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LARGE N SADDLE POINTS

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A B S T R A C T

We use the saddle point method to study the large N limit in quantum mechanics. In a model with $O(N)$ symmetry this yields the correct answer for the ground state energy. However, in a (quaternionic) matrix model with $Sp(N)$ symmetry that is not the case.

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I. INTRODUCTION

In this work we examine the large N limit in quantum mechanics by the saddle point method. We present the calculation of the ground state energy of a $N \times N$ quaternionic matrix model for which the saddle point result (to leading order in N) does not yield the correct answer. Corrections to the saddle point value prove to be of the same order in N . We contrast this with the case of a $O(N)$ symmetric model where the saddle point result does provide the correct answer and corrections are suppressed by one power of N . In both cases we use the exactly soluble harmonic oscillator potential to establish our conclusions.

In Section II the $O(N)$ model is discussed. In Section III we apply the same approach to the matrix model and point out the differences between the two cases. Our results are relevant for recent discussions ¹⁾ of the role of classical solutions in the large N limit of various field theories, in which zero-dimensional matrix models are used as prototypes.

II. THE $O(N)$ MODEL

Consider the $O(N)$ invariant classical Euclidean Lagrangian given by :

$$L = \frac{1}{2} \sum_{i=1}^N \dot{\vec{q}}_i^2 + V(\vec{q}^2) \quad (1)$$

The quantum Hamiltonian is obviously :

$$H = -\frac{1}{2} \nabla^2 + V(\vec{q}^2) \quad (2)$$

The Laplace operator can be decomposed into radial and angular parts

$$\nabla^2 \equiv \sum_{i=1}^N \frac{\partial^2}{\partial q_i^2} = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \nabla_\Omega^2 = H_0 + \frac{1}{r^2} \nabla_\Omega^2 \quad (3)$$

where $r = \sqrt{\vec{q}^2}$ and ∇_Ω^2 is the Beltrami operator. Since the ground state of the model is $O(N)$ symmetric the Hamiltonian for that state is just H_0 .

The ground state energy is given by

$$E_0 = \text{Min} \frac{\int [d^N q] \langle \psi_0 | H | \psi_0 \rangle}{\int [d^N q] \langle \psi_0 | \psi_0 \rangle} \quad (4)$$

where $[d^N q] \equiv r^{N-1} dr d\Omega$. The angular integrations cancel out and if we define $\chi_0 \equiv r^{(N-1)/2} \psi_0$ the equation for χ_0 is :

$$-\frac{1}{2} \frac{\partial^2 \chi_0}{\partial r^2} + \frac{N^2 - 1}{8r^2} \chi_0 + V(r^2) \chi_0 = E_0 \chi_0 \quad (5)$$

If N is large we may neglect the derivative term and the minimization condition just amounts to finding the extrema of the effective potential :

$$V_{\text{eff}}(r^2) = \frac{N^2 - 1}{8r^2} + V(r^2) \quad (6)$$

For a harmonic potential, $V(r^2) = \frac{1}{2} r^2$, we obtain :

$$\left. \frac{\partial V_{\text{eff}}}{\partial r} \right|_{r_0} = 0 \Rightarrow r_0 = \frac{1}{2} \sqrt{N^2 - 1} ; V_{\text{eff}}(r_0) = V_{\text{min}} = \frac{1}{2} \sqrt{N^2 - 1} \quad (7)$$

For large N this agrees with the exact answer $E_0 = N/2$. The corrections to the saddle point result can be obtained by expanding V_{eff} around r_0 ; since $(\partial^2 V_{\text{eff}} / \partial r^2)|_{r_0}$ is $O(1)$ they will be down by one power of N with respect to the saddle point value.

III. THE MATRIX MODEL

Let Q be a $N \times N$ matrix whose entries are quaternions :

$$Q_{ij} = Q_{ij}^\alpha e_\alpha = Q_{ij}^0 \mathbb{1} + Q_{ij}^a e_a ; Q_{ij}^\alpha \in \mathbb{C}$$

$$e_1 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} ; e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} ; e_3 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} ; e_a e_b = \epsilon_{abc} e_c$$

Any $2N \times 2N$ complex matrix can be written in this form. Let us define the quaternionic operations of :

$$\begin{aligned}
 \text{CONJUGATION :} \quad \bar{Q}_{ij} &\equiv Q_{ij}^0 \mathbb{1} - Q_{ij}^a e_a \\
 \text{COMPLEX CONJUGATION :} \quad Q_{ij}^* &\equiv Q_{ij}^{0*} \mathbb{1} + Q_{ij}^{a*} e_a \\
 \text{HERMITEAN CONJUGATION :} \quad Q_{ij}^+ &\equiv Q_{ij}^{0*} \mathbb{1} - Q_{ij}^{a*} e_a = \bar{Q}_{ij}^*
 \end{aligned}$$

Using these one can define the quaternion matrix operations of :

$$\begin{aligned}
 \text{HERMITEAN CONJUGATION } (Q \rightarrow Q^+) : (Q^+)_{ij} &\equiv Q_{ji}^+ \\
 \text{DUALITY } (Q \rightarrow \tilde{Q}) : (\tilde{Q})_{ij} &\equiv \bar{Q}_{ji}
 \end{aligned}$$

