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PERTURBATION AROUND THE MIGDAL TRANSFORMATION FOR THE
NON-LINEAR $O(n)$ SIGMA MODEL IN TWO DIMENSIONS

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A B S T R A C T

The perturbation around the Migdal transformation proposed by Martinelli and Parisi, has been computed to first order in the shifting parameter for the $O(n)$ sigma model in two dimensions, in the weak coupling limit. The correction has the correct sign, but to first order heavily overshoots the exact result of the momentum space perturbation theory.

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1. INTRODUCTION

Recently¹⁾ a systematic improvement of the Migdal renormalization transformation^{2,3)} has been proposed by means of the introduction of a shifting parameter ϵ , and by establishing a systematic procedure to compute an asymptotic series in such a parameter which connects the approximate (zero order) Migdal transformation with the exact theory. Such a procedure has been applied with success to the Ising model for a square lattice in two and three dimensions, to the percolation model¹⁾, and to the triangular Ising model in two dimensions⁴⁾. Here we have applied the same technique to the $O(n)$ sigma model in two dimensions.

The model is interesting by itself and also for its suggested connection with the gauge theory in four dimensions. We have concentrated on the weak coupling region limit which we think is the most critical and at the same time the most important in connection with the continuum limit of the theory^{*)}.

The zero-order Migdal transformation works very well for the $O(n)$ model in two dimensions, as it gives, through a very simple calculation, the weak coupling beta function with an error of about 5%. This success has to be contrasted, for example, with the beta function for gauge theories in four dimensions, where the result obtained by using the pure Migdal approximation in the weak coupling region is too large by a factor of 1.8. One wonders whether such a good result for the $O(n)$ non-linear sigma model in two dimensions, obtained with such elementary means, is purely accidental. Here, following the general procedure described in Ref. 1, we have computed the first-order correction, in the shifting parameter, to the Migdal result. The computation, although not trivial, can be performed in a completely algebraic way, giving the result for the beta function in closed form.

We recall that the spirit of the pure Migdal transformation is to allow a maximal complexity for the simple bond interaction, while neglecting all other kinds of effective interactions. The perturbation procedure proposed in Ref. 1,

*) The formulation of the theory on the lattice simplifies the treatment in the strong coupling region. The importance of providing a real-space renormalization transformation which works also in the weak coupling region needs hardly to be emphasized.

when applied to the square lattice $O(n)$ sigma model, increases the number of effective couplings to 12 in the weak coupling region, which by symmetry relation are reduced to 8 independent couplings when the decimation parameter $(\lambda-1) \rightarrow 0$ ¹⁾. This would appear to be a sufficiently high complexity to treat the model, with a good approximation, over the whole range of the bare coupling constant. It is true (see Section 3) that such couplings are confined within the elementary cell, but they include, beside the usual nearest neighbour interaction, the diagonal, three-spin, and four-spin interactions. The result, which we shall discuss in detail in Section 3, is that the first-order correction moves the value of the beta function in the correct direction but overshoots the exact result by 37%. We have also computed the derivative of the beta function in the shift parameter ε for the triangular lattice. Here the calculation is much simpler, as we have to first order in ε only three effective couplings. The computed derivative turns out to be rather large also in this case. However, a computation to order ε^2 appears feasible. The result for the square lattice is rather intriguing, as it points to a very slow effective convergence; or worse, to an instability of such an ε -expansion in the weak coupling region. It goes without saying that the computation of the ε^2 correction is far more complex.

2. THE METHOD

The general procedure described in Ref. 1 can be summarized as follows. If $\mathcal{L}$ is the Lagrangian of the system, let us consider $\mathcal{L} + (1-\varepsilon)\Delta$, where Δ is the shift operator as defined by Migdal²⁾ and by Kadanoff³⁾. Such a shift operator Δ (which can be chosen from a whole class) has the property of moving bonds in the Lagrangian. Thus we can very easily perform the integration of $e^{\mathcal{L}+\Delta}$ in a decimation procedure which transforms the original lattice spacing a into λa without producing interactions more complex than single-bond interactions for global gauge theories (single-plaquette interactions for local gauge theories). The "zero-order" pure Migdal approximation is obtained for $\varepsilon = 0$, while $\varepsilon = 1$ corresponds to the exact theory. Thus a systematic expansion in ε provides an asymptotic series which

formally connects the pure Migdal approximation with the exact theory. In other words, we can compute the derivatives of the various physical quantities with respect to ε at $\varepsilon = 0$, where the theory is simply computable, and then by a suitable continuation procedure put $\varepsilon = 1$. From the practical point of view, what happens is that such an expansion in the shift parameter ε generates more and more complex effective interactions, like diagonal, three-spin, four-spin interactions, etc. Every new effective interaction is formally of a well-defined order in ε and has to be treated consistently as such. As we shall see, such a systematic procedure preserves all the invariance properties of the system.

