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A TRANSFORMATION FORMULA FOR INTEGRALS OVER SUPERFIELDS

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A B S T R A C T

We give an intrinsic definition of an integral over superfields and show that there is no Jacobi determinant in the transformation formula for such integrals.

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Many calculations in supersymmetric theories are considerably facilitated by the use of superfields^{1),2)}. This is in particular true for the computation of quantum corrections in $N = 1$ supersymmetric models³⁾. More recently, $N = 1$ superfields have been successfully employed to establish the vanishing of the β function in the $N = 4$ supersymmetric Yang-Mills theory⁴⁾ up to three loops⁵⁾. For these quantum calculations, functional integrals over superfields were introduced in a somewhat formal manner; the subsequent manipulations could be justified by termwise comparison with component field calculations.

In this note we demonstrate that there does exist an intrinsic (and hence more rigorous) definition of an integral over superfields which permits us to deduce various interesting consequences all of which are invariably characterized by the absence of the determinant factors that one commonly encounters. For instance, a Gaussian integral over a superfield always equals one, regardless of the quadratic form on which it depends; this result is implicitly contained in some of the references cited above. Here we will concentrate on the proof of a transformation formula for superfield integrals, according to which the transformed measure has no Jacobi determinant for all transformations which leave the internal θ structure untouched (the class of admissible transformations may be larger however). The eventual usefulness of the formalism most probably lies in the realm of future calculations in quantum supergravity: as is well known, renormalization in quantum gravity requires non-linear field redefinitions⁶⁾ and thus a knowledge of the behaviour of the associated functional measure under such transformations.

To set up the basic framework, we consider a real Grassmann algebra $\mathcal{G}$ which is generated by n elements $\theta_1 \dots \theta_n$ ⁷⁾. A superfield $g \in \mathcal{G}^{(1)}$ may be expressed as

$$g \equiv g(\theta_1 \dots \theta_n) = \sum_{\nu=0}^{2^n} \sum_{\alpha_1 \dots \alpha_\nu} g_{\alpha_1 \dots \alpha_\nu}^{(\nu)} \theta_{\alpha_1} \dots \theta_{\alpha_\nu} \quad (1)$$

where the summation ranges over all totally antisymmetric index combinations. The expansion coefficients $\{g_{\alpha_1 \dots \alpha_\nu}^{(\nu)}\}$ will henceforth be referred to as "components". An integration measure dg is defined as the product measure over all components

$$dg \equiv \prod_{\nu=0}^{2^n} \prod_{\alpha_1 \dots \alpha_\nu} dg_{\alpha_1 \dots \alpha_\nu}^{(\nu)} \quad (2)$$

The integral over odd components is understood to be in the sense of Berezin⁷⁾. We will now consider transformations on the components $\{g_{\alpha_1 \dots \alpha_v}^{(v)}\}$ which are induced by transformations on the superfield g . If $f: \mathbb{R} \rightarrow \mathbb{R}$ is any differentiable function, the components $\{\tilde{g}_{\alpha_1 \dots \alpha_v}^{(v)}\}$ of $\tilde{g} = f(g)$ are computed as follows. First, we approximate the function f by a power series and evaluate $f(g)$ term by term (knowing how to multiply superfields). We then reexpress all quantities in terms of derivatives of f and define the resulting expression as $f(g)$ for any sufficiently differentiable f . In this way, we obtain

$$\tilde{g}^{(0)} = f(g^{(0)}), \quad \tilde{g}_{\alpha}^{(1)} = f'(g^{(0)}) g_{\alpha}^{(1)}, \dots \quad (3)$$

The transformation formula for integrals over Grassmann algebras contains a superdeterminant (sometimes called Berezinian)^{7),8)} instead of the usual Jacobi determinant. In the case at hand, it is a determinant of a $2^n \times 2^n$ matrix and will be abbreviated by

$$\text{sdet} \frac{\partial \tilde{g}}{\partial g} \equiv \text{sdet} \frac{\partial (\{\tilde{g}_{\alpha_1 \dots \alpha_v}^{(v)}\})}{\partial (\{g_{\alpha_1 \dots \alpha_v}^{(v)}\})} \quad (4)$$

Theorem: if $\tilde{g} = f(g)$, then $\text{sdet} \partial \tilde{g} / \partial g = 1$ or

$$d\tilde{g} = dg \quad (5)$$

for any sufficiently differentiable $f: \mathbb{R} \rightarrow \mathbb{R}$.