Matrices which are Hermitean ($Q = Q^+$) and self-dual ($Q = \tilde{Q}$) are of course quaternion real ($Q_{ij}^\alpha \in \mathbb{R}$). Furthermore, it can be shown ²⁾ that, for such matrices, there exists a symplectic matrix S such that :

$$Q = S \Lambda S^{-1} ; \quad S \in Sp(N)$$

where Λ is diagonal, real and scalar :

$$\Lambda_{ij} = \Lambda_i \delta_{ij} = (\lambda_i e_0) \delta_{ij} ; \quad \lambda_i \in \mathbb{R} ; e_0 \equiv \mathbb{1}$$

Consider now the $Sp(N)$ invariant Euclidean Lagrangian given by :

$$L = \frac{1}{2} \text{Tr} \dot{Q}^2 + \text{Tr} V(Q^2) \quad (8)$$

The trace is taken with respect to the matrix indices as well as the 2×2 quaternions. The quantum Hamiltonian is then

$$H = -\frac{1}{4} \nabla^2 + \text{Tr} V(Q^2) \quad (9.a)$$

$$\nabla^2 \equiv \sum_{i=1}^N \frac{\partial^2}{\partial Q_{ii}^2} + \frac{1}{2} \sum_{i < j}^N \left[\frac{\partial^2}{\partial Q_{ij}^2} + \frac{\partial^2}{\partial Q_{ji}^2} \right] \quad (9.b)$$

Since Q can be diagonalized by a $Sp(N)$ transformation, the Laplace operator can be expressed ²⁾ as the sum of a term involving the eigenvalues λ_i plus an "angular" part :

$$\nabla^2 \equiv \frac{1}{\Delta^4} \sum_{i=1}^N \frac{\partial}{\partial \lambda_i} \left[\Delta^4 \frac{\partial}{\partial \lambda_i} \right] + \frac{1}{\Delta^4} \nabla_\Omega^2 ; \quad \Delta \equiv \prod_{i < j}^N (\lambda_i - \lambda_j) \quad (10)$$

The ground state is $Sp(N)$ symmetric so that we can restrict ourselves to the "radial" term H_o . Its wave function ψ_o is symmetric in the λ_i 's. Its energy is given by (4) with $[d^N q]$ replaced by $[dQ] \equiv \prod_i d\lambda_i \Delta^4 d\Omega$. Angular integrations will once again cancel out and the Δ^4 can be absorbed into $\chi_o \equiv \Delta^2 \psi_o$. We note that χ_o is also symmetric in the λ_i 's [this would not be so if we had $U(N)$ rather than $Sp(N)$ invariance] so that we can try to apply the ordinary saddle point method. The equation for χ_o is simply :

$$-\frac{1}{4} \sum_{i=1}^N \frac{\partial^2 \chi_o}{\partial \lambda_i^2} + \left[\sum_{i < j}^N \frac{1}{(\lambda_i - \lambda_j)^2} + V(\lambda_i) \right] \chi_o = E_o \chi_o \quad (11)$$

In order to arrive at (11) we have used :

$$\frac{\partial \ln \Delta}{\partial \lambda_k} = \frac{1}{\Delta} \frac{\partial \Delta}{\partial \lambda_k} = \sum_{\substack{i=1 \\ i \neq k}}^N \frac{1}{(\lambda_k - \lambda_i)} \quad (12.a)$$

$$\sum_{i < k}^N \frac{1}{(\lambda_k - \lambda_i)^2} = \frac{1}{2} \sum_{k=1}^N \left[\sum_{\substack{i=1 \\ i \neq k}}^N \frac{1}{(\lambda_k - \lambda_i)} \right]^2 \quad (12.b)$$

If we were to proceed as in the case of the $O(N)$ model, the derivative term would be neglected and the problem would reduce to extremizing the effective potential :

$$V_{\text{eff}}(\lambda_i) = \sum_{i < j}^N \frac{1}{(\lambda_i - \lambda_j)^2} + V(\lambda_i) \quad (13)$$

Introducing the density of eigenvalues $\varphi(x)$:

$$\varphi(x) \equiv \sum_{i=1}^N \delta(x - \lambda_i) \quad ; \quad \int_{-\infty}^{\infty} dx \varphi(x) = N \quad (14)$$

the effective potential can be rewritten as :

$$V_{\text{eff}}(\varphi) = \int_{-\infty}^{\infty} dx \left[\frac{\pi^2}{6} \varphi^3(x) + V(x) \varphi(x) \right] \quad (15)$$

where we have made use of the identity ³⁾ :

$$\int_{-\infty}^{\infty} dx \varphi(x) \left[\mathbb{P} \int_{-\infty}^{\infty} dy \frac{\varphi(y)}{x-y} \right]^2 = \frac{\pi^2}{3} \int_{-\infty}^{\infty} dx \varphi^3(x) \quad (16)$$

From the stationarity requirement $(\delta V_{\text{eff}}/\delta \varphi)|_{\varphi_0} = 0$, subject to the constraint (14), one obtains :

$$\varphi_0(x) = \frac{\sqrt{2}}{\pi} \sqrt{\mu - V(x)} \Theta(\mu - V(x)) \quad (17)$$

where μ is a Lagrange multiplier. Inserting this saddle point expression back into the constraint equation yields the value of μ as a function of N and allows us to compute $V_{\min} = V_{\text{eff}}(\varphi_0)$.

