the observables over these configurations. For other measurements, aimed at greater statistical accuracy, some of the 27 configurations have been taken as initial states and averages have been made over several further MC iterations, performed now at fixed β . The quantities which we have measured are the following:

- i) the expectation value $E \equiv \langle S \rangle$ of the action of an individual plaquette;
- ii) the expectation values of Wilson factors $\bar{W}(m,n) \equiv \langle W_Y \rangle$ for m by n rectangular loops, extending up to $m = n = 6$;
- iii) the plaquette-plaquette correlation function

$$C(d) \equiv \langle W_{\square} W_{\square'} \rangle - \langle W_{\square} \rangle^2, \quad (2.4)$$

where $\square$ and $\square'$ stand for two plaquettes facing each other and separated by d lattice links;

- iv) the expectation value

$$Q(d) \equiv \langle W_{\lambda} W_{\lambda'} \rangle \quad (2.5)$$

of the product of loop factors associated with parallel straight segments λ and λ' running through the lattice and separated by d lattice links. These segments define closed loops by virtue of the periodic boundary conditions. The topology of the lattice is indeed that of a torus, and the paths λ and λ' wind around the torus once.

Whenever we say that an average has been taken over a lattice, we mean that the quantity has been averaged over all possible locations and orientations within the lattice. Thus, for instance, a measurement of $C(d)$ over a single lattice configuration involves an average over $6(\text{orientations of the plaquettes}) \times 2(\text{directions of separation}) \times 16^4(\text{locations within the lattice}) = 786\,432$ numbers. In the graphs we shall frequently reproduce error bars. When the quantity has been measured averaging over many MC iterations, the error has been inferred from the

fluctuations in the averages taken over the individual lattices. When the averages have been taken over a single configuration, the errors have been estimated partitioning the lattice into four sublattices and evaluating the fluctuation of the averages over the sublattices.

In the next section, we describe our results and discuss their physical implications.

3. RESULTS AND INTERPRETATION

i) The expectation value of the action

Figure 1 gives an over-all view of the function $E(\beta)$ for the gauge model under consideration. The numbers have been obtained with a rather small lattice (4^4), doing a simulation of a thermal cycle, where β is first decreased from 6 to 0 (x), and then increased back to 6 (•) in small steps after each MC iteration, and from mixed phase runs. In these mixed phase runs, we start from a lattice which is half ordered ($U_{ij} = 1$) and half disordered (U_{ij} chosen at random), and perform several iterations at fixed β until E appears to converge to a definite value. Mixed phase runs overcome some of the problems associated with metastabilities, especially near first order phase transitions. The purpose of Fig. 1 is to show that the system undergoes a discontinuous transition at $\beta_c \approx 5.9$ [see also Ref. 4)], without noticeable critical points for lower values of β .

A more refined measurement of $E(\beta)$ is presented in Fig. 2. The values are those obtained averaging over the individual lattices of size 16^4 encountered while increasing β from 0 to 4.5, as described in Section 2. The continuous lines show Padé's approximants of the strong coupling expansion carried to order J^{20} , where

$$\begin{aligned}
 J &= \frac{\int dU \left(\frac{1}{2} \text{tr} U \right) e^{\beta \frac{\text{tr} U}{2}}}{\int dU e^{\beta \frac{\text{tr} U}{2}}} = \\
 &= \frac{\partial}{\partial \beta} \ln \left[\int_0^{2\pi} e^{\beta \cos \vartheta} \sin^2 \vartheta d\vartheta \right].
 \end{aligned}
 \tag{3.1}$$

The strong coupling series

$$\begin{aligned}
 E = 1 - J \{ & 1 + 4J^4 - 8J^6 + \frac{188}{3} J^8 - \\
 & - 196.9975309 J^{10} + 1349.23786 J^{12} - \\
 & - 5939.062747 J^{14} + 34280.32143 J^{16} - \\
 & - 175554.005 J^{18} + 977457.232 J^{20} - \dots \} \quad (3.2)
 \end{aligned}$$

has been derived by K. Wilson, who has kindly allowed us to reproduce his results. Figure 2a illustrates the Padé's approximants obtained directly from Eq. (3.2), while in Fig. 2b the approximants after the conformal mapping $J^2 = Z/(1-2Z)$ are displayed. Series and numerical results agree very well up to $\beta \approx 2$, after which value the results of the MC computation exhibit a rather swift departure from the predictions of the strong coupling expansion.

ii) Wilson loop factors

We have measured $\bar{W}(m,n)$ for rectangular loops of size up to 6 by 6. The measurements have been done over the individual lattices recorded on magnetic tape at the 27 preselected values of β and also, for $\beta = 2.25, 2.5, 2.75, 3$ and 3.25, over the last 30 lattices of long MC runs (50 iterations) at fixed β (starting from the configurations on tape at the same or the closest value of β). These last runs were made to obtain results with good statistical accuracy.