Proof: we prove the theorem by induction on the number of generators $\theta_1 \dots \theta_n$ of $\mathcal{G}$. For $n = 1$, there is only one θ and we have $g = g^{(0)} + \theta g^{(1)}$ and $f(g) = f(g^{(0)}) + \theta f'(g^{(0)}) g^{(1)}$. The relevant superdeterminant is easily found to be

$$\text{sdet} \frac{\partial \tilde{g}}{\partial g} = \text{sdet} \begin{pmatrix} f'(g^{(0)}) & 0 \\ f''(g^{(0)}) g^{(1)} & f'(g^{(0)}) \end{pmatrix} = \frac{1}{f'(g^{(0)})} f'(g^{(0)}) = 1 \quad (6)$$

If $n > 1$, we assume that the theorem holds for some n and decompose g and $\tilde{g} = f(g)$ according to their dependence on θ_{n+1} :

$$\begin{aligned} g(\theta_1 \dots \theta_{n+1}) &= p(\theta_1 \dots \theta_n) + \theta_{n+1} q(\theta_1 \dots \theta_n) \\ \tilde{g}(\theta_1 \dots \theta_{n+1}) &= \tilde{p}(\theta_1 \dots \theta_n) + \theta_{n+1} \tilde{q}(\theta_1 \dots \theta_n) = \\ &= f(p(\theta_1 \dots \theta_n)) + \theta_{n+1} f'(p(\theta_1 \dots \theta_n)) q(\theta_1 \dots \theta_n) \end{aligned} \quad (7)$$

The integration measure is factorized accordingly

$$dg = dp dq = \prod_{v=0}^{2^n} \prod_{\alpha_1 \dots \alpha_v} dp_{\alpha_1 \dots \alpha_v}^{(v)} dq_{\alpha_1 \dots \alpha_{v,n+1}}^{(v+1)} \quad (8)$$

We now observe that $\tilde{p} = f(p)$ does not depend on q and therefore

$$\text{sdet} \frac{\partial \tilde{g}}{\partial g} = \text{sdet} \begin{pmatrix} \frac{\partial \tilde{p}}{\partial p} & 0 \\ \frac{\partial \tilde{q}}{\partial p} & \frac{\partial \tilde{q}}{\partial q} \end{pmatrix} = \text{sdet} \frac{\partial \tilde{p}}{\partial p} \text{sdet} \frac{\partial \tilde{q}}{\partial q} \quad (9)$$

Because p and $\tilde{p}$ are superfields over n generators, the induction hypothesis tells us that $\text{sdet} \partial \tilde{p} / \partial p = 1$. Furthermore, $\tilde{q}$ is obtained from q by multiplication with the superfield $f'(p)$ which is constant with respect to q . Hence it remains to be shown that $\text{sdet} \partial \tilde{q} / \partial q = 1$ for

$$\tilde{q}(\theta_1 \dots \theta_n) = M(\theta_1 \dots \theta_n) q(\theta_1 \dots \theta_n) \quad (10)$$

where $M(\theta_1 \dots \theta_n)$ is any "constant" superfield. The proof that this is indeed true is again by induction of n ; since it is exactly analogous to the argument given above it will not be reproduced here. The formula (9) then yields the desired result.

There are at least two generalizations of our theorem which we will simply state without proof since the proofs follow the same pattern as the one given above: the crucial ingredient is always the triangularity of the matrix $\partial \tilde{g}/\partial g$ as evident in (6) or (9). First, the superfield g may carry an index i running from 1 to N ; if the index $x \equiv i$ is allowed to be continuous, $g(x, \theta_1 \dots \theta_n)$ becomes a superfield in the conventional sense of the word. The theorem remains valid if dg in (2) is replaced by

$$dg \equiv \prod_{i=1}^N \prod_{v=0}^{2^n} \prod_{\alpha_1 \dots \alpha_v} dg_{i\alpha_1 \dots \alpha_v}^{(v)} \quad (11)$$

and $f: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is any sufficiently differentiable function.