3. O(n) ON THE SQUARE LATTICE

The bare Lagrangian $\mathcal{L}$ is defined by

$$\mathcal{L} = \beta_0 \sum_{\underline{i}} \left(\vec{\sigma}_{\underline{i}} \cdot \vec{\sigma}_{\underline{i}+\underline{1}} + \vec{\sigma}_{\underline{i}} \cdot \vec{\sigma}_{\underline{i}+\underline{2}} \right), \quad \vec{\sigma}_{\underline{i}}^2 = 1 \quad (3.1)$$

For $\lambda = 2$ we define the shift operator by

$$\Delta = \beta_0 \sum_{\underline{i}} (\pm) \vec{\sigma}_{\underline{i}} \cdot \vec{\sigma}_{\underline{i}+\underline{2}} \quad (3.2)$$

where we have to choose the + when the x coordinate of $\underline{i}$ is even and - when it is odd ($\underline{1}$ and $\underline{2}$ denote the lattice unit vectors in the x and y directions). The action of the bond shift operator Δ is shown in Fig. 1.

In the weak coupling region ($\beta_0 \gg 1$) the spins $\vec{\sigma}_{\underline{i}}$ belonging to some finite region of space (e.g. the elementary cell) differ slightly from each other and thus it is convenient for computational purposes to represent them in the form $(\sqrt{1-\underline{\pi}^2}, \underline{\pi})$, where $\underline{\pi}$ is an (n-1)-dimensional vector. Thus in the intermediate stages of the computation we lose explicit O(n) invariance, which is however explicitly recovered in the final result. The integration measure is given by $\delta(\underline{\pi}^2 + \pi_n^2 - 1) d^{n-1} \underline{\pi} d\pi_n = d^{n-1} \underline{\pi} / 2\sqrt{1-\underline{\pi}^2}$. In the following we shall work consistently up to order $O(\beta^{-1})$ both in the Lagrangian and in the measure; this is sufficient to compute the first non-trivial term of the beta function. It is convenient also to expand the effective bond interaction in powers of θ^2 (the angle between the spins), keeping in mind that $\underline{\pi}^2 = O(\beta^{-1})$:

$$\beta \theta_{12}^2 = \beta \left[(\pi_1 - \pi_2)^2 + \frac{1}{12} ((\pi_1 - \pi_2)^2)^2 + \frac{1}{4} (\pi_1^2 - \pi_2^2)^2 + O(\beta^{-3}) \right]. \quad (3.3)$$

To this order in β we can write the integration measure as

$$\frac{1}{2} \left(1 + \frac{1}{2} \pi^2 \right) d^{n-1} \pi. \quad (3.4)$$

The pure Migdal potential moving renormalization group equations for β_x and β_y are defined for $\lambda = 2$ by^{2,3)}

$$\begin{aligned} e^{-\beta'_x \theta_{13}^2} &= \int e^{-\beta_x (\theta_{12}^2 + \theta_{23}^2)} \frac{1}{2} \left(1 + \frac{1}{2} \pi^2 \right) d^{n-1} \pi_2 = \\ &= \text{const} e^{-\frac{\beta_x}{2} \theta_{13}^2} \left[1 + \frac{n-2}{24} \theta_{13}^2 - \frac{(n-1)(n-2)}{24 \beta_x} + O(\beta_x^{-2}) \right] = \\ &= \text{const} e^{-\left[\frac{\beta_x}{2} - \frac{n-2}{24} \right] \theta_{13}^2} + O(\beta_x^{-2}) \end{aligned} \quad (3.5)$$

and

$$\beta'_y = 2 \beta_y \quad (3.6)$$

Now the shift operator in the y direction has to be applied, giving

$$\beta_x \rightarrow \frac{\beta_x}{2} - \frac{n-2}{24} \rightarrow \beta_x - \frac{n-2}{12} \quad (3.7)$$

$$\beta_y \rightarrow 2 \beta_y \rightarrow \beta_y - \frac{n-2}{24} \quad (3.8)$$

The generalization to arbitrary integer λ is straightforward. We get

$$\beta_x \rightarrow \frac{\beta_x}{\lambda} - \frac{(n-2)(\lambda-1)}{12 \lambda} \rightarrow \beta_x - \frac{(n-2)(\lambda-1)}{12} \quad (3.9)$$

$$\beta_y \rightarrow \lambda \beta_y \rightarrow \beta_y - \frac{(n-2)(\lambda-1)}{12 \lambda} \quad (3.10)$$

This decimation procedure, as that proposed in Ref. 1, is asymmetric with respect to the x and y axis, unless we make the decimation parameter $\lambda - 1$ tend to zero^{*)}. In order to perform such a limit (also when computing the perturbation around the Migdal result), we have first to compute the recursion formulae for the generic integer λ . The results are simple rational functions of λ which have an obvious analytic continuation for real λ , thus allowing the limit $\lambda \rightarrow 1$ to be taken. In the limit $\lambda \rightarrow 1$, Eqs. (3.9) and (3.10) give:

$$\left. \frac{\partial \beta}{\partial \ln \lambda} \right|_{\lambda=1} = - \frac{n-2}{12} \quad (3.11)$$

for both β_x and β_y , which compares rather well with the exact result obtained from momentum space perturbation theory⁵⁾ $\partial \beta / \partial \ln \lambda = -1/4\pi$. The error is about +5%.

