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PERTURBATIVE QCD CALCULATION OF JET CROSS-SECTIONS

IN e^+e^- ANNIHILATION

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A B S T R A C T

We calculate the α_s^2 corrections to three-jet cross-sections in e^+e^- annihilation in a scheme that uses regulator masses for the infra-red and collinear divergences. Using various methods to define the three-jet region, we find that the corrections to the $O(\alpha_s)$ cross-sections are large.

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1. - INTRODUCTION

QCD as a theory of strong interactions has met with several successes. Combined with the operator product expansion ¹⁾ it has given a good description of scaling violations in deep inelastic scattering. It is also possible to reproduce the results obtained with the operator product expansion by summing leading logarithms in perturbation theory ²⁾.

One particular process for which many results exist is the annihilation of e^+e^- into hadrons. The total cross-section has been calculated including terms of order α_s^2 ³⁾. It turns out that the order α_s^2 contribution to the total cross-section is small in the $\overline{MS}$ subtraction scheme, suggesting that perturbation theory may give sensible answers here. If one makes the assumption, based on experimental evidence, that quarks and gluons will show up as jets in the final state, it is also possible to calculate the cross-section for multi-jet production as a perturbation series in α_s . This had been pointed out ⁴⁾ immediately after the discovery ⁵⁾ of a two-jet structure in the hadronic final state.

By now also the occurrence of three jets in e^+e^- annihilation has been established and compared with the theoretical prediction in order α_s . This comparison has given a value for the QCD coupling constant $(g_s^2/4\pi) = \alpha_s \approx 0.17$ at $\alpha_s \approx 30$ GeV ⁶⁾. It is, of course, important to know whether the order α_s^2 contribution to the three-jet cross-section will affect this determination of α_s to a significant extent. We will therefore present in this paper the results of such a calculation and make a comparison with two similar calculations which have appeared recently ⁷⁾⁻⁹⁾.

It is possible to make this study of higher order contributions for all events excepting the two-jet region or for only the three-jet events specifically leaving out events with a larger number of jets. The existence of four-jet events was predicted by DeGrand, Ng and Tye ¹⁰⁾ and has already been calculated ^{11),12)} in lowest order. Using a cut on jet-jet invariant masses we can separate them from the three-jet events. This allows for a much more meaningful comparison between the various orders in perturbation theory.

To regularize the infra-red and collinear divergences we introduce a mass-like parameter for the gluon using the Kinoshita ^{13),14)} prescription while the ultra-violet divergences are renormalized using the $\overline{MS}$ scheme. The details of this regularization method are explained in Section 3 after an inventory of the problem has been made in Section 2. In Section 4 we describe how we calculated the

amplitude of all virtual diagrams using SCHOONSCHIP ¹⁵⁾. We also present the contribution of the virtual diagrams to the matrix element in analytical form. This formula is surprisingly compact.

As the mass scales in jet physics are typically of the order of 5 GeV we noted that a regularization mass parameter of the order of 100 MeV should give answers that are very close to the real values. This enabled us to do the entire radiative integral by Monte Carlo technique without having to split this integral up in a soft piece that is usually integrated analytically and a hard piece of which the integration method is less important. The details of this integration of the radiative diagrams are explained in Section 5.

The results of the full calculation are presented in Section 6 where we show among other things that the Monte Carlo integration does give a good cancellation of the infra-red and collinear divergences without unacceptable loss of accuracy. The conclusions can be found in Section 7.

2. - AN INVENTORY OF THE PROBLEM

To calculate the reaction $e^+e^- \rightarrow \text{hadrons}$ up to order α_s^2 we have to expand the amplitude $\mathcal{M}$ for this reaction up to order g^4 where g is the strong interaction coupling constant with $g^2 = 4\pi\alpha_s$. Then we can take $|\mathcal{M}|^2$, keep all terms up to α_s^2 and integrate. Since we are not interested in the orientation of three-jet configurations with respect to the initial e^+ and e^- beam directions, we integrate over $d\Omega$ in the centre-of-mass system and find that the matrix element obtained in this way is equivalent to that of a heavy unpolarized virtual photon decaying into hadrons except for over-all constants. This has already been explained in Ref. 12). We will therefore only study the reaction $\gamma^* \rightarrow \text{hadrons}$.