The second generalization concerns the case of a Grassmann algebra with real generators $\theta_1 \dots \theta_n$ and complex components. If we denote the complex conjugate of $g_{i\alpha_1 \dots \alpha_v}^{(v)}$ by $\bar{g}_{i\alpha_1 \dots \alpha_v}^{(v)}$, the measure (11) becomes

$$dg d\bar{g} \equiv \prod_{i=1}^N \prod_{v=0}^{2^n} \prod_{\alpha_1 \dots \alpha_v} dg_{i\alpha_1 \dots \alpha_v}^{(v)} d\bar{g}_{i\alpha_1 \dots \alpha_v}^{(v)} \quad (12)$$

where, for even v , $dg^{(v)} d\bar{g}^{(v)}$ is defined as $d(\operatorname{Re} g^{(v)}) d(\operatorname{Im} g^{(v)})$. Again the theorem stays valid with one important modification: the function $f: \mathbb{C}^N \rightarrow \mathbb{C}^N$ must be holomorphic. This is so because the following fundamental property of holomorphic functions^{*)},

$$\left| \det \frac{\partial f_i}{\partial \bar{z}_j} \right|^2 = \det \begin{pmatrix} \frac{\partial u_i}{\partial x_j} & \frac{\partial v_i}{\partial x_j} \\ \frac{\partial u_i}{\partial y_j} & \frac{\partial v_i}{\partial y_j} \end{pmatrix} \quad (13)$$

which is a consequence of the Cauchy-Riemann equations⁹⁾, enters the proof in an essential way.

^{*)} Of course, $z_j = x_j + iy_j$ and $f_j = u_j + iv_j$.

It is surprising that the transformation formula for integrals over superfields as defined here does not contain the conventional Jacobi determinant. However, this result is presumably related to the vanishing of the vacuum energy in supersymmetric theories¹⁰⁾, at least if there is no spontaneous breakdown of supersymmetry. In the case of a spontaneously broken theory, the global properties of the transformation will come into play even though local properties such as (5) remain unaffected. This connection has been discussed in Ref. 11).

Finally, we mention that an analogous theorem holds for δ functions of superfields. Restricting ourselves to the complex case for convenience^{*}), we define

$$\delta(g) = \prod_{v=0}^{2^n} \prod_{\alpha_1 \dots \alpha_v} \delta(g_{\alpha_1 \dots \alpha_v}^{(v)}) \delta(\bar{g}_{\alpha_1 \dots \alpha_v}^{(v)}) \quad (14)$$

The δ functions of bosonic and fermionic components are the usual ones. Applying the same methods as before, we find

$$\delta(f(g)) = \sum_{f(g_s)=0} \delta(g - g_s) \quad (15)$$

for any differentiable f . The formula (15) should be contrasted with the corresponding one for ordinary δ functions which has an additional factor depending on the derivative of f .

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^{*}) Otherwise, we would have to worry about additional sign factors.

REFERENCES

- 1) A. Salam and S. Strathdee, Nucl. Phys. B76 (1974) 477;
S. Ferrara, J. Wess and B. Zumino, Phys. Lett. 51B (1974) 239.
- 2) P. Fayet and S. Ferrara, Phys. Rep. C32 (1977) 249;
B. Zumino, "Superspace" in the Proceedings of the Erice Workshop, March 1980,
Eds S. Ferrara, J. Ellis and P. Van Nieuwenhuizen (Plenum Press, New York,
March 1980), p. 101.
- 3) K. Fujikawa and W. Lang, Nucl. Phys. B88 (1975) 61;
F. Honerkamp, F. Krause, M. Scheunert and M. Schlindwein, Nucl. Phys. B95
(1975) 397;
S. Ferrara and O. Piguet, Nucl. Phys. B93 (1975) 261;
D.M. Capper, Nuovo Cim. 25A (1975) 646;
T. Clark, O. Piguet and K. Sibold, Ann. Phys. 109 (1977) 418.
- 4) L. Brink, J.H. Schwarz and J. Scherk, Nucl. Phys. 121 (1977) 77;
F. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. B122 (1977) 252.
- 5) M. Grisaru, M. Rocek and W. Siegel, Nucl. Phys. B159 (1979) 429; Phys. Rev.
Letters 45 (1980) 1063.
- 6) G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincaré 20 (1974) 69;
S. Deser and P. Van Nieuwenhuizen, Phys. Rev. D10 (1974) 401, 411;
S. Deser, P. Van Nieuwenhuizen and H.S. Tsao, Phys. Rev. D10 (1974) 3337.
- 7) F.A. Berezin, "The method of second quantization", (Academic Press, New York
1966).
- 8) R. Arnowitt, P. Nath and B. Zumino, Phys. Lett. 56B (1975) 81.
- 9) B.A. Fuks, "Introduction to the theory of analytic functions of several com-
plex variables", (American Math. Soc., Providence, R.I. 1963).
- 10) B. Zumino, Nucl. Phys. B89 (1975) 535.
- 11) H. Nicolai, Nucl. Phys. B176 (1980) 419.

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