Insertion to first order in ϵ of a vertical bond¹⁾, as shown in Fig. 2, gives rise to 10 new effective couplings [to order $O(\beta^{-1})$], which have to be added to the original simple $\beta_x \theta_x^2$, $\beta_y \theta_y^2$. They are shown graphically in Fig. 3. Each line in the figure stands for the square of the angle between the two spins belonging to the end points of the bond. The next job is to compute how these new effective couplings, which are all formally of order ϵ , when inserted in the lattice and after performing the integration on the decimated spins, influence the renormalization equations for β_x and β_y couplings and for themselves. We have to keep in mind that to be consistent with the expansion in ϵ , these new couplings have not to be shifted but explicitly integrated, keeping obviously only the first order in ϵ .

The final result is a renormalization group transformation given by a (non-linear) operator acting in the 12-dimensional space of the couplings:

$$\vec{\beta} = \begin{pmatrix} \beta_x \\ \beta_y \\ \vdots \\ \beta_{20} \end{pmatrix} \quad (3.12)$$

*) It is also possible to perform a symmetric decimation. However, in this case the couplings which originate in the expansion in the shift parameter ϵ are far more complicated.

We have computed such an operator algebraically for all values of λ . The explicit form of the operator is given in the Appendix. It has the following structure:

$$\vec{\beta}(\lambda) = \begin{array}{c} 3 \\ \begin{array}{|c|c|} \hline E(\varepsilon) + \frac{1}{\beta_x} F(\varepsilon) & \frac{1}{\beta_x} G \\ \hline H(\varepsilon) & K \\ \hline \end{array} \end{array} \vec{\beta}(1) = \mathcal{R}_x(\lambda) \vec{\beta}(1) \quad (3.13)$$

where the numbers indicate the dimensions of the various blocks. The dependence of E, F, H on ε is linear and, in addition, only F and G depend on n. All elements obviously depend on λ .

The transformation (3.13) has to be followed by the analogous transformation obtained by exchanging the role of the x and y directions, and finally the derivative with respect to $\ln \lambda$ has to be computed. Such a derivative for $\lambda = 1$ turns out to be symmetric with respect to x and y, and as a consequence the independent couplings are reduced to eight, i.e.

$$\beta_x (= \beta_y), \beta_D, \beta_{xx} (= \beta_{yy}), \beta_{DD}, \beta_{xy}, \beta_{xD} (= \beta_{yD}), \beta_{2x} (= \beta_{2y}), \beta_{2D} \quad (3.14)$$

The derivative with respect to $\ln \lambda$ for $\lambda = 1$ of such an 8-dimensional vector is given by the following evolution equation:

$$\vec{\beta} = \begin{array}{c} 2 \\ \begin{array}{|c|c|} \hline A(\varepsilon) + \frac{1}{\beta_x} B(\varepsilon) & \frac{1}{\beta_x} C \\ \hline D(\varepsilon) & M \\ \hline \end{array} \end{array} \vec{\beta} = T \vec{\beta} \quad (3.15)$$

Again, because of the appearance of $1/\beta_x$, the operator is non-linear. The explicit values of the matrix elements are also given in the Appendix.

Renormalizing the theory means going to distances much larger than the original lattice cut-off a . In other words, it means iterating the transformation $R_y(\lambda)R_x(\lambda)$ of Eq. (3.13) N times, with $N \ln \lambda \rightarrow \infty$, or giving the solution of Eq. (3.15) for a very large value of the logarithmic scale t [$\dot{\vec{\beta}} = \partial \vec{\beta} / \partial t$]. As we shall see, starting from a large value of β_x , the effective couplings $\vec{\beta}$ eventually tend to decrease, as a result of the asymptotic freedom of the system. Thus we have to consider the solution of (3.15) for large values of t , but always such that β_x remains large. Such a procedure can be carried out numerically, for example, by starting from a generic set of values for $\vec{\beta}$ (e.g. $\beta_x \gg 1$, and all the other couplings set to zero as the initial condition). However, the exact solution of (3.15) can be given explicitly in the asymptotic regime for large values of β , neglecting the term $O(\beta^{-1})$, by keeping in mind that, owing to the renormalizability of the theory, at large distances all effective couplings will depend on a single parameter, which we can choose, e.g. β_x . Thus a possible choice of the solution is

