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A PROGRAM FOR COMPUTING THE CONICAL FUNCTIONS OF THE FIRST KIND

$P_{-\frac{1}{2}+i\tau}^m(x)$ FOR $m = 0$ AND $m = 1$

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PROGRAM SUMMARY

Title of program: FCONIC

Catalogue number:

Program obtainable from:

Computer: CDC 7600, Installation: CERN, Geneva

Operating system: CDC scope

Programming languages used: FORTRAN IV

High speed store required: 1164 words. No. of bits in a word: 60.

Is the program overlaid? No

No. of magnetic tapes required: None

What other peripherals are used? Line Printer

No. of cards in combined program and test deck: 853

Card punching code: BCD

Keywords: Conical function, Legendre function, spherical functions, boundary value problems, integral transforms, Mehler-Fock transform, asymptotic expansions.

Nature of the physical problem: The conical functions $P_{-\frac{1}{2}+i\tau}^m(x)$ appear in a wide range of physical applications. Here we mention boundary value problems for the Laplace equation in regions bounded by cones, by two intersecting spheres, or by one or two confocal hyperboloids of revolution. Further, these functions are used in the Mehler-Fock integral transform for problems in potential and heat theory, and in elementary particle physics.

Method of solution: Series expansions based on Gaussian hypergeometric functions evaluated by direct summation or rational approximations are used together with asymptotic expansions for real $x > -1$ and real $\tau \geq 0$.

Accuracy: As the tests show, an accuracy of at least approximately 10 significant digits is normally obtained. The x and τ not too large, the accuracy increases to about 12-13 significant digits.

Running time: A running time of about 300 μ s on a CDC 7600 was found on the average. For $x > 0$ and $\tau > 20$, the time reduces to about 200 μ s on the average.

1. INTRODUCTION

The special cases $\nu = -\frac{1}{2} + i\tau$ of the Legendre functions of the first kind $P_{\nu}^m(z)$ are often called conical functions [1-4]. They appear in a number of physical applications, mainly when integrating the Laplace equation over regions bounded by cones, by two intersecting spheres, or by one or two confocal hyperboloids of revolution.

Another field where the conical functions occur is the Mehler-Fock transform [5,6]:

$$\begin{aligned}\psi(x) &= \int_0^{\infty} P_{-\frac{1}{2}+i\tau}^m(x) f(\tau) d\tau \quad (1 \leq x < \infty) \\ f(\tau) &= \frac{\tau}{\pi} \sinh \pi\tau \Gamma(\tfrac{1}{2} - m + i\tau) \Gamma(\tfrac{1}{2} - m - i\tau) \\ &\times \int_1^{\infty} P_{-\frac{1}{2}+i\tau}^m(x) \psi(x) dx .\end{aligned}\tag{1}$$

The conical functions $P_{-\frac{1}{2}+i\tau}^m(z)$ are solutions of the differential equation

$$(1 - z^2)u'' - 2zu' - \left(\tau^2 + \frac{1}{4} + \frac{m^2}{1 - z^2}\right)u = 0 .\tag{2}$$

For simplicity, we shall write $P_{-\frac{1}{2}+i\tau}^m(z)$ instead of $P_{-\frac{1}{2}+i\tau}^0(z)$ for $m = 0$.

These functions have been investigated by several authors. A recent treatise is that of Zhurina and Karmazina [7], originally published in Russian. This volume contains the theory and a large number of references as well as surfaces of the functions for real $z = x$ and $m = 0, 1$. Tables of the coefficients of the expansion for $\tau \rightarrow \infty$ (see below) and $\tau \rightarrow 0$ are also given. Tables of $P_{-\frac{1}{2}+i\tau}^m(x)$ have been published by the same authors, for $0 \leq \tau \leq 50$, $|x| < 1$, in [8] and for $0 \leq \tau \leq 50$, $x > 1$ in [9]. Our aim in this paper is to present a computer program for calculating the functions $P_{-\frac{1}{2}+i\tau}^m(x)$ for $m = 0, 1$ and real x and τ .

2. FORMULAE FOR THE CONICAL FUNCTIONS

Expressions for $P_{-\frac{1}{2}+i\tau}^m(x)$ can be obtained in principle by specialization of the well-known formulae for $P_\nu^\mu(z)$, given in the relevant handbooks [1-4]. They can also be found in [7]. For convenience, we give in the following those expressions which are used for computation or checking purposes in the program or its testing, together with asymptotic expansions containing higher terms apparently not published beforehand. If not otherwise specified, we restrict the value of m to $m = 0$ and $m = 1$.

