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MESSAGE

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HIGHER RANK REPRESENTATIONS OF LOWER SPIN

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A B S T R A C T

New examples for spins 0, 1 and $1/2$ are given of lower spin systems as gauge theories of higher rank fields. These are characterized by the interchange of rank of potentials and field strengths with respect to ordinary gauge theories. The spin 1 action is of fourth time derivative order but ghost-free. A self-coupled supersymmetric theory of this type is also constructed. Couplings to gravity are, however, inconsistent.

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1. INTRODUCTION

The rank of a field operator describing particles of a given spin is not uniquely determined by that spin; many different field representations may be possible. Various unconventional ones have become popular recently with impetus from supersymmetry models, and the fact that they may have inequivalent^{*} topological effects^{1),2)}. In particular, spin 0 has been represented by antisymmetric tensors³⁾ (for zero mass) and by vectors^{4),5)} (for non-zero mass) and a possible model of zero mass spin 1/2 as an antisymmetric tensor-spinor has been proposed⁶⁾. Among other unusual features, the massless models possess gauge invariance, even for spin 0 or 1/2. They also exhibit the phenomenon of "spin-jumping": when mass or higher derivative terms are added there is a discontinuity in the spin content. We develop in this paper some general techniques for constructing new field representations, and construct models, including supersymmetric ones, for spin 0, 1 and 1/2 using higher rank tensors. Finally, we consider their gravitational interactions.

To introduce our work we recall the example of the antisymmetric tensor gauge theory for spin 0 which has the action

$$I \sim -\frac{1}{2} \int G_{\mu\nu} G^{\mu\nu} \quad ; \quad G^{\mu\nu} \equiv -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\rho B_{\sigma} \quad (1.1)$$

It is invariant under $\delta B_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$. To show that (1.1) describes spin 0, note first that it is equivalent to the first order form,

$$I \sim \int \left[\frac{1}{4} B_{\rho\sigma} \epsilon^{\rho\sigma\mu\nu} F_{\mu\nu}(A) + \frac{1}{2} A_\mu A^\mu \right] \quad ; \quad F_{\mu\nu}(A) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \quad (1.2)$$

upon insertion of the A_μ field equation $A_\mu = G_\mu$. Since the $B_{\rho\sigma}$ field equation is $F_{\mu\nu}(A) = 0$ or equivalently $A_\mu = \partial_\mu \phi$, (up to topological questions that we ignore here), (1.2) becomes the usual $I \sim \int 1/2 (\partial_\mu \phi)^2$ spin 0 action.

Notice that this model is "complementary" to the Maxwell action in the following sense: the tensor structure of the gauge potential, A_μ , and field strength, $F_{\mu\nu}$, of the Maxwell action is reversed in the antisymmetric tensor theory, where $B_{\mu\nu}$ is the gauge potential and A_μ the invariant field strength. Indeed, Ogievetsky and Polubarinov³⁾, who introduced this model, called the antisymmetric tensor field a "notoph" to contrast it with the photon. This kind of complementarity is built into the theory by the basic construction of (1.2).

^{*}) But see W. Siegel, Caltech preprint CALT-68-803 (1980).

In general we will start by constraining to zero, via a Lagrange multiplier, the field strength of some gauge potential, and then add an appropriate "mass term". Since this constraint requires the potential to be pure gauge, the gauge parameters become the potentials of the new theory. Which of these actually appear in the theory is determined by the "mass term". In general there may be several choices of "mass term" which will produce final actions describing several different spins. The aim is then to rewrite the theory entirely in terms of the Lagrange multiplier. This step involves subtleties that we shall discuss in our examples, the first being those for which the symmetric tensor $h_{\mu\nu}$ is constrained to have vanishing Riemann or Weyl tensor. The gauge potential in this case is a fourth rank tensor with the symmetries of the Riemann tensor and describes spin 1 or spin 0 respectively. For spin 1/2, an analogous theory is obtained by constraining the fermionic analogue of the Weyl tensor to vanish. This model differs from that of Ref. 6) in that an additional constraint is included. The original model, without this constraint, does not describe pure spin 1/2 but contains, in addition, a spin 3/2 dipole.

Supersymmetry unifies several of these models and there is a supersymmetric theory in which the super Weyl tensor^{7),8),9)} is constrained to vanish. Its components in the linearized case⁷⁾ are, in the notation of Ref. 9),

$$\left\{ R_{ab\alpha} , -i(R_{ab}\gamma_5)_\alpha , (\sigma_{cd}W_{ab}{}^{cd} + iF^{cd}\gamma_5 T_{abcd})_{\beta\alpha} , -2T_{ab}{}^{cd}(\bar{\phi}_{cd})_\alpha , \right. \\ \left. -2T_{ab}{}^{cd}(i\bar{\phi}_{cd}\gamma_5)_\alpha \right\} \equiv W_{ab\alpha} . \quad (1.3)$$

T_{abcd} is a projection operator into γ traceless tensor spinors, while the individual field strengths are

$$R_{ab\alpha} = \partial_b \bar{\psi}_{a\alpha} - \partial_a \bar{\psi}_{b\alpha} + (\bar{\phi}_a \gamma_b - \bar{\phi}_b \gamma_a)_\alpha \\ F_{ab} = \partial_a A_b - \partial_b A_a \\ W_{abcd} = h_{da,cb} + h_{cb,da} - h_{db,ca} - h_{ca,db} + 2\gamma_{da} f_{cb} \\ + 2\gamma_{cb} f_{da} - 2\gamma_{db} f_{ca} - 2\gamma_{ca} f_{db} \\ \bar{\phi}_{ab\alpha} = (\partial_a \bar{\phi}_b - \partial_b \bar{\phi}_a)_\alpha . \quad (1.4)$$

The fields f_{ab} and ϕ_a appearing in this multiplet are not independent, but are chosen to be functions of h_{ab} , ψ_a , A_μ such that $(\gamma^a)_{\alpha\beta} W_{ab\beta} = 0$. This implies $(\gamma^a)_{\alpha\beta} R_{ab\beta}$ and $W_{abcd}\eta^{bc} = 0$. A $W_{ab\alpha}$ satisfying this constraint can be written in Weyl spinor notation as $W_{\alpha\beta\gamma}$ with total symmetry in $\alpha\beta\gamma$. When we come to the supersymmetric model this notation will be convenient. However, it will initially prove useful to keep f_{ab} and ϕ_a as independent variables. These fields have their origin in the superconformal group; they are respectively the gauge fields of proper conformal and S supersymmetry⁹⁾. But in our case they will have a quite different significance.

We have mentioned that one possible application of these models is in supersymmetry where "spin jumping" representations are sometimes useful. It is simple to see how this happens for the antisymmetric tensor gauge theory. If a mass term $\kappa B_{\mu\nu}^2$ is added to (1.2) then $B_{\mu\nu}$ can be trivially eliminated to arrive at the massive vector theory. If instead we modify (1.1) in a gauge invariant way by adding a higher derivative term $G_\mu \square G^\mu$, then a massive spin 1 ghost is propagated¹⁰⁾ along with the massless spin 0. To see this, recall that in the form (1.2) without higher derivatives, the field equations are $F_{\mu\nu}(A) = 0$ and $A_\mu = G_\mu$ with G_μ satisfying the Bianchi identity $\partial_\mu G^\mu \equiv 0$. Writing A_μ as $A_\mu^T + \partial_\mu \phi$ with $\partial_\mu A^{\mu T} \equiv 0$ we see that these two equations imply $\square \phi = 0$, $A_\mu^T = 0$. In the higher derivative case, however, the field equations imply

$$(\square + m^2)(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0, \quad A_\mu = G_\mu \Rightarrow \square \phi = 0 = (\square + m^2)A_\mu^T.$$

Analogous results will hold for our new low spin actions although the addition of a mass term will not in general produce a ghost-free theory or describe just one spin.