$$\vec{\beta} = \beta_x \vec{v} + \vec{w} + \beta_x^{-1} \vec{z} + \dots \quad (3.16)$$

where $\vec{v}$, $\vec{w}$, $\vec{z}$ are constant vectors, i.e. vectors depending only on n and ε ($v_1 = 1$, $w_1 = 0$). To order $O(\beta^0)$, Eq. (3.15) gives us

$$\dot{\beta}_x \vec{v} = \begin{array}{|c|c|} \hline A(\varepsilon) & 0 \\ \hline D(\varepsilon) & M \\ \hline \end{array} \beta_x \vec{v} \quad (3.17)$$

and thus we have to determine the eigenvalues, and the eigenvectors of the matrix obtained from the operator appearing in Eq. (3.15) when B and C are set to zero. Because of the structure of such a matrix it is natural to decompose the eigenvector $\vec{v}$ in a two-component vector h and in the remaining six-component vector k . Then we must have

$$\mu(\varepsilon) h = A(\varepsilon) h = \begin{pmatrix} -\frac{4}{3} \varepsilon & \frac{4}{3} \\ \frac{2}{3} \varepsilon & -\frac{2}{3} \end{pmatrix} h, \quad (3.18)$$

giving as an eigenvalue 0 with the corresponding eigenvector $h_0 = (1, \varepsilon)$; the second eigenvalue is $-\frac{2}{3} - \frac{4}{3} \varepsilon$, with eigenvector $(1, -\frac{1}{2})$. Only the first eigenvalue is relevant at large distances as the contribution of the other fades away. The other eigenvectors of the type $h = 0, k \neq 0$ do not correspond to the asymptotic solution of Eq. (3.15) with β_x large, because they would imply $\beta_x = 0$, making Eq. (3.15) itself, which was derived in the $\beta_x \gg 1$ hypothesis, meaningless. Coming back to the eigenvalue 0, the remaining six-component vector k_0 is obtained by solving the equation

$$D(\varepsilon) h_0 + M k_0 = 0, \quad (3.19)$$

which gives $k_0 = M^{-1} D(\varepsilon) h_0$. The explicit form of the column vector k_0 is given from the solution of Eq. (3.19) by $k_0 = \varepsilon(-\frac{3}{4}, -\frac{1}{15}, \frac{1}{12}, \frac{1}{15}, -\frac{1}{60}, \frac{1}{30})$. Thus we have, substituting in (3.13),

$$\dot{\beta}_x h_0 = B(\varepsilon) h_0 + C k_0 + A(\varepsilon) w \quad (3.20)$$

This is a system of two equations where the unknown are β_x and w_2 (because $w_1 \equiv 0$ by definition). The explicit form of Eq. (3.20) is

$$\dot{\beta}_x = (n-2) \left[-\frac{1}{12} + \frac{4\varepsilon}{15} \right] + \frac{4}{3} w_2 \quad (3.21a)$$

$$\varepsilon \dot{\beta}_x = -\frac{(n-2)}{10} \varepsilon - \frac{2}{3} w_2 \quad (3.21b)$$

which gives $w_2 = -[(n-2)/40]\varepsilon$ and^{*)}

*) For $n = 2$ the derivatives of all couplings vanish. This is consistent with the fact that for $O(2)$ the beta function vanishes to all order^{5,6)}. The variable n does not appear simply in the form $n-2$ in the operator (3.15) (see the Appendix). Thus this result is a rather strong check on the correctness of our algebraic computation.

$$\dot{\beta}_x = (n-2) \left[-\frac{1}{12} + \frac{7\varepsilon}{30} \right] \quad (3.22a)$$

$$\varepsilon \dot{\beta}_x = \dot{\beta}_D = -\frac{(n-2)}{12} \varepsilon + O(\varepsilon^2) \quad (3.22b)$$

Starting from generic initial conditions [with the exception of those with $\beta_x(t=0) = 0$], for large enough t , the vector $\vec{\beta}$ can always be written in the form (3.16). Asymptotically [but still with $\beta_x(t) \gg 1$] Eqs. (3.22) are, as expected, independent of the initial conditions. This algebraic procedure is exactly equivalent to the study of the asymptotic behaviour of $\vec{\beta}$ due to the evolution equation (3.15), for large $\vec{\beta}$. As a check, we have in fact numerically solved Eq. (3.15) by starting with a very large value of β_x , with all the other couplings initially set to zero, and looking at the solution for a large value of t . Such a numerical solution agrees perfectly with the above-mentioned exact algebraic treatment. In order to connect our effective couplings with the usual definition of the coupling constant in the continuum limit, we have to examine the continuum limit of our effective Lagrangian.