2.1 Series expansions

The following series define the conical functions for $m = 0, 1$; and real x and τ :

$$P_{-\frac{1}{2}+i\tau}^m(x) = \sqrt{\pi} \left(\frac{2}{\sqrt{1-x^2}} \right)^m \times \\ \times \left\{ \frac{1}{|\Gamma(\frac{3}{4} - \frac{1}{2}m - i\tau)|^2} \operatorname{Re} {}_2F_1\left(\frac{1}{4} - \frac{m}{2} - i\frac{\tau}{2}, \frac{1}{4} - \frac{m}{2} + i\frac{\tau}{2}; \frac{1}{2}; x^2\right) - \right. \\ \left. - \frac{2x}{|\Gamma(\frac{1}{4} - \frac{1}{2}m - i\tau)|^2} \operatorname{Re} {}_2F_1\left(\frac{3}{4} - \frac{m}{2} - i\frac{\tau}{2}, \frac{3}{4} - \frac{m}{2} + i\frac{\tau}{2}; \frac{3}{2}; x^2\right) \right\}^* (|x| < 1), \quad (3)$$

(see [4], formula 3.4 (11));

$$P_{-\frac{1}{2}+i\tau}^m(x) = \left[\frac{1}{2} \left(\tau^2 + \frac{1}{4} \right) \operatorname{sgn}(1-x) \sqrt{|x^2 - 1|} \right]^m \times \\ \times \operatorname{Re} {}_2F_1\left(\frac{1}{2} + m - i\tau, \frac{1}{2} + m + i\tau; m+1; \frac{1-x}{2}\right) \quad (-1 < x < 3), \quad (4)$$

(see [7]);

$$P_{-\frac{1}{2}+i\tau}^m(x) = \frac{2}{\sqrt{\pi}} \left(\frac{x}{\sqrt{x^2 - 1}} \right)^m \times \\ \times \operatorname{Re} \left\{ (2x)^{-\frac{1}{2}+i\tau} \frac{\Gamma(i\tau)}{\Gamma(\frac{1}{2} - m + i\tau)} {}_2F_1\left(\frac{1}{4} - \frac{m}{2} - \frac{1}{2}i\tau, \frac{3}{4} - \frac{m}{2} - \frac{1}{2}i\tau; 1 - i\tau; \frac{1}{x^2}\right) \right\}, \\ (|x| > 1), \quad (5)$$

*) Note the misprints in formula [7, (3.9)]; where one should read " $\frac{3}{2}; x^2$ " instead of " $\frac{3}{2}; x^2$ ", and in formula [8, (8)], where the factor x is missing in the second term.

(see [7]), where

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c1!} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \dots \quad (6)$$

is the Gaussian hypergeometric function.

2.2 Integral representations

Integral representations have been used for a long time in the theory of special functions. It is, however, worthwhile mentioning that the availability of accurate high-speed computers often makes them useful for the numerical evaluation of functions as well; in particular when good integration procedures are available. Numerical integration is often the only reliable way of testing function values obtained by other methods. For $P_{-\frac{1}{2}+i\tau}^m(x)$, one has the following representations [7], often called Mehler integrals:

$$P_{-\frac{1}{2}+i\tau}^m(\cos \theta) = (-1)^m \frac{2^{m+\frac{1}{2}}}{\pi(2m-1)!!} \frac{\Gamma\left(\frac{1}{2} + m + i\tau\right)}{\Gamma\left(\frac{1}{2} - m + i\tau\right) \sin^m \theta} \int_0^\theta \frac{\cosh \tau t dt}{(\cos t - \cos \theta)^{\frac{1}{2}-m}} \quad (7)$$

$$= \frac{(2m-1)!!}{2^{m-\frac{1}{2}} \pi} \cosh \pi \tau \int_0^\infty \frac{\cos \tau t dt}{(\cosh t + \cos \theta)^{m+\frac{1}{2}}} \quad (8)$$

for $0 < \theta < \pi$, $m = 0, 1, 2, \dots$, and

$$P_{-\frac{1}{2}+i\tau}^m(\cosh \eta) = \frac{2^{m+\frac{1}{2}}}{\pi(2m-1)!!} \frac{\Gamma\left(\frac{1}{2} + m + i\tau\right)}{\Gamma\left(\frac{1}{2} - m + i\tau\right) \sinh^m \eta} \int_0^\eta \frac{\cos \tau t dt}{(\cosh \eta - \cosh t)^{\frac{1}{2}-m}} \quad (9)$$

for $0 < \eta < \infty$, $m = 0, 1, 2, \dots$.