Another possible application that we have mentioned earlier is to the topological inequivalence of different field representations, which can manifest itself either through differences in the trace or axial anomalies¹⁾, or through the appearance of a cosmological term^{2),1)}. In order to discuss such possibilities we would first have to couple these different field representations to gravity. That this is in general not possible is something of a surprise, but one that was anticipated by the similar inconsistencies⁵⁾ of gravitational coupling to the original antisymmetric tensor spinor model. Now we find inconsistencies even for pure spin 1/2 and 0 and most likely for spin 1 as well.

Finally, as we use changes of variables involving time derivatives quite freely in the text, it is important to check that these manipulations are allowed. This we do in Appendices A, B and C for our new spin 0, 1 and 1/2 actions. [That changes of variables involving derivatives are not generally legitimate is shown, e.g., by inserting the complete decomposition $A^\mu = \partial^\mu \phi + \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho,\sigma}$ into the trivial action $\int A_\mu^2$. The resulting action $\int [(\partial_\mu \phi)^2 + (\partial_\mu H_{\nu\rho}^*)^2]$ clearly describes a pair of massless scalars.]

2. THE $B^{\mu\nu\rho\sigma}$ FIELD: SPIN 1 AND SPIN 0

Consider an action which constrains the linearized Riemann tensor $R_{\mu\nu\rho\sigma}(h) = (h_{\sigma\mu,\rho\nu} + \text{symmetrizations})$ to vanish, and whose "mass term" is of the Pauli-Fierz form *)

$$\mathcal{I} = \int \left\{ \frac{1}{4} B^{\mu\nu\rho\sigma} (h_{\sigma\mu,\rho\nu} + h_{\rho\nu,\sigma\mu} - h_{\sigma\nu,\rho\mu} - h_{\rho\mu,\sigma\nu}) - \frac{1}{2} (h_{\mu\nu} h^{\mu\nu} - (h^\alpha_\alpha)^2) \right\} . \quad (2.1)$$

The constraint imposed by $B_{\mu\nu\rho\sigma}$ has the pure gauge solution $h_{\mu\nu} = \xi_{\mu,\nu} + \xi_{\nu,\mu}$ in terms of which (2.1) reduces to the Maxwell action for ξ_μ . Alternatively, elimination of $h_{\mu\nu}$ in (2.1) by means of its field equation leads to the following action for $B_{\mu\nu\rho\sigma}$

$$\mathcal{I} = \frac{1}{2} \int \left[\left(B_{\mu}{}^{\rho\sigma}{}_{,\rho\sigma} \right)^2 - \frac{1}{3} \left(B_{\mu}{}^{\rho\sigma}{}_{,\rho\sigma} \right)^2 \right] . \quad (2.2)$$

It is remarkable that such a peculiar higher order (also in time derivatives) action should describe pure spin 1, but indeed it does as we demonstrate in Appendix A.

Consider next the weaker constraint that the (linearized) Weyl tensor, $W_{\mu\nu\rho\sigma}(h)$, vanish which implies that $h_{\mu\nu} = \xi_{\mu,\nu} + \xi_{\nu,\mu} + \eta_{\mu\nu}\phi$. In principle, one could now attempt to find, by appropriate choice of mass term, an action for either spin 1 or spin 0. To obtain a spin 1 theory the mass term would have to depend only on the traceless part of $h_{\mu\nu}$ in which case, however, the theory also contains a spin 0 admixture. A spin 0 action results from the choice

$$\mathcal{I} \sim \int [C^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma}(h) + \mathcal{R}(h)] \quad (2.3)$$

where $\mathcal{R}(h)$ is the usual linearized Einstein action. Since $\int \mathcal{R}(h)$ is gauge invariant, it reduces to $\int \phi \square \phi$ when the Weyl constraint is used.

The Lagrange multiplier $C_{\mu\nu\rho\sigma}$ has the algebraic symmetries of the Weyl tensor, and so has ten components. Therefore, from $C_{\mu\nu\rho\sigma}$ and $h_{\mu\nu}$ one can construct a twenty component $B_{\mu\nu\rho\sigma}$ with the symmetries of the Riemann tensor,

*) Any other choice of "mass term" leads to lower spin ghosts, or, for the choice $(h^\alpha_\alpha)^2$, no excitations at all.

$$B_{\mu\nu\rho\sigma} \equiv G_{\mu\nu\rho\sigma} + \frac{1}{2}(\gamma_{\nu\rho}h_{\mu\sigma} - \gamma_{\mu\rho}h_{\nu\sigma} + \gamma_{\mu\sigma}h_{\nu\rho} - \gamma_{\nu\sigma}h_{\mu\rho}) - \frac{1}{2}(\gamma_{\nu\rho}\gamma_{\mu\sigma} - \gamma_{\mu\rho}\gamma_{\nu\sigma})h_{\lambda}^{\lambda}. \quad (2.4)$$

In terms of $B_{\mu\nu\rho\sigma}$, (2.3) takes the form

$$I = \frac{1}{2} \int \left[\frac{1}{3} B_{\lambda}^{\nu\eta\lambda} B_{\rho\sigma}^{\sigma\rho},_{\eta\nu} - B^{\mu\nu\rho\sigma} B_{\lambda\sigma\mu}^{\lambda},_{\rho\nu} \right] \quad (2.5)$$

which will prove useful for later comparison with the spin 1 action (2.2). In Appendix B a canonical analysis of (2.5) is given demonstrating its pure spin 0 content.

An alternative derivation of (2.5) that is analogous to the derivation of the antisymmetric tensor/spin 0 action is to use the form given in (1.4) for the Weyl tensor but keeping the auxiliary field $f_{\mu\nu}$ independent. That the action,

$$I = \int \left\{ \frac{1}{8} B^{\mu\nu\rho\sigma} (h_{\sigma\mu,\rho\nu} + h_{\rho\nu,\sigma\mu} - h_{\sigma\nu,\rho\mu} - h_{\sigma\mu,\rho\nu} + 2\gamma_{\sigma\mu}f_{\rho\nu} + 2\gamma_{\rho\nu}f_{\sigma\mu} - 2\gamma_{\sigma\nu}f_{\rho\mu} - 2\gamma_{\rho\mu}f_{\sigma\nu}) + (f_{\mu\nu}h^{\mu\nu} - f_{\mu}^{\mu}h_{\nu}^{\nu}) \right\} \quad (2.6)$$

is equivalent to (2.3) follows upon eliminating $f_{\mu\nu}$ by using the trace of the $B_{\mu\nu\rho\sigma}$ constraint. On the other hand, notice that the "mass term" has no derivatives here and so $f_{\mu\nu}$ and $h_{\mu\nu}$ can be directly eliminated by using their field equations. In particular $h_{\mu\nu}$ is a Lagrange multiplier imposing the condition

$$f_{\mu\nu} = -\frac{1}{2} (B_{\mu}^{\rho\sigma\nu} - \frac{1}{3} \gamma_{\mu\nu} B_{\lambda}^{\rho\sigma\lambda}),_{\rho\sigma} \quad (2.7)$$

which yields (2.5) when inserted into the action (2.6). This quantity is actually the gauge invariant field strength of both the spin 0 and 1 models and, therefore, will play a central role. To make this clear we discuss the gauge invariances of (2.5) and (2.2). Consider first the spin 0 action; it has nine invariances. Five occur because of the well-known identity on the Weyl tensor

$$\epsilon^{\mu\nu\rho\sigma} \partial_\rho \partial_\lambda W_{\mu\nu}{}^{\lambda\gamma} \equiv 0. \quad (2.8)$$