We have

$$\vartheta_{i,i+1}^2 = \text{const} - 2 \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} + O(\vartheta_{i,i+1}^4) \quad (3.23)$$

$$\begin{aligned} \text{and } 2\beta_x \sum_i (\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} + \vec{\sigma}_i \cdot \vec{\sigma}_{i+2}) + 2\beta_D \sum_i (\vec{\sigma}_i \cdot \vec{\sigma}_{i+1+2} + \vec{\sigma}_i \cdot \vec{\sigma}_{i+1-2}) = \\ = \text{const} + \beta_x \sum_i \{ (\vec{\sigma}_i - \vec{\sigma}_{i+1})^2 + (\vec{\sigma}_i - \vec{\sigma}_{i+2})^2 \} + \\ + \beta_D \sum_i \{ (\vec{\sigma}_i - \vec{\sigma}_{i+1+2})^2 + (\vec{\sigma}_i - \vec{\sigma}_{i+1-2})^2 \} = \\ \simeq \text{const} + (\beta_x + 2\beta_D) a^2 \sum_i (\nabla \vec{\sigma})^2 \simeq \text{const} + (\beta_x + 2\beta_D) \int d^2x (\nabla \vec{\sigma})^2 \end{aligned} \quad (3.24)$$

The remaining terms all involve higher powers of the derivatives (such as $\beta_{xx} \theta_{i,i+1}^2 \theta_{i,i+1}^2$ or $\beta_{xy} \theta_{i,i+1}^2 \theta_{i,i+2}^2$) and thus do not contribute in the formal continuum limit which is used to define the coupling constants in the weak coupling limits. Following now the procedure of Ref. 1 which exploits the information that $\partial\beta/\partial\varepsilon|_{\varepsilon=1} = 0$ (because of the symmetry properties of the bond-shifting operator), we obtain

$$\begin{aligned} \frac{\partial\beta}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{1}{g^2} \right) = \frac{\partial}{\partial t} (\beta_x + 2\beta_D) = (n-2) \left[-\frac{1}{12} + \varepsilon \left(1 - \frac{\varepsilon}{2} \right) \left(\frac{4}{15} - \frac{2}{10} \right) \right] \\ &= (n-2) \left[-\frac{1}{12} + \varepsilon \left(1 - \frac{\varepsilon}{2} \right) \frac{1}{15} \right] \end{aligned} \quad (3.25)$$

which for $\varepsilon = 1$ gives $\dot{\beta} = -(n-2)/20$. As we see, the first-order correction shifts the value of $\dot{\beta}$ in the correct direction, but with respect to the exact result^{5,6)} $\beta = -1/4\pi$ the error goes from -4.7% to +37%. Thus we have a heavy overshooting.

4. DISCUSSION

The rather large overshooting of the first-order correction reported in the previous section is rather surprising in view of the exceptionally good performance (5% of error) of the pure Migdal transformation for the non-linear sigma model. We recall that for other models such as gauge theory in four dimensions (where unlike the Ising-like system there exists no duality transformation constraints), the pure Migdal result is much worse (80% of error). This seems to indicate that the 5% error for the two-dimensional sigma model is essentially accidental. The only good feature of the computed correction is that it has the correct sign. The computed $\Delta\beta/\beta$ is of the same order as $\Delta v/v$ for the two-dimensional Ising model¹⁾; but in the latter, as the starting point is much worse, the correction results in a real improvement. On the whole, one could say that this is the first case in which the correction scheme fails even though the sign of the correction is right. The choice of the value of $\lambda = 1$ appears to be the best one for several reasons. First, the transformation is explicitly x-y symmetric. Secondly, moving λ towards 2 worsens the situation because at $\lambda = 2$ also for infinitesimal ε the $O(\beta^0)$ order solution corresponding to the zero eigenvalue has

the structure $h_0 = (0, 1)$, i.e. there is an exchange in the role of the leading parameter between β_x and β_D which appears to make the ϵ -expansion unstable. Moreover, for $\lambda > 2$ there is a lower bound on ϵ in order to have non-positive eigenvalues, which forbids the evaluation of the derivatives with respect to ϵ for $\epsilon = 0$. The triangular lattice gives to the order ϵ a large value of the derivative and a too poor structure of effective couplings. In both cases one could attempt an ϵ^2 computation; this is far more complex and, owing to the poor performance of the first-order correction, even if it should result in a definite improvement it would not answer the question whether the Migdal approach and its related perturbative expansion give a reliable answer for weak coupling.