Apart from formula (8), these representations are not well suited for numerical integration. It is, however, possible to derive from them representations which appear to be more complicated, but which are in fact better adapted for numerical evaluation for fixed θ or η , and τ not too large. We follow Zhurina and Karmazina [7] and obtain for $m = 0, 1$, using the substitutions

$$t = 2 \arcsin \left(\sin \frac{1}{2} \theta \sin \phi \right) \quad \text{and} \quad t = 2 \operatorname{Arsh} \left(\sinh \frac{1}{2} \eta \cos \phi \right)$$

in (7) and (9), respectively:

$$P_{-\frac{1}{2}+i\tau}^m(\cos \theta) = \frac{2}{\pi} \left[2 \left(\tau^2 + \frac{1}{4} \right) \tan \frac{1}{2} \theta \right]^m \int_0^{\pi/2} \frac{\cosh \left\{ 2\tau \arcsin \left(\sin \frac{1}{2} \theta \sin \phi \right) \right\} \cos^{2m} \phi}{\sqrt{1 - \sin^2 \frac{1}{2} \theta \sin^2 \phi}} d\phi \quad (10)$$

and

$$P_{-\frac{1}{2}+i\tau}^m(\cosh \eta) = \frac{2}{\pi \cosh \frac{1}{2} \eta} \left[-2 \left(\tau^2 + \frac{1}{4} \right) \tanh \frac{1}{2} \eta \right]^m \int_0^{\pi/2} \frac{\cos \left\{ 2\tau \operatorname{Arsh} \left(\sinh \frac{1}{2} \eta \cos \phi \right) \right\} \sin^{2m} \phi}{\sqrt{1 - \tanh^2 \frac{1}{2} \eta \sin^2 \phi}} d\phi. \quad (11)$$

From these integrals we can easily derive expressions for $\tau = 0$, namely

$$P_{-\frac{1}{2}}(\cos \theta) = \frac{2}{\pi} K \left(\sin \frac{1}{2} \theta \right) \quad (12)$$

$$P_{-\frac{1}{2}}(\cosh \eta) = \frac{2}{\pi \cosh \frac{1}{2} \eta} K \left(\tanh \frac{1}{2} \eta \right) \quad (13)$$

$$P_{-\frac{1}{2}}^1(\cos \theta) = \frac{2}{\pi \sin \theta} \left[E \left(\sin \frac{1}{2} \theta \right) - \cos^2 \frac{1}{2} \theta K \left(\sin \frac{1}{2} \theta \right) \right] \quad (14)$$

$$P_{-\frac{1}{2}}^1(\cosh \eta) = \frac{2 \cosh \frac{1}{2} \eta}{\pi \sinh \eta} \left[E \left(\tanh \frac{1}{2} \eta \right) - K \left(\tanh \frac{1}{2} \eta \right) \right], \quad (15)$$

where

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \quad (16)$$

and

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi} d\phi \quad (17)$$

are the complete elliptic integrals of the first and second kind.

2.3 Asymptotic expansions

Here again, asymptotic expansions of the conical functions can be obtained for large values of τ by specializing the general expansions for the Legendre functions. Zhurina and Karmazina [7] present, instead, series of the form

$$P_{-\frac{1}{2}+i\tau}^m(\cos \theta) = \sum_{n=-1}^{N-1} A_n^m(\theta) I_n(\tau\theta) \tau^{-n} + O(\tau^{-N-\frac{1}{2}}), \quad (0 < \theta < \pi), \quad (18)$$

and

$$P_{-\frac{1}{2}+i\tau}^m(\cosh \eta) = \sum_{n=-1}^{N-1} \bar{A}_n^m(\eta) J_n(\tau\eta) \tau^{-n} + O(\tau^{-N-\frac{1}{2}}), \quad (0 < \eta < \infty), \quad (19)$$

where $J_n(z)$ and $I_n(z)$ are the Bessel functions. As they point out, the advantage of these expansions lies in the fact that for large τ they are applicable even when θ and η are small, down to $\theta = 0$, $\eta = 0$. In fact, they are even convergent for $0 \leq \theta \leq \pi - \varepsilon$ ($0 < \varepsilon < \pi$), and $\eta < 2\pi \sqrt{2 + \sqrt{5}}$ [7].

The development of the series (19) for $P_{-\frac{1}{2}+i\tau}^m(\cosh \eta)$ is presented in a paper (in Russian) by Karmazina [10], but the method goes back to earlier work by Fock and by Szegő (for references, see [7]). In [7], the four coefficients $A_n^m(\theta)$ and $\bar{A}_n^m(\eta)$ are given for $n \leq 3$. Because the calculation of these coefficients for higher n is both tedious and error-prone when carried out by hand, they provide a good opportunity for demonstrating the usefulness of algebraic manipulation by computer. We therefore present here complete sets of $A_n^m(\theta)$ and of $\bar{A}_n^m(\eta)$ for $-1 \leq n \leq 6 - m$ ($m = 0, 1$). These coefficients have been obtained with the help of the algebraic formula manipulation system REDUCE developed by Hearn [11], and the use of terms beyond the first four coefficients does indeed contribute to higher accuracy.