Just as the identities $\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma}(A) = 0$ imply three gauge invariances for $B_{\mu\nu}$ in (1.2), so does (2.8) imply gauge invariance of (2.5) under the following transformation of the traceless part of $B_{\mu\nu\rho\sigma}$:

$$\delta B_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu}{}^{\lambda\gamma} \partial_\lambda (\partial_\rho \omega_{\gamma\sigma} - \partial_\sigma \omega_{\gamma\rho}) + ((\mu, \nu) \leftrightarrow (\rho, \sigma)) \quad (2.9a)$$

$$\omega_{\mu\nu} = \omega_{\nu\mu}, \quad \omega_{\mu}{}^\mu = 0, \quad \omega_{\mu\nu}{}^{,\nu} = 0. \quad (2.9b)$$

However, unlike in the antisymmetric tensor case, there are four more gauge invariances of (2.5) that arise because the "mass term" in (2.3) is invariant under $\delta h_{\mu\nu} = \xi_{\mu,\nu} + \xi_{\nu,\mu}$. In terms of the B-notation this invariance takes the form

$$\delta \bar{B}_{\mu\nu} \equiv \delta (B_{\mu\lambda}{}^\lambda{}_\nu - \frac{1}{3} \eta_{\mu\nu} B_\rho{}^\rho{}^\lambda{}_\lambda) = \xi_{\mu,\nu} + \xi_{\nu,\mu}. \quad (2.10)$$

The tensor $f_{\mu\nu}$ of (2.7) is invariant under both (2.9) and (2.10), and the spin 0 and spin 1 actions can be constructed from it directly. It is more convenient to use instead the combination

$$\bar{f}_{\mu\nu} \equiv f_{\mu\nu} - \eta_{\mu\nu} f = -\frac{1}{2} B_\mu{}^\rho{}^\sigma{}_\nu{}_{,\rho\sigma} \quad (2.11)$$

which satisfies the simple Bianchi identity

$$\partial_\mu \bar{f}^{\mu\nu} \equiv 0. \quad (2.12)$$

There is only one gauge invariant second order action that we can construct from $\bar{f}_{\mu\nu}$:

$$I_0 \sim \int \bar{B}_{\mu\nu} \bar{f}^{\mu\nu}(B) \quad (2.13)$$

As in the Einstein action, gauge invariance is ensured by the invariance of the field strength, and its Bianchi identity (2.12). It is, of course, the spin 0 action of (2.5).

It is obvious that a gauge invariant fourth order action can be built from bilinears in $\bar{f}_{\mu\nu}$. In particular the analogue of the Weyl action,

$$I_1 \sim \int \left[\bar{f}_{\mu\nu} \bar{f}^{\mu\nu} - \frac{1}{3} (f_r^r)^2 \right] \quad (2.14)$$

is the unique combination which possesses the additional Weyl type invariance

$$\delta B_{\mu\rho\sigma\nu} = (\gamma_{\mu\nu}\gamma_{\rho\sigma} - \gamma_{\rho\nu}\gamma_{\mu\sigma}) \lambda(x) \quad (2.15)$$

and is therefore independent of $B_{\mu\nu\nu\mu}$. Remarkably, the action (2.14) is none other than the spin 1 action (2.2), which therefore possesses the ten gauge invariances (2.9), (2.10) and (2.15).

Notice that the tensor structure of potential and field strength in these models is reversed with respect to that of spin 2. Following the example of Ref. 3) we could call the $B_{\mu\rho\sigma\nu}$ field the "notivarg". The table summarizes this complementarity.

3. SPIN 1/2

To construct a higher rank tensor theory for spin 1/2, we proceed in analogy with the previous section by constraining the fermionic analogue of the Weyl tensor to vanish. There are two fermionic field strengths^{*)} in the super Weyl multiplet (1.2), so following the example of the previous section we take the action

^{*)} We thank B. Zumino for pointing out the relevance of the second fermionic constraint imposed by $\xi_{\mu\nu}$ in (3.1).

$$\begin{aligned} \mathcal{I} = \int \left\{ \frac{1}{2} \bar{\chi}^{\mu\nu} (\partial_\mu \psi_\nu - \partial_\nu \psi_\mu + \gamma_\mu \phi_\nu - \gamma_\nu \phi_\mu) + \frac{1}{2} \tilde{\xi}^{\mu\nu} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \right. \\ \left. + \bar{\psi}_\mu \sigma^{\mu\nu} \phi_\nu \right\} \end{aligned} \quad (3.1)$$

where, because of the projection operator T_{abcd} in (1.2), the Lagrange multiplier, $\tilde{\xi}_{\mu\nu}$ satisfies $\gamma_\mu \tilde{\xi}_{\mu\nu} = 0$. If the constraint imposed by $\gamma_\mu \chi^{\mu\nu}$ is solved for ϕ_μ the constraints become equivalent to the vanishing of the fermionic components of the usual super Weyl tensor $W_{\alpha\beta\gamma}$, and the "mass term" $\psi_\mu \sigma^{\mu\nu} \phi_\nu$ becomes the Rarita-Schwinger action for ψ_μ . If we can then show that the solution to the constraints on ψ_μ and $\phi_\mu(\psi)$ is of the form

$$\begin{aligned} \psi_\mu &= \partial_\mu \alpha + \gamma_\mu \chi \\ \phi_\mu &= \partial_\mu \chi \end{aligned} \quad (3.2)$$

then because of the gauge invariance of the "mass term", α cancels and we are left with a Dirac action for the (Majorana) spinor χ . This was the strategy of Ref. 6) but there the $\tilde{\xi}_{\mu\nu}$ constraint was missing, which allows a more general solution than (3.2), as we now show by analyzing (3.1) with the $\tilde{\xi}_{\mu\nu}$ constraint absent.

Firstly, we eliminate ϕ_μ by imposing the γ trace of the $\chi_{\mu\nu}$ constraint. This leaves the following constraint on ψ_μ :

$$\psi_{\mu\nu} - \frac{1}{2} \gamma_\mu (\gamma^\lambda \psi_{\lambda\nu}) + \frac{1}{2} \gamma_\nu (\gamma^\lambda \psi_{\lambda\mu}) - \frac{1}{3} \sigma_{\mu\nu} (\sigma^{\rho\sigma} \psi_{\rho\sigma}) = 0 ; \quad (3.3)$$

$$\psi_{\mu\nu} \equiv \partial_\mu \psi_\nu - \partial_\nu \psi_\mu .$$

The field ψ_μ can be further decomposed as

$$\left\{ \psi_0 , \psi_i \equiv \psi_i^{Tt} + \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \gamma_j \chi + \partial_i \alpha \right\} \quad (3.4)$$

where ψ_i^{Tt} is doubly transverse ($\gamma_i \psi_i^{Tt} = 0, \partial_i \psi_i^{Tt} = 0$). These projections of ψ_i are all uniquely invertible in terms of ψ_i itself (assuming appropriate fall-off at spatial infinity). Substitution of the above decomposition into the

remaining constraint (3.3) yields two independent conditions. One relates ψ_0 to χ and α , the other imposes the following condition on ψ_i^{Tt}