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REFERENCES

- 1) G. Martinelli and G. Parisi, CERN preprint Ref. TH.2882 (to be published in Nuc. Phys. B).
- 2) A.A. Migdal, Soviet Phys. JETP 42 (1976) 413; 42 (1976) 743.
- 3) L.P. Kadanoff, Ann. Phys. (NY) 100 (1976) 359.
- 4) S. Caracciolo, private communication.
- 5) E. Brezin and J. Zinn-Justin, Phys. Rev. Lett. 36 (1976) 691; Phys. Rev. B14 (1976) 311.
- 6) W.A. Bardeen, B.W. Lee and R. Shrock, Phys. Rev. D 14 (1976) 985.

APPENDIX

We give first the explicit form of the matrices $E + (1/\beta_x) F(\epsilon)$, G , $H(\epsilon)$, and K appearing in Eq. (3.13) of the text.

We next give the explicit form of the matrices $A(\epsilon) + (1/\beta_x) B(\epsilon)$, C , $D(\epsilon)$, and M appearing in Eq. (3.15) of the text.

Matrix $E(\epsilon) + \frac{1}{\beta_x} F(\epsilon)$

$\frac{1}{\lambda} - \frac{(n-2)(\lambda-1)}{12\lambda\beta_x}$	$\left[-\frac{\lambda^2-1}{3\lambda} + \frac{(n-2)(\lambda^2-1)(7\lambda^2+2)}{180\lambda^2\beta_x} \right] \epsilon$	$-\frac{2(\lambda^2-1)}{3\lambda} + \frac{(n-2)(\lambda^2-1)(7\lambda^2-8)}{90\lambda^2\beta_x}$
0	$\lambda + \left[-\frac{\lambda^2-1}{3\lambda} - \frac{(n-2)(\lambda^2-1)}{36\beta_x} \right] \epsilon$	$\frac{4(\lambda^2-1)}{3\lambda} - \frac{(n-2)(\lambda^2-1)}{18\beta_x}$
0	$\left[\frac{\lambda^2-1}{6\lambda} - \frac{(n-2)(\lambda^2-1)}{72\beta_x} \right] \epsilon$	$1 + \frac{(\lambda-1)(\lambda-2)}{3\lambda} - \frac{(n-2)(\lambda^2-1)}{36\beta_x}$

Matrix G

$\frac{2(n+1)(\lambda-1)}{\lambda^2}$	$-\frac{4(n+1)}{15\lambda^2} \cdot (\lambda^4-1)$	$-\frac{2(n+1)}{15\lambda^2} \cdot (2\lambda^4-5\lambda^2+3)$	$\frac{2}{3} \left[\frac{\lambda^2-3\lambda+2}{\lambda} - \frac{n(\lambda-2)(\lambda^2-1)}{\lambda^2} \right]$	$\frac{2}{3} \left[\frac{\lambda^2-3\lambda+2}{\lambda} - \frac{n(\lambda-2)(\lambda^2-1)}{\lambda^2} \right]$	$\frac{2(\lambda-1)(-4\lambda^3+6\lambda^2-9\lambda-4)}{15\lambda^2} + \frac{4n(\lambda^2-1)(3-2\lambda^2)}{15\lambda^2}$	$\frac{n-1}{\lambda^2} \cdot (\lambda-1)$	$\frac{2}{3} \frac{(\lambda-1)(\lambda-2)}{\lambda} - \frac{(n+1)(\lambda^4-5\lambda^2+4)}{15\lambda^2}$	$\frac{2\lambda^3-4\lambda^2+4\lambda-2}{3\lambda^2} - \frac{(n+1)(\lambda^2-1)(2\lambda^2-3)}{15\lambda^2}$
0	$\frac{2(n+1)}{15\lambda^2} \cdot (3\lambda^4-5\lambda^2+2)$	$\frac{2}{15} \frac{(n+1)}{\lambda^2} \cdot (3\lambda^4-5\lambda^2+2)$	$\frac{2}{3} \left[\frac{(\lambda^2-1)(1-2\lambda)}{\lambda^2} + \frac{n(\lambda-1)(2\lambda^2+1)}{\lambda^2} \right]$	$\frac{2}{3} \left[\frac{-2\lambda^3+\lambda^2-\lambda+2}{\lambda^2} + \frac{2n(\lambda-1)(\lambda^2-1)}{\lambda^2} \right]$	$\frac{2(\lambda-1)(6\lambda^3-14\lambda^2-4\lambda+1)}{15\lambda^2} + \frac{2n(\lambda^2-1)(6\lambda^2+1)}{15\lambda^2}$	0	$\frac{(\lambda^2-1)}{15\lambda^2} \left[-20\lambda + (n+1)(3\lambda^2+8) \right]$	$-\frac{2(\lambda-1)(2\lambda-1)}{3\lambda} + \frac{(n+1)(3\lambda^4-5\lambda^2+2)}{15\lambda^2}$
0	$\frac{2(n+1)}{15\lambda^2} \cdot (\lambda^4-1)$	$\frac{2}{15} \frac{(n+1)}{\lambda^2} \cdot (\lambda^4-1)$	$\frac{1-\lambda}{3\lambda^2} \left[\lambda^2-\lambda+1 - n(\lambda^2-1) \right]$	$-\frac{(\lambda^2-1)(\lambda-2)}{3\lambda^2} + \frac{n(\lambda-1)(\lambda^2+2)}{3\lambda^2}$	$\frac{(\lambda-1)(4\lambda^3-6\lambda^2+4\lambda-1)}{15\lambda^2} + \frac{n(\lambda^2-1)(4\lambda^2-1)}{15\lambda^2}$	0	$-\frac{(\lambda-1)(\lambda-2)}{3\lambda} + \frac{(n+1)(\lambda^4-5\lambda^2+4)}{15\lambda^2}$	$-\frac{(\lambda^2-1)}{3\lambda} + \frac{(n+1)(\lambda^4-1)}{15\lambda^2}$