We take the case $P_{-\frac{1}{2}+i\tau}^1(\cosh \eta)$, and follow the method given for $P_{-\frac{1}{2}+i\tau}(\cosh \eta)$ in [10]. Starting from the integral (9), e.g.

$$P_{-\frac{1}{2}+i\tau}^1(\cosh \eta) = -\frac{2}{\pi} \frac{\left(\tau^2 + \frac{1}{4}\right)}{\sinh \eta} \int_0^\eta \cos \tau t \sqrt{2(\cosh \eta - \cosh t)} dt, \quad (20)$$

we write

$$\sqrt{2(\cosh \eta - \cosh t)} = \sqrt{\frac{\sinh \eta}{\eta}} \sqrt{\eta^2 - t^2} \sum_{n=0}^{\infty} F_n(\eta) (\eta^2 - t^2)^n \quad (21)$$

and develop

$$\cosh \eta - \cosh t = - \sum_{n=1}^{\infty} c_n(\eta) (\eta^2 - t^2)^n \quad (22)$$

as a Taylor series with coefficients

$$c_n(\eta) = \frac{(-1)^n}{n! \eta^{2n}} \left. \frac{d^n \cosh \eta \sqrt{x}}{dx^n} \right|_{x=1} \quad (23)$$

Setting $u = \eta^2 - t^2$, and equating the coefficients of equal powers of u in

$$- \frac{1}{2} \frac{\sinh \eta}{\eta} u \left(\sum_{n=0}^{\infty} F_n(\eta) u^n \right)^2 = \sum_{n=1}^{\infty} c_n(\eta) u^n, \quad (24)$$

we obtain the following system of non-linear equations:

$$\sum_{v=0}^n F_v(\eta) F_{n-v}(\eta) + \frac{2\eta}{\sinh \eta} c_n(\eta) = 0, \quad (n = 0, 1, 2, \dots), \quad (25)$$

which can be solved successively, giving

$$F_0(\eta) = 1, \quad F_1(\eta) = (1 - \eta \coth \eta)/(8\eta^2), \dots$$

Introducing this solution together with (21) into the integral (20), and using

[2, formula 3.7718]

$$\int_0^{\eta} (\eta^2 - t^2)^{n+\frac{1}{2}} \cos \tau t \, dt = \frac{\pi}{2} (2n+1)!! \left(\frac{\eta}{\tau} \right)^{n+1} J_{n+1}(\tau \eta), \quad (26)$$

we obtain the expression

$$\begin{aligned}
 P_{-\frac{1}{2}+i\tau}^1(\cosh \eta) &= - \frac{\tau^2 + \frac{1}{4}}{\sqrt{\eta \sinh \eta}} \sum_{n=0}^{\infty} (2n+1)!! F_n(\eta) \eta^{n+1} \frac{J_{n+1}(\tau\eta)}{\tau^{n+1}} \\
 &= - \sqrt{\frac{\eta}{\sinh \eta}} \left(\tau J_1(\tau\eta) + 3F_1(\eta)\eta J_2(\tau\eta) + \right. \\
 &\quad + \sum_{n=1}^{\infty} (2n+3)!! F_{n+1}(\eta) \eta^{n+1} J_{n+2}(\tau\eta) \tau^{-n} \\
 &\quad \left. + \frac{1}{4} \sum_{n=1}^{\infty} (2n-1)!! F_{n-1}(\eta) \eta^{n-1} J_n(\tau\eta) \tau^{-n} \right).
 \end{aligned} \tag{27}$$

Using the well-known recurrence relation

$$J_{n+1}(z) = \frac{2n}{z} J_n(z) - J_{n-1}(z) \tag{28}$$

for the Bessel functions, (27) can be written as

$$\begin{aligned}
 P_{-\frac{1}{2}+i\tau}^1(\cosh \eta) &= - \sqrt{\frac{\eta}{\sinh \eta}} \left(\tau J_1(\tau\eta) + \frac{3}{8} \frac{\eta \coth \eta - 1}{\eta} J_0(\tau\eta) \right. \\
 &\quad + \sum_{n=1}^{\infty} \eta^n \left[2n(2n+1)!! F_n(\eta)/\eta + \frac{1}{4} (2n-1)!! F_{n-1}(\eta)/\eta \right. \\
 &\quad \left. \left. - (2n+3)!! F_{n+1}(\eta)\eta \right] J_n(\tau\eta) \tau^{-n} \right) \\
 &= \sum_{n=-1}^{\infty} \bar{A}_n^1(\eta) J_n(\tau\eta) \tau^{-n}.
 \end{aligned} \tag{29}$$

The other coefficients in (18) and (19) can be found in a similar way. The whole problem can then be programmed in REDUCE, in particular the procedure of solving (25) and the corresponding somewhat more complicated equation for $m = 0$.