A signal of confinement is the presence in $-\ln \bar{W}$ of a term growing like the area enclosed by the loop. To isolate such a term, it is convenient to compare the expectation values of W_Y for square loops and rectangular loops with the same perimeter, and differing only by one lattice unit of area. Thus we measured the quantities

$$K(I, \beta) = -\ln \frac{\bar{W}(I, I)}{\bar{W}(I+1, I-1)} \quad (3.3)$$

In so far as $\bar{W}$ is dominated by a term growing like the area, $K(I, \beta)$ gives the coefficient of this term and is therefore directly related to the physical string tension:

$$K = a^2 \sigma \quad (3.4)$$

a being the lattice spacing. In the scaling regime, the whole dependence of K on β should be in the relation between β and a , as predicted by the renormalization group:

$$a^2(\beta) = \frac{1}{\Lambda_0^2} \left(\frac{6\pi^2}{11} \beta \right)^{\frac{102}{121}} \exp \left\{ - \frac{6\pi^2}{11} \beta \right\}, \quad (3.5)$$

Λ_0^2 being the lattice regularization parameter.

Indeed, one expects Eq. (3.4) to be true only for sufficiently large I at any given β . If we keep I fixed and increase β , thus decreasing the lattice spacing, eventually the physical size of the loop becomes small and the expectation value of W_Y can be computed in perturbation theory. One then finds a large β behaviour given by a power series in $1/\beta$. Thus we expect the scaling behaviour expressed by Eqs. (3.4) and (3.5) to appear only as an envelope of the curves $K(I, \beta)$ determined for different values of I .

In Fig. 3a we plot all the values we have measured for $K(I, \beta)$ up to $I = 5$, whereas in Fig. 3b we reproduce the points (and errors) determined through the longer MC runs. From Fig. 3a (where the first term in the strong coupling expansion of the string tension is also reproduced) it is apparent that all the curves $K(I, \beta)$ agree, for low β , with the strong coupling expansion, then follow for a while the scaling behaviour until the weak coupling perturbative regime takes over. Of the points with better statistics, (Fig. 3b), we have selected those which appear to lie closer to the envelope of the curves $K(I, \beta)$ (i.e., the points with $I = 2, 3, 4$ at $\beta = 2.25$, $I = 4, 5$ at $\beta = 2.5$, $I = 5$ at $\beta = 2.75$) and have determined the ratio σ/Λ_0^2 fitting the scaling curve, [Eqs. (3.4) and (3.5)], through these points. We find

$$\Lambda_0 = (0.011 \pm 0.002) \sqrt{\sigma}, \quad (3.6)$$

which agrees with a previous measurement by Creutz²⁾.

iii) The glueball mass

On general grounds, the plaquette-plaquette correlation function $C(d)$ [see Eq. (2.4)] is expected to behave as

$$C(d) \sim c_0 \exp \{ -\mu(\beta) d \} \quad (3.7)$$

for large separation.

In the scaling limit ($\beta \rightarrow \infty$) the dimensionless mass gap $\mu(\beta)$ should go to zero with $a(\beta)$, in such a way that the physical mass

$$M = \mu a^{-1} \quad (3.8)$$

approaches a finite limit. This is interpreted as the mass of the lowest quantum state (the glueball) coupled to the $F_{\mu\nu}^a F_a^{\mu\nu}$ gauge invariant operator.

In practice, we are limited to measurements with finite values of d . This is partly because of the finite extent of the lattice (d cannot be larger than 8 on a 16^4 lattice), but especially because the number to be measured decreases very rapidly as d increases and the signal soon becomes buried in the noise of statistical fluctuations. Thus we have measured a d dependent mass, defined according to the equation

$$\mu(d, \beta) = -\ln \frac{C(d)}{C(d-1)} \quad (3.9)$$

Repeating the discussion made for the string tension, the limit of μ for $d \rightarrow \infty$ should follow the scaling behaviour

$$\mu(\infty, \beta) \sim M a(\beta) \quad (3.10)$$

for sufficiently large β . But for any finite d , $\mu(d, \beta)$ should eventually

behave perturbatively as β becomes large. Thus, the scaling curve would appear only as the envelope of the curves $\mu = \mu(d, \beta)$.