Matrix H(ϵ)

0	$-\frac{\lambda^2-1}{72\lambda} \epsilon$	$-\frac{\lambda^2-1}{36\lambda}$
0	$-\frac{(\lambda^2-1)(4\lambda^2-1)}{360\lambda^3} \epsilon$	$-\frac{\lambda^4-1}{45\lambda^3}$
0	$-\frac{(\lambda^2-1)(4\lambda^2-1)}{360\lambda^3} \epsilon$	$-\frac{\lambda^4-1}{45\lambda^3}$
0	$\frac{\lambda^2-1}{72\lambda} \epsilon$	$\frac{\lambda^2-1}{36\lambda}$
0	$\frac{\lambda^2-1}{72\lambda} \epsilon$	$\frac{\lambda^2-1}{36\lambda}$
0	$\frac{(\lambda^2-1)(3\lambda^2-2)}{360\lambda^3} \epsilon$	$\frac{(\lambda^2-1)(3\lambda^2+8)}{180\lambda^3}$
0	$-\frac{\lambda^4-1}{90\lambda^3} \epsilon$	$-\frac{(\lambda^2-1)(\lambda^2-4)}{45\lambda^3}$
0	$\frac{\lambda^4-1}{180\lambda^3} \epsilon$	$\frac{(\lambda^2-1)(\lambda^2-4)}{90\lambda^3}$
0	$\frac{\lambda^4-1}{180\lambda^3} \epsilon$	$\frac{(\lambda^2-1)(\lambda^2-4)}{90\lambda^3}$