We now list the coefficients $A_n^m(\theta)$ and $\bar{A}_n^m(\eta)$ so obtained, thus extending (and correcting) those given in [7]. In order to save space, we introduce for notational purposes a new variable t , which can take the value θ or η , and define:

for $0 < \theta < \pi$

$$f = \cot \theta, \quad \sigma = -1, \quad A_n^m(\theta) = \sqrt{\frac{\theta}{\sin \theta}} C_n^m;$$

for $0 < \eta < \infty$

$$f = \coth \eta, \quad \sigma = +1, \quad \bar{A}_n^m(\eta) = \sqrt{\frac{\eta}{\sinh \eta}} C_n^m,$$

and obtain for $m = 0$ ($C_n \equiv C_n^0$)

$$C_{-1} = 0$$

$$C_0 = 1$$

$$C_1 = (tf - 1)/(8t^2)$$

$$C_2 = (9t^2f^2 + 6tf - 8\sigma t^2 - 15)/(128 t^2)$$

$$C_3 = 5(15t^3f^3 + 27t^2f^2 + 21tf - 16\sigma t^3f - 24\sigma t^2 - 63)/(1024t^3)$$

$$C_4 = 7(525t^4f^4 + 1500t^3f^3 + 2430t^2f^2 + 1980tf - 720\sigma t^4f^2 - 1600\sigma tf + 192t^4 - 2160\sigma t^2 - 6435)/(32768t^4) \quad (30)$$

$$C_5 = 21(2835t^5f^5 + 11025t^4f^4 + 24750t^3f^3 + 38610t^2f^2 + 32175tf - 4800\sigma t^5f^3 - 15120\sigma t^4f^2 - 26400\sigma t^3f + 1984t^5f + 4032t^4 - 34320\sigma t^2 - 109395)/(262144t^5)$$

$$C_6 = 11(218295t^6f^6 + 1071630t^5f^5 + 3009825t^4f^4 + 6142500t^3f^3 + 9398025t^2f^2 + 7936110tf - 441000\sigma t^6f^4 + 254016t^6f^2 - 1814400\sigma t^5f^3 + 749952t^5f - 4127760\sigma t^4f^2 - 6552000\sigma t^3f - 31232\sigma t^6 + 1100736t^4 - 8453800\sigma t^2 - 27776385)/(4194304t^6)$$

and for $m = 1$

$$C_{-1}^1 = -1$$

$$C_0^1 = 3(1 - tf)/(8t)$$

$$C_1^1 = (-15t^2f^2 + 6tf + 8\sigma t^2 + 9)/(128t^2)$$

$$C_2^1 = 3(-35t^3f^3 - 15t^2f^2 + 15tf + 32\sigma t^3f + 8\sigma t^2 + 35)/(1024t^3)$$

$$C_3^1 = (-4725t^4f^4 - 6300t^3f^3 - 3150t^2f^2 + 3780tf + 6000\sigma t^4f^2 + 5760\sigma t^3f - 12616t^4 + 1860\sigma t^2 + 10395)/(32768t^4) \quad (31)$$

$$C_4^1 = 7(-10395t^5f^5 - 23625t^4f^4 - 28350t^3f^3 - 14850t^2f^2 + 19305tf + 16800\sigma t^5f^3 - 6336t^5f + 30000\sigma t^4f^2 + 25920t^3f - 6080t^4 + 7920\sigma t^2 + 57915)/(262144t^5)$$

$$C_5^1 = (-2837835t^6f^6 - 9168390t^5f^5 - 16372125t^4f^4 - 18918900t^3f^3 - 10135125t^2f^2 + 13783770tf + 5556600\sigma t^6f^4 - 3044160t^6f^2 + 14817600\sigma t^5f^3 - 5588352t^5f + 20790000\sigma t^4f^2 + 17297280t^3f + 323072\sigma t^6 - 4213440t^4 + 5405400\sigma t^2 + 43648605)/(4194304t^6).$$

In [7], these coefficients are given up to $n = 3$. Note, however, that the coefficients A_{-3}^1 in formula (7.8) and $\bar{A}_{-3}^1$ in formula (7.12) of [7], and the corresponding tables, are incorrect and should be replaced by C_3^1 as given above.

Introducing these coefficients C_n^m into (18) and (19), we finally obtain, with the help of the recurrence relation (28), for $m = 0, 1$

$$P_{-\frac{1}{2}+i\tau}^m(\cos \theta) = \sqrt{\frac{\theta}{\sin \theta}} \left[V_0(\theta, \tau) I_0(\tau\theta) + V_1(\theta, \tau) I_1(\tau\theta) \right] \quad (32)$$

$$P_{-\frac{1}{2}+i\tau}^m(\cosh \eta) = \sqrt{\frac{\eta}{\sinh \eta}} \left[V_0(\eta, \tau) J_0(\tau\eta) + V_1(\eta, \tau) J_1(\tau\eta) \right]$$

with

$$\begin{aligned}
 V_0 = & C_0^m + \sigma \frac{-C_2^m}{\tau^2} + \frac{-4C_3^m/t + C_4^m}{\tau^4} + \\
 & + \sigma \frac{-24C_4^m/t^2 + 12C_5^m/t - C_6}{\tau^6} + \\
 & + \frac{-192C_5^m/t^3 + 144C_6^m/t^2}{\tau^8} + \sigma \frac{-1920C_6^m/t^4}{\tau^{10}} + \dots,
 \end{aligned} \tag{33}$$