$$\not\partial(\gamma^0 \psi_i^{Tt}) = 0 \quad . \quad (3.5)$$

The net effect is that the $\chi_{\mu\nu}$ constraint alone requires ψ_μ to be of the form $(\partial_0 \alpha + \gamma_0 \eta, \partial_i \alpha + \gamma_i \chi + \psi_i^{Tt})$ (subject to (3.5)). Insertion of this result into the $\bar{\psi}_\mu \sigma^{\mu\nu} \phi_\nu$ term yields the action

$$\mathcal{I} = \int \left\{ \bar{\chi} \not\partial \chi + \bar{\psi}_i^{Tt} \not\partial \psi_i^{Tt} + \bar{\lambda}_i^{Tt} (\not\partial \gamma^0 \psi_i^{Tt}) \right\} \quad (3.6)$$

where the last term imposes the condition (3.5) with a Lagrange multiplier. The χ part is the expected spin 1/2 action. To see what extra modes ψ_i^{Tt} and λ_i^{Tt} represent, we use their field equations to deduce that $\gamma_0 \psi = -\frac{1}{2} \nabla^{-2} \not\partial \lambda$ while also $\not\partial \gamma_0 \psi = 0$. Thus, λ is the independent variable satisfying $\not\partial \not\partial \lambda_i^{Tt} = 0$, and from this it follows that $\tilde{\not\partial} \lambda = 0$ with $\tilde{\not\partial} = \gamma_0 \not\partial \gamma_0$. An equivalent action for ψ_i^{Tt} and λ_i^{Tt} is therefore

$$\mathcal{I}_{\text{equiv.}} = \int \frac{1}{2} \bar{\lambda}_i^{Tt} \tilde{\not\partial} \not\partial \lambda_i^{Tt} \quad . \quad (3.7)$$

The propagator is $(p^2)^{-2} \tilde{\not\partial} \not\partial$. With the choice $p_\mu = (|p|, 0, 0, \pm |p|)$ the residue of the double pole is $2|p|^2(1 \pm \gamma_0 \gamma_3)$. As for the residue $\not\partial$ of the Dirac propagator $(p^2)^{-1} \not\partial$, $(1 \pm \gamma_0 \gamma_3)$ has two zero eigenvalues so that only two of the four components of λ are propagated. But unlike the Dirac propagator, λ has a double pole. From this we conclude that the additional modes propagated by (3.6) are those of a spin 3/2 dipole ghost. A form equivalent to (3.6) was obtained by C. Aragone¹¹⁾, by canonical decomposition of the action of Ref. 6); he thereby first noticed the spin 3/2 content.

To show that the additional constraint on ϕ_μ removes these extra modes we first rewrite it as

$$\partial_\mu \phi_\nu - \partial_\nu \phi_\mu + \delta_\mu \rho_\nu - \delta_\nu \rho_\mu = 0 \quad . \quad (3.8)$$

The auxiliary variable ρ_μ is eliminated by the γ -trace of (3.8). What is left is just the $\tilde{\xi}_{\mu\nu}$ constraint of (3.1), so it is equivalent to consider (3.8). The advantage is that we have already solved this constraint, but for (ψ_μ, ϕ_μ)

in place of (ϕ_μ, ρ_μ) . Therefore, we must have $\phi_\mu = \partial_\mu \chi' + \gamma_\mu \beta + \phi_\mu^{Tt}$, with $\phi_\mu^{Tt} = (0, \phi_1^{Tt})$. On the other hand, ϕ_μ is already determined by the $\chi_{\mu\nu}$ constraint to be $\phi_\mu = \partial_\mu \chi - \frac{1}{2} \partial \psi_\mu^{Tt}$, with $\psi_\mu^{Tt} = (0, \psi_1^{Tt})$. Compatibility requires $\chi' = \chi$, $\beta = 0$ and $\phi_\mu^{Tt} = -\frac{1}{2} \partial \psi_\mu^{Tt}$. Since ϕ_μ^{Tt} must also satisfy $\tilde{\partial} \phi_\mu^{Tt}$ in order to be a solution of (3.8), as must ψ_μ^{Tt} , we discover that $\tilde{\partial} \psi_\mu^{Tt} = 0$. But given that $\tilde{\partial} \psi_\mu^{Tt} = 0$, this implies that $\nabla^2 \psi_\mu^{Tt} = 0$, and therefore that $\psi_\mu^{Tt} = 0$. We conclude that (3.2) is the most general solution of the two constraints of (3.1) and consequently that (3.1) describes pure spin 1/2. A derivation not involving time derivative constraints is given in Appendix C.

To obtain the Lagrange multiplier form of (3.1) we follow the procedure of Section 2. The field $\phi_\mu(\psi_\mu)$ appears only linearly and acts as a constraint on $\psi_\mu(\phi_\mu)$ that determines it as a function of $(\chi_{\mu\nu}, \tilde{\xi}_{\mu\nu})$. Performing this elimination of ϕ_μ and ψ_μ we obtain

$$I = \left(\left(2 \partial_\mu \bar{\chi}^{\mu\nu} \left(\gamma^\lambda \chi_{\lambda\nu} + \frac{1}{3} \gamma_\nu \sigma^{\rho\sigma} \chi_{\rho\sigma} \right) - 2 \tilde{\xi}^{\rho\sigma} \left(\partial_\rho \partial^\sigma \chi_{\mu\sigma} \right) \right) \right) \quad (3.9)$$

which differs (up to conventions) from that of Ref. 6) in that it involves the additional $\tilde{\xi}_{\mu\nu}$ field.

4. SUPERSYMMETRY AND SELF-COUPLING

The gauge fields that appear in the Weyl supermultiplet, (1.3), are those of conformal supergravity, $e_{a\mu}, \psi_\mu, A_\mu$. Constraining this multiplet to vanish, sets these fields to be pure gauge functions of conformal supergravity. Let us set to zero the parameters of the subgroup of super Poincaré gauge transformations. What is left is

$$\begin{aligned} e_{a\mu} &= \gamma_{a\mu} + \kappa \phi \gamma_{a\mu} \equiv \gamma_{a\mu} \kappa^2 A^2 \\ A_\mu &= \partial_\mu B \\ \psi_\mu &= \gamma_\mu \chi \end{aligned} \quad (4.1)$$

Therefore, by constraining the super Weyl tensor $W_{\alpha\beta\gamma}$, symmetric in its (spinor) indices, to vanish via a Lagrange multiplier $\chi^{\alpha\beta\gamma}$ and adding a "mass term" that is super Poincaré invariant, we can obtain a theory that is equivalent to the massless Wess-Zumino model for A, B, χ (again up to topological questions that we ignore). The Lagrange multiplier form of this model is one that unites our spin 0 and spin 1/2 actions with the antisymmetric tensor/spin 0 action.

The model is most easily constructed in superspace where the action is the usual one of non-linear supergravity, subject to the Weyl constraint:

$$\mathcal{I} = \int d^4x d^4\theta \det(E_M^A) + \left(\int d^4x d^2\theta \chi^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \text{h.c.} \right). \quad (4.2)$$

Using the $\chi^{\alpha\beta\gamma}$ field equation, $W_{\alpha\beta\gamma} = 0$, the only remaining components of the super vielbein E_M^A are those of a chiral superfield ϕ , [as in (4.1) except that now we include auxiliary fields]. The action becomes

$$\mathcal{I} = \int d^4x d^4\theta \bar{\phi} \phi \quad (4.3)$$

which is the free massless Wess-Zumino action for a chiral superfield, despite the non-linearity of (4.2).