Matrix K

$\frac{1}{\lambda^3}$	$\frac{\lambda^4-1}{15\lambda^3}$	$\frac{\lambda^4-1}{15\lambda^3}$	$-\frac{\lambda^2-1}{3\lambda^3}$	$-\frac{\lambda^2-1}{3\lambda^3}$	$\frac{(\lambda^2-1)(2\lambda^2-3)}{15\lambda^3}$	0	$\frac{(\lambda^2-1)(\lambda^2-4)}{30\lambda^3}$	$\frac{(\lambda^2-1)(\lambda^2-4)}{30\lambda^3}$
0	$1 + \frac{(\lambda-1)(6\lambda^3-9\lambda^2+\lambda+1)}{15\lambda^3}$	$\frac{2(\lambda^2-1)(3\lambda^2-2)}{15\lambda^3}$	0	0	$\frac{2(\lambda^2-1)(6\lambda^2+1)}{15\lambda^3}$	0	$\frac{(\lambda^2-1)(3\lambda^2-2)}{15\lambda^3}$	$\frac{(\lambda^2-1)(3\lambda^2-2)}{15\lambda^3}$
0	$\frac{\lambda^4-1}{15\lambda^3}$	$\frac{15+(\lambda^2-1)(\lambda^2+11)}{15\lambda^3}$	0	0	$\frac{2(\lambda^4-1)}{15\lambda^3}$	0	$\frac{(\lambda^2-1)(\lambda^2-4)}{30\lambda^3}$	$\frac{(\lambda^2-1)(\lambda^2-4)}{30\lambda^3}$
0	$-\frac{(\lambda^2-1)(3\lambda^2-2)}{15\lambda^3}$	$-\frac{(\lambda^2-1)(3\lambda^2-2)}{15\lambda^3}$	$\frac{3+2(\lambda^2-1)}{3\lambda^3}$	$\frac{2(\lambda^2-1)}{3\lambda^3}$	$-\frac{(\lambda^2-1)(6\lambda^2+1)}{15\lambda^3}$	0	$-\frac{(\lambda^2-1)(3\lambda^2+8)}{30\lambda^3}$	$-\frac{(\lambda^2-1)(3\lambda^2-2)}{30\lambda^3}$
0	$-\frac{2(\lambda^4-1)}{15\lambda^3}$	$-\frac{2(\lambda^4-1)}{15\lambda^3}$	$\frac{\lambda^2-1}{3\lambda^3}$	$\frac{3+(\lambda^2-1)}{3\lambda^3}$	$-\frac{(\lambda^2-1)(4\lambda^2-1)}{15\lambda^3}$	0	$-\frac{(\lambda^2-1)(\lambda^2-4)}{15\lambda^3}$	$-\frac{\lambda^4-1}{15\lambda^3}$
0	$\frac{(\lambda^2-1)(3\lambda^2-2)}{15\lambda^3}$	$\frac{(\lambda^2-1)(3\lambda^2+8)}{15\lambda^3}$	0	0	$\frac{15+(\lambda^2-1)(6\lambda^2+11)}{15\lambda^3}$	0	$\frac{(\lambda^2-1)(3\lambda^2+8)}{30\lambda^3}$	$\frac{(\lambda^2-1)(3\lambda^2+8)}{30\lambda^3}$
0	$\frac{2(\lambda^4-1)}{15\lambda^3}$	$\frac{2(\lambda^2-1)(\lambda^2-4)}{15\lambda^3}$	$-\frac{2(\lambda^2-1)}{3\lambda^3}$	$-\frac{2(\lambda^2-1)}{3\lambda^3}$	$\frac{2(\lambda^2-1)(2\lambda^2-3)}{15\lambda^3}$	$\frac{1}{\lambda^3}$	$\frac{(\lambda^2-1)(\lambda^2-4)}{15\lambda^3}$	$\frac{\lambda^4-1}{15\lambda^3}$
0	$\frac{2(\lambda^4-1)}{15\lambda^3}$	$\frac{2(\lambda^2-1)(\lambda^2-4)}{15\lambda^3}$	0	0	$\frac{4(\lambda^4-1)}{15\lambda^3}$	0	$\frac{15+(\lambda^2-1)(\lambda^2+11)}{15\lambda^3}$	$\frac{(\lambda^2-1)(\lambda^2-4)}{15\lambda^3}$
0	$\frac{2(\lambda^4-1)}{15\lambda^3}$	$\frac{2(\lambda^2-1)(\lambda^2-4)}{15\lambda^3}$	0	0	$\frac{4(\lambda^4-1)}{15\lambda^3}$	0	$\frac{(\lambda^2-1)(\lambda^2-4)}{15\lambda^3}$	$\frac{15+(\lambda^2-1)(\lambda^2+11)}{15\lambda^3}$

$$A(\varepsilon) = \begin{pmatrix} -\frac{4}{3}\varepsilon & \frac{4}{3} \\ \frac{2}{3}\varepsilon & -\frac{2}{3} \end{pmatrix}; \quad B(\varepsilon) = (n-2) \begin{pmatrix} -\frac{1}{12} + \frac{2\varepsilon}{45} & -\frac{2}{15} \\ -\frac{\varepsilon}{18} & -\frac{1}{9} \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{6(n+1)}{5} & ; & \frac{8}{15}(n+1) & ; & \frac{10}{3}n-2 & ; & \frac{56}{15}n-\frac{104}{15} & ; & \frac{49}{15}n-\frac{31}{15} & ; & \frac{4}{15}(n+1) \\ \frac{16}{5}(n+1) & ; & \frac{16}{15}(n+1) & ; & -\frac{2}{3} & ; & \frac{14}{5}n+\frac{22}{15} & ; & -\frac{4n}{5}-\frac{2}{15} & ; & \frac{8}{15}n-\frac{4}{5} \end{pmatrix}$$

$$D(\varepsilon) = \begin{pmatrix} -\frac{2\varepsilon}{45} & -\frac{13}{90} \\ -\frac{\varepsilon}{30} & -\frac{8}{45} \\ \frac{\varepsilon}{18} & \frac{1}{9} \\ \frac{\varepsilon}{30} & \frac{8}{45} \\ -\frac{\varepsilon}{45} & \frac{1}{15} \\ \frac{2\varepsilon}{45} & -\frac{2}{15} \end{pmatrix}$$

$$M = \frac{1}{15} \begin{pmatrix} -42 & 8 & -10 & 16 & -1 & -1 \\ 8 & -42 & 0 & 16 & -6 & -6 \\ -4 & -4 & -50 & 12 & -22 & -2 \\ -6 & 14 & 10 & -52 & 17 & 7 \\ 16 & -24 & -20 & -8 & -72 & -2 \\ 16 & -24 & 0 & 32 & -12 & -42 \end{pmatrix}$$

Figure captions

- Fig. 1 : Bond shifting in the simple Migdal approximation.
- Fig. 2 : Bond shifting in the expansion in the bond shift parameter ϵ .
- Fig. 3 : β_x and β_y are the original couplings. The others are the new effective couplings.

Fig. 1

Fig. 2

Original couplings :

Additional effective couplings :

Fig. 3