and

$$\begin{aligned}
 V_1 = & \sigma C_{-1}^m \tau + \frac{C_1^m}{\tau} + \sigma \frac{2C_2^m/t - C_3}{\tau^3} \\
 & + \frac{8C_3^m/t^2 - 8C_4^m/t + C_5^m}{\tau^5} + \sigma \frac{48C_4^m/t^3 - 72C_5^m/t^2 + 18C_6^m/t}{\tau^7} + \\
 & + \frac{384C_5^m/t^4 - 768C_6^m/t^3}{\tau^9} + \sigma \frac{3840C_6^m/t^5}{\tau^{11}} + \dots,
 \end{aligned}$$

where we have neglected coefficients C_n^m with $n > 6$ and taken $C_6 = 0$ for $m = 0$.

3. THE COMPUTATION OF THE HYPERGEOMETRIC FUNCTION ${}_2F_1(a, b; c; z)$

The hypergeometric function can be computed in several ways. The most straightforward, if $|z| < 1$ is small enough, is the evaluation of the defining series (6). It is, however, often useful -- in particular for other z , but also for reasons of accuracy -- to use rational approximations, e.g. those developed by Luke [12], i.e.

$${}_2F_1(a, b; c; -z) = A_n(z)/B_n(z) + R_n(z), \tag{35}$$

where $A_n(z)$ and $B_n(z)$ are defined by the same recurrence relation,

$$B_n(z) = (1 + F_1 z)B_{n-1}(z) + (E + F_2 z)zB_{n-2}(z) + F_3 z^3 B_{n-3}(z), \quad (n \geq 3), \tag{36}$$

where

$$\begin{aligned}
 F_1 &= \frac{3n^2 + (a + b - 6)n + 2 - ab - 2(a + b)}{2(2n - 3)(n + c - 1)}, \\
 F_2 &= - \frac{[3n^2 - (a + b + 6)n + 2 - ab](n + a - 1)(n + b - 1)}{4(2n - 1)(2n - 3)(n + c - 2)(n + c - 1)} \\
 F_3 &= \frac{(n + a - 2)(n + a - 1)(n + b - 2)(n + b - 1)(n - a - 2)(n - b - 2)}{8(2n - 3)^2(2n - 5)(n + c - 3)(n + c - 2)(n + c - 1)}, \\
 E &= - \frac{(n + a - 1)(n + b - 1)(n - c - 1)}{2(2n - 3)(n + c - 2)(n + c - 1)};
 \end{aligned} \tag{37}$$

$$\begin{aligned}
 B_0(z) &= 1, \quad B_1(z) = 1 + \frac{(a + 1)(b + 1)}{2c} z, \\
 B_2(z) &= 1 + \frac{(a + 2)(b + 2)}{2(c + 1)} z + \frac{(a + 1)(a + 2)(b + 1)(b + 2)}{12c(c + 1)} z^2;
 \end{aligned} \tag{38}$$

$$\begin{aligned}
 A_0(z) &= 1, \quad A_1(z) = B_1(z) - \frac{ab}{c} z, \\
 B_2(z) &= B_2(z) - \frac{ab}{c} \left[1 + \frac{(a + 2)(b + 2)}{2(c + 1)} z \right] z + \frac{a(a + 1)b(b + 1)}{2c(c + 1)} z^2.
 \end{aligned} \tag{39}$$

4. THE PROGRAM

In order to calculate the conical functions $P_{-\frac{1}{2}+i\tau}^m(x)$ ($m = 0, 1$) for $x > -1$ and $\tau \geq 0$, a function subprogram FCONIC (X,TAU,M) has been written which makes use of some of the formulae given in previous sections. In some regions it is not easy to choose formulae for given values of x and τ in such a way that speed and accuracy are optimized. Extensive test computation suggests that the following scheme is reasonable, placing somewhat stronger emphasis on accuracy than on speed. It is, however, not claimed that this choice is the best of all possible ones, and a different choice of boundaries might well result in higher speed or slightly higher accuracy, or both, for given values m, x, τ .

The x, τ -plane is divided in the following way:

- i) $-1 < x \leq 0$
 $0 < x \leq 0.1, \quad 0 < \tau \leq 17$
 $0.1 < x \leq 0.2, \quad 0 < \tau \leq 5$ } Formula (3).
- ii) $0.1 < x \leq 0.2, \quad 5 < \tau \leq 17$
 $0.2 < x \leq 1.5, \quad 0 < \tau \leq 20$ } Formula (4)
- iii) $1.5 < x, \quad 0 < \tau \leq \max(20, x)$; Formula (5).
- iv) $\tau = 0$; Formulae (12)-(15).
- v) All other cases, Formula (32).