We can easily show that in the present case the saddle point result is incorrect. We restrict our attention to the harmonic oscillator potential $V(\lambda_i) = \sum_{i=1}^N \lambda_i^2$. If we carry out the procedure outlined in the preceding paragraph we obtain $\mu = \sqrt{2} N$ and

$$V_{\min} = V_{\text{eff}}(\varphi_0) = \int_{-\infty}^{\infty} dx \left[\frac{\pi^2}{6} \varphi_0^3(x) + x^2 \varphi_0(x) \right] = \frac{N^2}{\sqrt{2}} \quad (18)$$

However, from the exact ground state solution :

$$\psi_0 = C \exp \left[-\frac{1}{2} \text{Tr} Q^2 \right] = C \exp \left[-\sum_{i=1}^N \lambda_i^2 \right] \quad (19)$$

we can compute E_0 , either by direct use of Eq. (11) or by simply noting that we just have a collection of $N(2N-1)$ uncoupled oscillators (the number of independent Q_{ij} 's). The answer is :

$$E_0 = \frac{1}{2} N(2N-1) = N^2 - \frac{N}{2} \quad (20)$$

clearly in disagreement with the saddle point result. It is not justified to leave out the derivative term since it contributes to the same order in N as the terms we have kept. It is instructive to go back to (13) and work directly in terms of the eigenvalues $\{\lambda_i\}$. From the inequality :

$$\sum_{k=1}^N \Omega_k^2 \equiv \sum_{k=1}^N \left[\frac{1}{\sqrt{2}} \sum_{\substack{i=1 \\ i \neq k}}^N \frac{1}{\lambda_k - \lambda_i} - \lambda_k \right]^2 \geq 0 \quad (21)$$

and using (12.b) one immediately obtains :

$$V_{\text{eff}}(\lambda_i) \geq \frac{N(N-1)}{\sqrt{2}} \quad (22)$$

The value on the right-hand side is $V_{\min}$ since differentiating (21) with respect to λ_k and looking for the extrema yields :

$$\lambda_k^{(0)} = \frac{1}{\sqrt{2}} \sum_{\substack{i=1 \\ i \neq k}}^N \frac{1}{\lambda_k^{(0)} - \lambda_i^{(0)}} \quad (23)$$

This saturates the inequality so that $V_{\min} = V_{\text{eff}}(\lambda_i^{(0)}) = N(N-1)/\sqrt{2}$. Taking a second derivative :

$$M_{ek} \equiv \frac{\partial^2 V_{\text{eff}}}{\partial \lambda_e \partial \lambda_k} \Big|_{\lambda^{(0)}} = \sum_{j=1}^N \left[\sqrt{2} \frac{\partial \Omega_j}{\partial \lambda_e} \right]_{\lambda^{(0)}} \left[\sqrt{2} \frac{\partial \Omega_j}{\partial \lambda_k} \right]_{\lambda^{(0)}} \quad (24)$$

Defining :

$$m_{je} = m_{ej} = \sqrt{2} \frac{\partial \Omega_j}{\partial \lambda_e} \Big|_{\lambda^{(0)}} = - \left\{ \delta_{je} \left[\sum_{\substack{i=1 \\ i \neq j}}^N \frac{1}{(\lambda_j^{(0)} - \lambda_i^{(0)})^2} + \sqrt{2} \right] - \frac{1 - \delta_{je}}{(\lambda_j^{(0)} - \lambda_e^{(0)})^2} \right\} \quad (25)$$

Then (24) is just $M = m^2$. If we rescale the $\lambda_k^{(0)}$'s, so that $\bar{\lambda}_k = 2^{1/4} \lambda_k^{(0)}$, Eq. (23) can be satisfied if we choose for the $\bar{\lambda}$'s the zeros of the N th order Hermite polynomial ⁴⁾. Furthermore, as shown by Calogero in Ref. 4), the matrix :

$$\delta_{je} \sum_{\substack{i=1 \\ i \neq j}}^N \frac{1}{(\bar{\lambda}_j - \bar{\lambda}_i)^2} - \frac{1 - \delta_{je}}{(\bar{\lambda}_j - \bar{\lambda}_e)^2} \quad (26)$$

has integer eigenvalues $n=0,1,\dots,N-1$. Thus, the eigenvalues of m are given by $n' = -\sqrt{2}(n+1)$. The matrix $M = m^2$ has therefore eigenvalues $2(n+1)^2$. Since these are the square of the normal frequencies of oscillation around the saddle point, we conclude that there will be corrections of order N^2 , i.e., of the same order as the result itself, unlike the case of the $O(N)$ model.

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