If we expand the Weyl superfield as

$$W_{(\alpha\beta\gamma)}(\theta) = W_{(\alpha\beta\gamma)} + \theta_\alpha F_{\beta\gamma} + \theta^\delta W_{(\alpha\beta\gamma\delta)} + \theta^2 W'_{(\alpha\beta\gamma)} \quad (4.4)$$

then the component version of (4.2) is found to be

$$\begin{aligned} \mathcal{I} = & \left\{ \left(\mathcal{L}_2(e_\mu) + \mathcal{L}_{\frac{3}{2}}(\psi_\mu) + e(A_\mu^2 - S^2 - P^2) \right) \right. \\ & + \left(\chi^{(\alpha\beta\gamma)} W_{\alpha\beta\gamma} + B^{(\alpha\beta)} F_{\alpha\beta} + B^{(\alpha\beta\gamma\delta)} W_{\alpha\beta\gamma\delta} \right. \\ & \left. \left. + \xi^{(\alpha\beta\gamma)} W'_{\alpha\beta\gamma} \right) \right\} \quad (4.5) \end{aligned}$$

where A_μ , S and P are the minimal auxiliary fields of supergravity. We can separate the various terms in (4.5) as follows:

$$\mathcal{L}_2(e_{\alpha\mu}) + B^{(\alpha\beta\gamma\delta)} W_{(\alpha\beta\gamma\delta)} \rightarrow \text{spin } 0 \text{ of Section 2}$$

$$\mathcal{L}_{\frac{3}{2}}(\psi) + \chi^{(\alpha\beta\gamma)} (W_{\alpha\beta\gamma}) + \xi^{(\alpha\beta\gamma)} W_{(\alpha\beta\gamma)} \rightarrow \text{spin } 1/2 \text{ of Section 3}$$

$$A_\mu^2 + B^{(\alpha\beta)} F_{(\alpha\beta)} \rightarrow \text{usual spin } 0/\text{antisymmetric tensor action of Section 1}$$

$$S^2 + P^2 \rightarrow F, G, \quad \text{auxiliary fields of chiral multiplet}$$

So, we can use the results of Sections 2 and 3 to write the entire supersymmetric theory in its Lagrange multiplier form, although we do not know how to do this directly in terms of superfields.

The identification of the F, G auxiliary fields with the S, P supergravity auxiliary fields is reminiscent of the superconformally extended version of supergravity via a compensating chiral supermultiplet¹²⁾. In fact if we replace the usual supergravity action in (4.2) by the superconformally extended action, we have

$$I = \int d^4x d^4\theta \det(E_M^A) \bar{\phi} \phi + \left(\int d^4x d^2\theta \chi^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \text{h.c.} \right) \quad (4.6)$$

where ϕ is the compensating chiral superfield. Then using $W_{\alpha\beta\gamma} = 0$ we can choose a gauge for which $\det(E_M^A) = 1$ and (4.6) becomes the standard free action for ϕ . This is the converse of the procedure of Ref. 12) because here we are able to gauge away everything except ϕ , while there it was ϕ that could be gauged away.

With the action in the form (4.6) it is easy to add interactions. We add a term of the form

$$I_{\text{int.}} = \frac{1}{3!} \lambda \int d^4x d^2\theta \phi^3 + \text{h.c.} \quad (4.7)$$

in terms of the compensating field ϕ . In the case of supergravity this becomes the (super) cosmological term after fixing the superconformal gauge, whereas in this case it becomes the Wess-Zumino self-interaction term upon solving the constraint $W_{\alpha\beta\gamma} = 0$. This way of obtaining self-interactions might be considered something of a cheat as we have had to introduce into the action from the start the chiral superfield that we are ultimately left with on solving $W_{\alpha\beta\gamma} = 0$.

However, the action in the form (4.2) without the compensating field, although apparently non-interacting as it can be reduced to (4.3), actually contains an integration constant in the equations of motion that, in the case of supergravity, can be interpreted as a cosmological constant^{2),1)}. In the present case, it can be interpreted as a self-interaction term for ϕ . In Feynman diagram language we could obtain these interactions by the addition of a total derivative term to (4.2). Thus all the interactions can be interpreted as arising from a total derivative term !

To make the above remarks less obscure, consider the self-interacting version of the spin 0 model of Section 2 for which the full non-linear Weyl tensor is constrained to vanish and the mass term is the full Einstein action, to which we add a cosmological constant

$$I \sim \left\{ C^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma}(g) + \frac{1}{2\kappa^2} \sqrt{g} R(g) + \sqrt{g} \Lambda \right\} . \quad (4.8)$$

Up to co-ordinate transformations, under which the "mass term" is invariant, the constraint reduces $g_{\mu\nu}$ to $\eta_{\mu\nu} \kappa^2 A^2$ and hence the action to $\frac{1}{2} \int [(\partial_\mu A)^2 - \lambda A^4]$, which includes self-interaction^{*)}. The reduction of the supersymmetric action (4.2) to its component form yields, however, an action without cosmological term, but with an additional auxiliary field term:

$$I \sim \left\{ C^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma}(g) + \frac{1}{2\kappa^2} \sqrt{g} R(g) + \sqrt{g} (\partial_\mu S^\mu)^2 \right\} . \quad (4.9)$$

Here S^μ is an auxiliary vector field appearing only through the combination $\partial_\mu S^\mu$. The S^μ field equation contains an integration constant that can be interpreted as a cosmological constant and so (4.9) is equivalent to (4.8), but with arbitrary Λ . Alternatively, we may add to (4.9) the total derivative term $\partial_\mu (\sqrt{-g} S^\mu)$ which contributes to Feynman diagrams in the same way as a cosmological constant.

It should be emphasized that although we have made use of constrained gravity and supergravity fields, no actual interactions to gravity have yet been included in any of our models.

^{*)} Note that the non-linearity of (4.8) is essential for self-interaction since there is no linearized analogue to the (co-ordinate invariant) cosmological constant.

5. COUPLING TO GRAVITY

The question of consistency of gravitational coupling is most easily answered by considering how such a coupling affects gauge invariances. For example, (3.1) has the gauge invariance⁶⁾

$$\delta\psi_\mu = \partial_\mu \alpha(x), \quad \delta\chi^{\mu\nu} = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \partial_\rho \delta\varsigma \delta_\sigma \alpha(x). \quad (5.1)$$

If all derivatives in the action (3.1) and the transformations (5.1) become gravitationally covariant derivatives, it is immediate to see that the variation of the action is (on-shell),

$$\delta I = \int \frac{1}{2} \bar{\chi}_{\mu\nu} \sigma^{ab} \alpha(x) R^{\mu\nu}{}_{ab}(g) \quad (5.2)$$

where $R_{\mu\nu ab}(g)$ is the (unconstrained) Riemann tensor for the metric $g_{\mu\nu}$. This term cannot be cancelled by additional variations of the fields proportional to curvatures nor by additions to the action proportional to curvatures, on dimensional grounds, and so the coupling is inconsistent. This is true independently of the presence or absence of the $\tilde{\xi}_{\mu\nu}$ constraint. Therefore, we reobtain the inconsistency⁵⁾ of gravitational coupling to the original $\chi_{\mu\nu}$ action (which is, however, not pure spin 1/2), and we sharpen this result to apply also to the $(\chi_{\mu\nu}, \tilde{\xi}_{\mu\nu})$ action, which is pure spin 1/2.