The hypergeometric function ${}_2F_1$ occurring in formulae (3) to (5) is evaluated as follows:

In formula (3), $0 < \tau \leq 10$; Series (6).

Otherwise, rational approximation (35).

In formula (4), $1 < x$
 $x \leq 1, \quad 0 < \tau \leq 5$ } Series (6)

Otherwise, rational approximation (35).

In formula (5), rational approximation (35).

The computation of the expressions (6) and (35) is carried out until the absolute value of the difference of two consecutive terms in (6) or ratios in (35) becomes less than EPS ($=10^{-14}$). For $x \rightarrow -1$, this usually needs a rather large number of terms in (6), but no way has been found to overcome this problem (apart perhaps from numerical integration). As a matter of fact, this might be not so serious, since often only $x \geq 0$ or even $x \geq 1$ is needed, e.g. in the Mehler-Fock transform (1). In any case, an error exit is provided which limits the number of terms to NMAX ($=200$).

The gamma function $\Gamma(z)$ for complex arguments is evaluated by the function subprogram CLOGAM [13], which computes $\ln \Gamma(z)$. The Bessel functions in formula (32) are calculated with the help of two function subprograms BESJY and BESIK [14],

based on expansions given by Clenshaw [15] and Luke [16]. The complete elliptic integrals $K(k)$ and $E(k)$ are obtained from a function subprogram with entries ELLICK and ELLICE [17], based on work by Cody [18].

5. THE TESTS

The program FCONIC has been tested on a CDC 7600 computer at CERN. The tests included the computation of $P_{-\frac{1}{2}+i\tau}^m(x)$ for $x = -0.9(0.1)0.9, 0.95(0.01)1.05, 1.1(0.1)2.5(0.5)10(1)20(5)100(100)1000$, and $\tau = 10^{-n}$, $n = 2(1)13$, $\tau = 0(0.1)1(1)25(5)100$. For the same values of x and τ , the integrals (10) and (11) were computed using an adaptive numerical integration routine. Relative errors of about 10^{-12} to 10^{-13} were obtained in a large proportion of the tests, in particular for τ not too large (≤ 10). For other cases, the results are somewhat less accurate, but 10 digits were usually found to be correct. For large τ , it is difficult to find accurate test values for comparison, and it may be that FCONIC gives values which are in fact correct to more digits than was revealed by the tests, since the check values are themselves only approximations. For $x > 1000$ and for $\tau > 100$ (if ever needed in applications), the method used to compute the functions suggests that no significant loss in accuracy should occur. The values for $m = 0$ are often slightly more accurate than for $m = 1$. It also turns out that writing

$$\frac{\Gamma(i\tau)}{\Gamma\left(\frac{1}{2} - m + i\tau\right)} = \frac{\frac{1}{2} - m + i\tau}{i\tau} \frac{\Gamma(1 + i\tau)}{\Gamma\left(\frac{3}{2} - m + i\tau\right)} \quad (40)$$

for the ratio of the gamma functions in formula (5) increases the otherwise bad accuracy of (5) near $\tau = 0$ to about 12 digits.

The examples given in [7-9] have also been checked. Note that the example $P_{-\frac{1}{2}+60i}^{(1.8)} = -0.0101178$ in [9] is incorrect. The correct value is -0.0848158 .

6. ERROR EXITS

Two error exits are provided:

- i) If m has any value other than 0 or 1, the message

FCONIC ... ILLEGAL VALUE FOR $M = (m)$

is printed on the line printer.

- ii) If, during computation of the hypergeometric function, the value $NMAX$ is reached, the message

FCONIC ... CONVERGENCE DIFFICULTIES FOR HYPERGEOMETRIC FUNCTION,

$X = (x)$, $TAU = (\tau)$, $M = (m)$

is printed on the line printer.

In both cases, FCONIC returns the value zero.

7. TEST RUN

The results of the test run are self-explanatory (see p. 17).

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REFERENCES

- [1] M. Abramowitz and I.A. Stegun, eds., Handbook of mathematical functions, with formulas, graphs and mathematical tables. Natl. Bur. Std. Appl. Math. Series 55, U.S. Govt. Printing Office, Washington, 1965.
- [2] I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series and products. Academic Press New York 1980.
- [3] W. Magnus, F. Oberhettinger, and R.P. Soni, Formulas and theorems for the special functions of mathematical physics, 3rd edn., Springer Verlag Berlin 1966.
- [4] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, Higher transcendental functions, vol. I, McGraw-Hill, New York 1953.
- [5] I.N. Sneddon, The use of integral transforms, McGraw-Hill New York 1972.
- [6] V. Ditkine and A. Proudnikov, Transformations intégrales et calcul opérationnel, Editions Mir Moscou 1978.
- [7] M.I. Zhurina and L.N. Karmazina, Tables and formulae for the spherical functions $P_{-\frac{1}{2}+i\tau}^m(z)$. Math. Tables Series 40, Pergamon Press Oxford 1966.
- [8] M.I. Zhurina and L.N. Karmazina, Tables of the Legendre functions $P_{-\frac{1}{2}+i\tau}(x)$, Part I, Math. Tables Series 22, Pergamon Press Oxford 1964.
- [9] M.I. Zhurina and L.N. Karmazina, Tables of the Legendre functions $P_{-\frac{1}{2}+i\tau}(x)$, Part II, Math. Tables Series 38, Pergamon Press Oxford 1965.
- [10] L.N. Karmazina, Vychisl. Mat. 6 (1960) 3.
- [11] A.C. Hearn, REDUCE Users Manual, Univ. of Utah, Salt Lake City, Utah, USA, 2nd edn., 1973.
- [12] Y.L. Luke, Algorithms for the computation of mathematical functions, Academic Press New York 1977.
- [13] K.S. Kölbig, Computer Phys. Comm. 4 (1972) 221.