Our results for the mass gap are reproduced in Fig. 4 [the axis is labelled $aM(d)$, rather than $\mu(d, \beta)$]. The numbers have been obtained averaging the correlations over one lattice to find $C(d)$ and then taking the logarithms of the ratios $C(d)/C(d-1)$ to evaluate $\mu(d, \beta)$. When many iterations at fixed β have been performed, the numbers obtained for $\mu(d, \beta)$ from the individual lattice configurations have been averaged. The results shown in Fig. 4 correspond to averages over the last 30 of 50 MC iterations for $\beta = 2.25, 3$ and 3.25 , over the last 70 of 100 iterations for $\beta = 2.5$ and 2.75 , to measurements over a single lattice configuration for all other values of β . The difficulty of the measurements, due to the statistical fluctuations, is indicated by the large errors, and by the fact that averages over 30 MC iterations have been necessary in order to obtain meaningful signals for $d = 3$ and over 70 iterations for $d = 4$.

The solid lines in Fig. 4 represent the lowest order strong coupling prediction and the (weak coupling) scaling behaviour of Eq. (3.10). Assuming that the points with $d = 2$ for $\beta = 2$, $d = 3$ for $\beta = 2.25$ and 2.5 , $d = 4$ for $\beta = 2.5$ and 2.75 , belong to the envelope of the functions $\mu(d, \beta)$, a fit of the scaling curve through these points gives:

$$M = (3 \pm 1) \sqrt{\sigma}. \quad (3.11)$$

iv) The interquark potential

The expectation value $\bar{W}(d, d')$ of Wilson factors for rectangular loops is expected to behave as

$$\bar{W}(d, d') \sim \exp \{-d' a \phi(d)\} \quad (3.12)$$

in the limit $d' \gg d$, $\phi(d)$ being the energy of two static sources (quark-anti-quark system) at a distance $r = da$. If we take $d' = 16$, the maximum length permissible with our lattice, $\bar{W}(d, 16)$ will not differ in an appreciable way from the expectation value $Q(d)$ of the product of Wilson factors for parallel loops at separation d running through the lattice. Hence one also expects

$$Q(d) \approx \exp \{ -16 a \varphi(d) \} \quad (3.13)$$

and a measurement of $Q(d)$ ought to allow a determination of the interquark potential at distances $r = da$.

The correct interpretation of $Q(d)$ is, however, more subtle. With a lattice extending for d' sites along the time direction and periodic boundary conditions, and of infinite extent along the spatial directions, one can demonstrate the relation

$$Q(d) = \exp \{ - V(d)/T \}, \quad (3.14)$$

where $T = 1/ad'$ is the physical temperature and $V(d)$ is the free energy of the system. Neglecting the entropy term in $V(d)$, $\phi(d)$ may still be interpreted as an interquark potential, but not at vanishing temperature. In particular, if T is large enough, the system may be in the Debye screened phase^{6,10)} and a measurement of $Q(d)$ would not reveal a confining force between static sources. An infinite extent of the lattice along the spatial directions is, in principle, necessary to produce a critical behaviour, but, as is often the case with Monte Carlo computations, a signal of the deconfining phase transition is observed even with a lattice of rather limited spatial extent.

Monte Carlo simulations give $T_c \approx 27\Lambda_0$ for the temperature of the decoupling phase transition in the $SU(2)$ model⁶⁾. Combining the requirement $T = 1/a(\beta)d' < T_c$ with the functional dependence of the lattice spacing a on β [Eq. (3.5)], one finds, for each value of d' , the maximum β (and therefore the lowest value of the bare coupling constant $g_0 = 2/\sqrt{\beta}$) which still avoids the phase transition. (In particular, the larger d' is, the lower is the minimum safe value of g_0 , which shows the advantage of working with a large lattice, as discussed in the Introduction.) On the other hand, for low β , the expectation value of the product of Wilson factors for loops running through the lattice is very small and, as soon as the loops are separated, it becomes indistinguishable from statistical fluctuations. To obtain meaningful numbers for $Q(d)$, we

measured this quantity at the largest value of β ($= 4.5$) met in our computation and the expectation values were determined averaging over 100 lattice configurations in a long MC run (at fixed β , with initial configuration the one recorded on magnetic tape, obtained, we recall, at the end of a cooling process of $450 + 2 \times 27 = 504$ MC iterations). The results for the interquark potential are reproduced in Fig. 5, where we plot the shifted potential $a\tilde{\phi}(d) = a(d) - c$ ($c = 0.161$).