For the spin 0 action it is equally easy to see that gravitational coupling is inconsistent. Consider the variation of the action in the form (3.13) under covariantization of the transformation $\delta\tilde{B}_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. The variation $\xi_\nu D_\mu \tilde{F}_{\mu\nu}$ contains the term

$$\xi_\nu D_\mu D_\rho D_\sigma C^{\mu\rho\sigma\nu} = \xi_\nu R_{\mu\rho}{}^\nu{}_\lambda D_\sigma C^{\mu\rho\sigma\lambda} \quad (5.3)$$

where $C_{\mu\rho\sigma\nu}$ is the traceless part of $B_{\mu\rho\sigma\nu}$, as in (2.4); it is the only variation containing $C_{\mu\rho\sigma\nu}$. Furthermore, non-minimal modifications do not help; by dimensions $\delta B_{\mu\nu\rho\sigma}$ cannot have non-minimal $R\xi$ terms, while additions to the action, which must be of the form $\sqrt{-g} R^2$, never produce bare ξ_μ 's as in (5.3).

For the spin 1 action (2.14) the Bianchi identity $\partial_\mu \bar{f}^{\mu\nu} = 0$ is not needed to establish invariance and so the analysis is more complicated. The variation of $\bar{f}_{\mu\nu}$ under the covariantization of (2.10) is

$$\delta \bar{f}_{\mu\nu} = \frac{1}{4} (R_{\sigma\lambda\mu\nu} + R_{\mu\lambda\nu\sigma}) D^\lambda \xi^\sigma. \quad (5.4)$$

Here we keep only terms involving the full uncontracted Riemann tensor, dropping all Ricci tensors. Then the variations of the $\bar{f}_{\mu\nu}^2$ action is

$$\delta I \propto \int \bar{f}^{\mu\nu} R_{\sigma\lambda\mu\nu} D^\lambda \xi^\sigma. \quad (5.5)$$

However, this could be cancelled by the addition of a non-minimal term

$\sim \bar{f}_{\mu\sigma} R_{\sigma\lambda\mu\rho} \bar{B}_{\lambda\rho}$ and the contribution from variation of $\bar{f}_{\mu\sigma}$ in this new term by the further addition $\sim (R_{\sigma\lambda\mu\rho} \bar{B}_{\lambda\rho})^2$. To proceed, we consider the other "invariance", (2.9), under which $\bar{f}_{\mu\nu}$ transforms as

$$\begin{aligned} \delta \bar{f}_{\mu\nu} = & \frac{1}{2} \Sigma_\mu^{\lambda\eta} D_\rho [D_\nu, D_\lambda] (D^\rho \omega_{\eta\sigma} - D_\sigma \omega_{\eta}{}^\rho) \\ & + \frac{1}{2} \Sigma_\sigma^{\rho\lambda\eta} [D_\rho, D_\lambda] D^\nu (D_\mu \omega_{\eta\nu} - D_\nu \omega_{\eta\mu}) + \Sigma_\sigma^{\rho\lambda\eta} D_\rho [D_\nu, D_\lambda] (D_\mu \omega_{\eta\nu} - D_\nu \omega_{\eta\mu}). \end{aligned} \quad (5.6)$$

This variation of $\bar{f}_{\mu\nu}$ contributes to δI from both the $\bar{f}_{\mu\nu}^2$ and the non-minimal $\bar{f} \bar{B}$ terms. The contribution from the latter specifically includes terms of the form $\bar{B} \bar{D} R D \omega$. To decide whether such a term can be cancelled one must consider all dimensionally allowed terms in the action and field variations. Schematically, these are

$$I(\text{non-minimal}) \sim (R + D^2) \bar{f} (\bar{B} + \bar{c}), \quad (R^2 + R D^2) (\bar{B}^2 + c^2 + \bar{B} c)$$

$$\delta \bar{B}(\text{non-minimal}) \sim R \omega$$

$$\delta c(\text{non-minimal}) \sim R \omega$$

These possibilities complicate the analysis and we have not established conclusively the inconsistency of gravitational coupling, although cancellation of the particular term $\bar{B} \bar{D} R D \omega$ seems highly unlikely.

6. COMMENTS

We have presented some fairly systematic techniques for generating actions for high rank fields which describe, as a consequence of gauge invariance, only lower spins. Such unorthodox field representations have potential applications in supersymmetry. Indeed, the fourth order spin 1 action (2.2) appears¹³⁾ in one formulation of linearized $N = 8$ supergravity¹⁴⁾. On the other hand, we have seen that the higher the rank of the representation, the more restrictions there are, in general, on possible interactions. In particular, gravitational coupling is already inconsistent beyond the spin 0/antisymmetric tensor model. Although different field representations may have different quantum effects, the absence of consistent gravitational coupling should eliminate most of them for any given spin. (Recall that if a system cannot couple to gravity it cannot couple to any matter that couples to gravity.) Ultimately, this could prove to be a useful restriction.

As gauge field theories, our higher rank models exhibit some unexpected possibilities. For example, the fourth order spin 1 action provides a counter-example to the folk-theorem that higher derivative actions necessarily contain ghosts. Also, the covariant quantization of these models seems problematical, even as free fields; it is difficult to see what will be the correct ghost action. Judging by the subtleties involved in covariant quantization of the antisymmetric tensor model¹⁵⁾, the Faddeev-Popov procedure is unlikely to be directly applicable. Finally, the supersymmetric model exhibited an unusual way for interactions to manifest themselves, namely by way of an integration constant in the field equations.

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APPENDIX A - CANONICAL ANALYSIS OF SPIN 1 ACTION

Because the fourth order action for spin 1 of (2.2) was obtained by solving a constraint that contains time derivatives, it is not guaranteed that it describes pure spin 1. We have checked that it does describe pure spin 1 in two ways: (i) by starting from the "first order" form (2.1) and solving only spatial constraints, and (ii) by direct canonical analysis of (2.2). We present the latter analysis here, which is longer but more instructive.

We define for future use the following variables

$$\begin{aligned} S_{ij} &\equiv S_{ji} \equiv B_{i00j}, \quad V_{ij} \equiv V_{ji} \equiv B_{iklj,kl}, \quad V \equiv V_{ii} \\ W_{ij} &\equiv W_{ji} \equiv (B_{oikj} + B_{ojki}),_{,k}, \quad \partial^2 S \equiv S_{ij,ij}. \end{aligned} \quad (A.1)$$

From these definitions it follows that

$$V_i \equiv V_{ij,j} \equiv 0, \quad W_{ij,ij} \equiv 0. \quad (A.2)$$

The action of (2.2) then reads,

$$\begin{aligned} \mathcal{I} = & \int \left\{ \left(V_{ij} + \dot{W}_{ij} + \ddot{S}_{ij} \right)^2 - \frac{1}{3} (V + \dot{W} + \ddot{S} - 2\partial^2 S) (V + \dot{W} + \ddot{S}) \right. \\ & \left. + \frac{2}{3} (\partial^2 S)^2 - 2(W_{ij,jj} + \dot{S}_{ij,jj})^2 \right\}. \end{aligned} \quad (A.3)$$

V_{ij} appears without time derivatives and yields constraints on the variation of (A.3). Since V, W and S are all independent components of $B^{\mu\nu\rho\sigma}$, (the cyclic identity is a constraint among the B_{oijk}), we may vary them independently of each other, but must regard each variation as the appropriate function of the $\delta B_{\mu\nu\rho\sigma}$. From $\delta V_{ij} \equiv \partial_k \partial_\ell \delta B^{iklj}$ we find

$$\begin{aligned} 0 &= \left[V_{ij} + \dot{W}_{ij} + \ddot{S}_{ij} - \frac{1}{3} \delta_{ij} (V + \dot{W} + \ddot{S} - 2\partial^2 S) \right]_{,kl} \delta B^{iklj} \\ &\equiv H_{ij,kl} \delta B^{iklj}, \end{aligned} \quad (A.4)$$