- [14] K.S. Kölbig, BESJY Bessel functions J_0, J_1, Y_0, Y_1 ; BESIK Bessel functions I_0, I_1, K_0, K_1 ; CERN Computer Centre Program Library C312, C313 (unpublished).
- [15] C.W. Clenshaw, Chebyshev series for mathematical functions, Nat. Phys. Lab., Math. Tables, Vol. 5, London 1962.
- [16] Y.L. Luke, The special functions and their approximations, Vol. II, Academic Press New York 1969.
- [17] K.S. Kölbig, ELLICK, ELLICE, Complete elliptic integrals K and E , CERN Computer Centre Program Library C308 (unpublished).
- [18] W.J. Cody, Math. Comp. 19 (1965) 105.

TEST RUN

X	TAU	P	P1
-.85	0.00	.17279236558972E+01	.10634850443812E+01
-.85	.10	.17605557979526E+01	.11142849272324E+01
-.85	3.00	.70807672035491E+03	.26201537781784E+04
-.85	15.00	.99535105138639E+16	.15706127164728E+18
-.85	50.00	.11485355572490E+56	.58343367209162E+57
.05	0.00	.11672134610493E+01	.25532417332859E+00
.05	.10	.11741611045757E+01	.26628448800769E+00
.05	3.00	.22300237225665E+02	.65308367551655E+02
.05	15.00	.83253989651497E+09	.12460262533419E+11
.05	50.00	.59563722748340E+32	.29765457135152E+34
.15	0.00	.11429022754169E+01	.22872118316056E+00
.15	.10	.11486107019880E+01	.23845619046937E+00
.15	3.00	.16657029480973E+02	.47881123536876E+02
.15	15.00	.18533934585491E+09	.27644302245895E+10
.15	50.00	.39258354249800E+30	.19598389080201E+32
.50	0.00	.10731820071494E+01	.14933621085537E+00
.50	.10	.10761621847241E+01	.15552007267513E+00
.50	3.00	.59474703054871E+01	.15651379836798E+02
.50	15.00	.73831046909638E+06	.10852960776347E+08
.50	50.00	.33335514569619E+22	.16570401429027E+24
1.25	0.00	.97077311174602E+00	-.82088592188165E-01
1.25	.10	.96961310976130E+00	-.85320624112363E-01
1.25	3.00	.17660142018852E+00	-.16600526140018E+01
1.25	15.00	-.23421048631849E+00	.81001961043664E+00
1.25	50.00	-.10050719697677E+00	-.40619023905969E+01
2.00	0.00	.90128629936045E+00	-.13666874968872E+00
2.00	.10	.89744673480279E+00	-.14182246318468E+00
2.00	3.00	-.34922606381428E+00	.17515275042365E+00
2.00	15.00	.15557903993725E+00	-.34710532119618E+00
2.00	50.00	-.52361042590678E-01	-.33644000250379E+01
50.00	0.00	.38144913332590E+00	-.12707370744169E+00
50.00	.10	.36360141095663E+00	-.12830117329959E+00
50.00	3.00	.59449089599972E-01	-.10971114921990E+00
50.00	15.00	.19605648731067E-01	.31351907406212E+00
50.00	50.00	-.15825970446446E-01	.11084061323428E+00
100.00	0.00	.30091748588200E+00	-.10544528203157E+00
100.00	.10	.28268342515387E+00	-.10537769391877E+00
100.00	3.00	-.36940442961803E-01	-.64101140132651E-01
100.00	15.00	-.20397827537169E-01	.53577734418444E-01
100.00	50.00	.10979320434022E-01	-.13572859431856E+00
500.00	0.00	.16697329348226E+00	-.63355020221548E-01
500.00	.10	.15055996077274E+00	-.61429101333079E-01
500.00	3.00	.10314132298113E-01	-.58657491769198E-01
500.00	15.00	-.60807108815433E-02	-.10078278758768E+00
500.00	50.00	.28302342329285E-02	.20747848941710E+00