$\beta = 4.5$ is larger than the value of β obtained from $16a(\beta) = 1/T_c$; thus we may be measuring the Debye screened potential rather than the confining potential among static quarks. But the Debye screening length is then much larger than the maximum separation allowed by our computation, and the string tension, on the other hand, has become so small that a confining part of the potential would also appear only at much larger distances. Indeed, we have fit our results for $a\phi(d)$ both to a Debye screened form

$$a\phi(d) = -\frac{\alpha e^{-\mu d}}{d} + c \quad (3.15)$$

and to a linear plus Coulomb form

$$a\phi(d) = -\frac{\alpha}{d} + a^2\sigma d + c, \quad (3.16)$$

obtaining

$$\alpha = 0.053, \mu = -0.052, c = 0.164 \quad (3.17)$$

and

$$\alpha = 0.052, c = 0.161 \quad (3.18)$$

in the two cases. Thus, whether the system is in the deconfined phase or not, we

are effectively measuring the Coulomb part of the potential at very small separation ($a(\beta) \approx 0.0019 \sigma^{-1/2}$, for $\beta = 4.5$), obtaining the result

$$\phi_{\text{Coulomb}}(r) \approx \frac{0.052}{r}, \quad (3.19)$$

$$r \approx (0.002 \div 0.016) \sigma^{-1/2}.$$

The smallness of the coefficient is not surprising, in view of the very short scale of distances.

4. CONCLUSIONS

The results presented in this article illustrate once again, we believe, the power of Monte Carlo computations. The confining properties of the SU(2) lattice gauge theory and its scaling behaviour toward the continuum limit emerge clearly. Moreover, various numbers of physical significance can be determined. We have confirmed previous measurements of the string tension and have evaluated the mass of the glueball and the parameters of the interquark potential at short distances.

But the limitations of the method should also be apparent. One computes averages over quantum mechanical fluctuations, suitably simulated. The presence of large errors of statistical nature is often unavoidable, especially when the quantities to be measured are small. The errors can be reduced only by averaging over longer Monte Carlo runs, but a reduction of the error by one order of magnitude would typically require a hundred times more data and would make the computation prohibitively costly.

There are many other quantities of physical interest which can be measured, and refinement of the results obtained up to now is still possible. But perhaps the most interesting future applications of the Monte Carlo method will consist of investigations where the power of computer simulations coupled with suitable analytical techniques will allow some insight into the mechanism of confinement.

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FIGURE CAPTIONS

- Fig. 1 : Relation between internal energy $E = \langle S_0 \rangle$ and inverse temperature β in the $\tilde{Y}$ gauge model (determined from a 4^4 lattice).
- Fig. 2a - b : The relation $E = E(\beta)$ determined with better statistical accuracy on a 16^4 lattice ($\bullet$) and fits (solid lines) obtained by Padé's approximants of the strong coupling series.
- Fig. 3a - b : Increments of the negative logarithm of $\langle W_\gamma \rangle$ as the area enclosed by the loop γ increases by one lattice unit, based on the comparison of $I \times I$ square loops with $(I + 1) \times (I - 1)$ rectangular loops. Fig. 3a reproduces all data, Fig. 3b only those with better statistical accuracy. The solid lines represent the leading term in the strong coupling expansion and the fit to the scaling behaviour.
- Fig. 4 : Measurements of the coefficient in the exponential fall-off of the plaquette-plaquette correlation function, used to estimate the glue-ball mass. The solid lines represent the leading term in the strong coupling expansion and the fit to the scaling behaviour.
- Fig. 5 : Estimate of the short distance behaviour of the potential between quark and antiquark, based on the analysis of parallel loops winding through the lattice.

Fig. 1

Fig. 2a

Fig. 2b

Fig. 3a

Fig. 3b

Fig. 4

Fig. 5