By virtue of the symmetries of δB , (A.4) is equivalent to the statement that the three dimensional curvature, built from H_{ij} as its metric, vanishes. The solution is that the "metric tensor" H_{ij} is pure gauge. To analyze this we

use the standard decomposition of a symmetric tensor

$$X_{ij} \equiv X_{ij}^{TT} + \frac{1}{2}(\delta_{ij} - \partial_i \partial_j) X^T + (X_{ij}^T + X_{ji}^T) + 2X_{,ij}^L \quad (A.5)$$

where X_{ij}^{TT} is transverse traceless, X_i^T is transverse, and all factors of ∇^2 are dropped for simplicity. As H_{ij} is pure gauge, i.e., $H_{ij} = \xi_{i,j} + \xi_{j,i}$ it follows that $H_{ij}^{TT} = H^T = 0$. That is, using (A.2),

$$V_{ij}^{TT} + \dot{W}_{ij}^{TT} + \ddot{S}_{ij}^{TT} = 0 \quad (A.6)$$

$$(V^T + \dot{W}^T + \ddot{S}^{TT}) = 4(\ddot{S}^L - S^L). \quad (A.7)$$

Upon inserting (A.6) and (A.9) into the action (A.3) the latter simply reduces to

$$\mathcal{I} = -\frac{1}{4} \int A_i^T \square A_i^T \quad ; \quad A_i^T \equiv 2\sqrt{-2\nabla^2} (W_i^T + \dot{S}_i^T). \quad (A.8)$$

These substitutions are allowed either because: (i) second order time derivative equations are constraints in fourth order actions, or (ii) the substitutions (A.6), (A.7) are equivalent to "Gaussian integration", and are therefore not equivalent to a change of variable involving time derivatives. For the same reason it is clear that the action of (A.8) is that of a free photon with (gauge invariant) transverse vector potential A_i^T .

To complete the argument we must show that varying (A.8) with respect to the fundamental variables B_{oijk} and B_{iooj} yields the same Maxwell equations, $\square A_i^T = 0$, as direct A_i^T variation. Varying with respect to B_{iooj} , i.e., $\delta S_i^T \equiv \dot{B}_{iooj,j}^T$ clearly yields $\partial_o \square A_i^T = 0$ which is equivalent. Next we note that for any transverse vector O_i^T ,

$$O_i^T \delta W_i^T = [O_{i,kj}^T - O_{k,ij}^T] \delta B_{ojki} \quad (A.9)$$

which respects the cyclic identity on B_{ojki} . Thus from δB_{ojki} we learn that

$$\partial_j \square (A_{i,k}^\top - A_{k,i}^\top) = 0 \quad (\text{A.10})$$

which is again equivalent to Maxwell's equations since the vanishing of the curl of a transverse vector is equivalent to the vanishing of the vector itself.

Although we do not investigate possible couplings of the photon to charges in this representation, it should be clear that no ultra-violet miracles will occur should the coupling be possible. The B propagators are indeed $\sim p^{-4}$ but the vector potential is effectively $\sim pB$ so the vertices would acquire a factor of a p to cancel the extra p^{-2} in the propagator. It is also worth noting that the folk theorem that quartic derivative actions have ghosts is evaded here because the propagator structure is a matrix whose entries are of the form $p^{-2} \times ((\vec{p}^2)^{-1}, p_0^{-2}, (|\vec{p} \cdot p_0|)^{-1})$.

APPENDIX B - CANONICAL ANALYSIS OF SPIN 0 ACTION

The action (2.8) contains only second derivatives but is (for this reason) rather more difficult to analyze than the spin 1 action. In the language of Appendix A the action reads

$$\begin{aligned} \mathcal{I} = & \int \left\{ \frac{1}{3} (\ddot{S} + 2\dot{W} + V - \partial^2 S) (B_{ikki} - 2S) - S \partial^2 S \right. \\ & \left. + 2(W_{ij,j} + \dot{S}_{ij,j}) B_{okki} - (\ddot{S}_{ij} + \dot{W}_{ij} + V_{ij}) (B_{ikkj} - S_{ij}) \right\}. \end{aligned} \quad (\text{B.1})$$

Unlike (A.3) this action depends on all components of the B field, as well as its divergences. It is therefore necessary to decompose all 20 parts of $B_{\mu\nu\rho\sigma}$. The $B_{oijk} = -B_{oikj}$ are a set of eight quantities since the cyclic identity $\epsilon^{ijk} B_{oijk} = 0$ provides one constraint. It can be expressed as follows

$$B_{oikj} \equiv \epsilon_{kje} \tilde{K}_{ie} + (\delta_{ij} z_k - \delta_{ik} z_j) ; \tilde{K}_{ii} \equiv 0 \quad (\text{B.2})$$

i.e., in terms of a vector z_i and a symmetric traceless tensor $\tilde{K}_{ij}$. It follows that

$$\begin{aligned} B_{okkj} &= -2z_j \\ W_{ij} &= \epsilon_{kje} \tilde{K}_{ie,k} + \epsilon_{kie} \tilde{K}_{je,k} + 2(\delta_{ij} - \partial^2_{ij}) z^L - (z^T_{i,j} + z^T_{j,i}) \\ W_{ii} &\equiv W^T = 4z^L \\ W_{ij,j} &\equiv \epsilon_{kie} \tilde{K}_{e,k}^T - z_i^T ; \partial^2 W \equiv 0 \end{aligned} \quad (\text{B.3})$$

where we have used the standard decomposition of (A.3). Throughout, we omit factors of ∇^2 , which are implicit by dimensions. Next we analyze the six components of the spatial tensor B_{ilkj} . These can be re-expressed in terms of a (six component) symmetric tensor R_{mn} as

$$\begin{aligned}
 B_{ikj} &\equiv \varepsilon_{ikm} \varepsilon_{kjm} R_{mn} \\
 B_{ikkj} &\equiv (R_{ij} - S_{ij} R) \\
 B_{ikki} &\equiv B \equiv -2R
 \end{aligned} \tag{B.4}$$

while

$$\begin{aligned}
 V_{ij} &\equiv \varepsilon_{ikm} \varepsilon_{kjm} \partial_{ik}^2 R_{mn} \quad ; \quad V_{ijjj} \equiv 0 \\
 V &\equiv V^T \equiv \partial^2 R - R \equiv -R^T .
 \end{aligned} \tag{B.5}$$

Our independent set of twenty variables now consists of the three symmetric tensors $(S_{ij}, \tilde{K}_{ij}, R_{ij})$ and the vector Z_i , which provide a generally useful parametrization of a curvature tensor. With the notation $S^T \equiv T$, $S^L \equiv L$ the four terms in (B.1) decompose as follows:

$$\begin{aligned}
 \mathcal{I} = & \int \left\{ -\frac{2}{3} (\ddot{T} + 2\ddot{L} - 2L - R^T + 4\dot{Z}^L) (R^T + 2R^L + T + 2L) \right. \\
 & - 2L(T + 2L) - 4 \left[2L\dot{Z}^L + Z_i^T (\dot{S}_i^T + \dot{W}_i^T) \right] \\
 & + \left[2(\ddot{S}_i^T + \dot{W}_i^T) (R_i^T - S_i^T) - (\ddot{S} + \dot{W} + V)_{ij}^{TT} (R_{ij}^{TT} - S_{ij}^{TT}) \right. \\
 & \left. \left. + \frac{1}{2} (\ddot{T} + 4\dot{Z}^L - R^T) (R^T + 4R^L + T) + 2\ddot{L} (R^T + 2L) \right] \right\} .
 \end{aligned} \tag{B.6}$$

It is easy to dispose of the spin 1 and spin 2 parts. The quantity $\dot{R}_i^T$ is clearly a Lagrange multiplier which states that $\dot{S}_i^T + \dot{W}_i^T = 0$, thus removing all transverse vector terms (without needing to write $\dot{W}_i^T$ explicitly in terms of $\tilde{K}, Z$). Likewise $\dot{W}^{TT}$ is a multiplier (entirely equivalent to $\dot{K}^{TT}$) which enforces $(R^{TT} - S^{TT})_{ij} = 0$. At this point (B.6) depends only on four scalars (T, L, R^T, R^L) together with the final (scalar) Lagrange multiplier $\dot{Z}^L$. The latter implies that

$$R^T = 4R^L - 20L - T \quad (B.7)$$

which we use to eliminate R^T . After some arithmetic one sees that all terms depending on R^L , T cancel, leaving only

$$I = 12 \int L \square L \quad (B.8)$$

which is the action for the massless scalar field $2\sqrt{6}L \equiv \sqrt{6} \nabla^{-2} \partial_{ij}^2 B_{iooj}$.

Finally we note that the pure spin 0 content of (2.8) may also be derived directly, but laboriously, from its original form in (2.3) by solving only time independent constraints and noting that all but the scalar field vanishes.

APPENDIX C - ANALYSIS OF SPIN 1/2 ACTION

For our spin 1/2 model we will not give here a canonical analysis of the action (3.9). Instead we will start from the first order form (3.1) and solve only the spatial constraints. The step from (3.1) to (3.9) is one of "Gaussian-integration" in the language of path integrals and is quite legitimate, so analysis of (3.1) is entirely equivalent to analysis of (3.9). We first write (3.1) in the form

$$\begin{aligned} \mathcal{I} = & \int \left\{ \bar{\chi}^{0i} (\dot{\psi}_i - \partial_i \psi_0 + \gamma_0 \phi_i - \gamma_i \phi_0) + \bar{\xi}^{0i} (\dot{\phi}_i - \partial_i \phi_0 + \gamma_0 \rho_i - \gamma_i \rho_0) \right. \\ & + \bar{\psi}_i \sigma_{ij} \phi_j + \frac{1}{2} \bar{\psi}_0 \gamma^\alpha \gamma_\alpha \phi + \frac{1}{2} \bar{\psi} \gamma^\alpha \gamma_\alpha \phi_0 \\ & \left. + \bar{\chi}^{ij} (\partial_i \psi_j - \partial_j \psi_i + \gamma_i \phi_j - \gamma_j \phi_i) + \bar{\xi}^{ij} (\partial_i \phi_j - \partial_j \phi_i + \gamma_i \rho_j - \gamma_j \rho_i) \right\}, \end{aligned} \quad (C.1)$$

in which we have introduced the variable ρ_μ that appeared in (2.8) rather than restrict $\xi_{\mu\nu}$ to be γ traceless. The fields ρ_0 , ϕ_0 and ψ_0 impose the following constraints:

$$\begin{aligned} \bar{\xi}^{0i} \gamma_i &= 0 \\ \frac{1}{2} \bar{\psi} \gamma \gamma^\alpha + \partial_i \bar{\xi}^{0i} - \bar{\chi}^{0i} \gamma_i &= 0 \\ \partial_i \bar{\chi}^{0i} + \frac{1}{2} \bar{\phi} \gamma \gamma^\alpha &= 0 \end{aligned} \quad (C.2)$$

while χ^{ij} , ξ^{ij} impose

$$\begin{aligned} \partial_i \psi_j - \partial_j \psi_i + \gamma_i \phi_j - \gamma_j \phi_i &= 0 \\ \partial_i \phi_j - \partial_j \phi_i + \gamma_i \rho_j - \gamma_j \rho_i &= 0. \end{aligned} \quad (C.3)$$

At this point the action is simply

$$\mathcal{I} = \int \left\{ \bar{\chi}^{0i} (\dot{\psi}_i + \gamma_0 \phi_i) + \bar{\xi}^{0i} (\dot{\phi}_i + \gamma_0 \rho_i) + \bar{\psi}_i \sigma_{ij} \phi_j \right\}. \quad (C.4)$$

subject to (C.2), (C.3).

Let us now decompose ψ_i into its orthogonal pieces in order to solve (C.3). Dropping factors of ∇^2 , this is¹⁶⁾

$$\begin{aligned} \psi_i &\equiv \partial_i \alpha + \kappa_i \beta + \psi_i^{Tt} ; \quad \gamma_i \psi_i^{Tt} = \partial_i \psi_i^{Tt} = \kappa_i \psi_i^{Tt} = 0 \\ \kappa_i &\equiv (\delta_{ij} - \partial_i \partial_j) \gamma_j ; \quad \partial \cdot \kappa = \kappa \cdot \partial = 0 . \end{aligned} \quad (C.5)$$

In terms of these variables, the solution of (C.3) is

$$\begin{aligned} \phi_i &= \partial_i \beta - \not\partial \psi_i^{Tt} \\ \gamma_i &= \psi_i^{Tt} . \end{aligned} \quad (C.6)$$

The action (C.4) is now

$$\begin{aligned} I = \int \left\{ \bar{\chi}^{oi} \left(\kappa_i \dot{\beta} + \dot{\psi}_i^{Tt} + \gamma_o \partial_i \beta - \gamma_o \not\partial \psi_i^{Tt} \right) \right. \\ \left. + \bar{\xi}^{oi} \left(\partial_i \dot{\beta} - \not\partial \dot{\psi}_i^{Tt} + \gamma_o \psi_i^{Tt} \right) + \left(-\bar{\beta} \kappa_i + \bar{\psi}_i^{Tt} \right) \sigma_{ij} \left(\partial_j \beta - \not\partial \psi_j^{Tt} \right) \right\} , \end{aligned} \quad (C.7)$$

As expected, the gauge parameter α cancels. In order to exploit the remaining constraints (C.2) we decompose χ^{oi} and ξ^{oi} according to the scheme (C.5).

The resulting arithmetic leads to

$$I = \int -\frac{3}{2} \bar{\beta} \not\partial \beta + (\bar{\chi}_i - \bar{\xi}_i \not\partial)^{Tt} \dot{\psi}_i^{Tt} - (\bar{\chi}_i + \bar{\xi}_i \not\partial)^{Tt} \gamma_o \not\partial \psi_i^{Tt} . \quad (C.8)$$

It is evident that the χ_i^{Tt} and ξ_i^{Tt} are Lagrange multipliers that require ψ_i^{Tt} to vanish. The action (C.8), therefore, reduces to the usual Dirac form for the Majorana spinor β . This analysis, which makes use only of spatial constraints, justifies the treatment in the text, where time derivatives constraints were solved.

Table

Complementarity of higher rank gauge theories for lower spin to lower rank gauge theories for higher spin.

	photon/notoph		graviton/notivarg			
potential	A_μ	$B_{\mu\nu}$	$h_{\mu\nu}$		$B_{\mu\nu\rho\sigma}$	
field strength	$F_{\mu\nu}$	G_μ	$R_{\mu\nu\rho\sigma}$		$\bar{F}_{\mu\nu}$	
actions	$F_{\mu\nu}^2$	G_μ^2	$h_{\mu\nu} R^{\mu\nu}$ or $R_{\mu\nu}^2 - \frac{1}{3}R^2$		$\bar{B}_{\mu\nu} \bar{F}^{\mu\nu}$ or $\bar{F}_{\mu\nu}^2 - \frac{1}{3}\bar{F}^2$	
spin content	1	0	2 or dipole 2		0 or 1	

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