Another simplification is made by not considering the α_s^2 corrections to the reaction $\gamma^* \rightarrow q\bar{q}$, the reason being that having performed the $d\Omega$ integral in the CM system, these corrections can be expressed by one number that can be obtained by subtracting the three-jet and four-jet cross-sections from the total cross-section in order α_s^2 . Consequently we only have to consider the lowest order diagrams of Fig. 1, the virtual one-loop diagrams of Fig. 2, the radiative diagrams for the $q\bar{q}q\bar{q}$ final state in Fig. 3 and the radiative diagrams for the $q\bar{q}q\bar{q}$ final state of Fig. 4.

The diagrams shown in Fig. 2 are divergent. Their divergences fall into three classes. First there are the ultra-violet divergences which have to be removed by the usual renormalization. Here we will adopt the $\overline{\text{MS}}$ scheme to accomplish this (see also Section 3). Second we have the infra-red divergences when the momentum through an internal gluon propagator becomes zero. Finally there is the divergence when the internal gluon momentum becomes parallel to one of the external lines. This divergence is called either a collinear divergence or a mass singularity.

The infra-red and collinear divergences must be compensated by similar divergences in the radiative diagrams of Figs. 3 and 4. The processes depicted by these diagrams have four particles in the final state. When one of the gluons of Fig. 3 becomes soft or is parallel to one of the other external particles, divergences of the same type as in the virtual diagrams develop. This also happens when the two quarks radiated by the gluon in Fig. 4 become parallel. The sum of the interferences of Figs. 1 and 2 with the radiative contributions of Figs. 3 and 4 has to be finite.

We turn now to a more detailed description of the regularization procedure.

3. - REGULARIZATION

Nowadays a very popular regularization scheme for QCD calculations is the n dimensional method¹⁶⁾. Among its advantages one counts preservation of gauge invariance and the fact that it regularizes both the ultra-violet, the infra-red and the collinear divergences simultaneously. Even though we will employ in this paper a hybrid scheme we will depend on these properties of n dimensional regularization. This hybrid scheme has already been used in its basic form by previous authors¹⁴⁾ but since the correctness of the calculation is critically dependent on various details we will describe it here to some depth.

First we introduce n dimensions with $n \neq 4$. This causes all diagrams to be finite. Because of this it is now possible to subtract the ultra-violet divergences by one of the various subtraction schemes. In the name of simplicity and standardization we decided to use the $\overline{\text{MS}}$ subtraction scheme¹⁷⁾. After this has been done we would like to introduce a mass parameter λ which will be used to regularize the infra-red and collinear divergences. To do this, however, after the

the Feynman parameter integrals have been done to extract the ultra-violet poles, is very difficult therefore we will introduce λ at the beginning of the calculation. The difference due to this will cause the appearance of some terms in $\lambda^2/n-4$; we should remember that these terms need some special attention.

The way λ is introduced is entirely in accordance with the Kinoshita double cut rule¹³⁾. In our case this means that we add a term $-\lambda^2$ to the denominator of each gluon propagator. As long as we keep $n \neq 4$ it is clear that the limit $\lambda \rightarrow 0$ is trivial and leads to the formulae that belong to the pure n dimensional regularization. It should also be noted that since λ does not really represent a gluon mass - it does not introduce new and unphysical polarizations - the only gauge breaking effects will be at most of order λ^2 , so they will vanish in the limit $\lambda \rightarrow 0$. At this point we would like to exchange the order of the limits $n \rightarrow 4$ and $\lambda \rightarrow 0$. Before we can do this, however, we have to deal with those terms for which the order of the limits makes a difference. These are the $\lambda^2/n-4$ terms. When we exchange the limits these terms will lead to mass counterterms. Actually, these terms should not be present in the first place as originally λ goes to zero first. Correspondingly the mass counterterms are not really present - they do not enter in the Lagrangian - and we will call them "pseudo mass counterterms".

After this has been sorted out it is possible to exchange the n and λ limits, do the ultra-violet subtraction and let n go to 4. Now the infra-red and collinear divergences of the individual diagrams are represented by terms proportional to powers of $\ln\lambda$. The double cut rule says that all terms with a $\ln\lambda$ in them should cancel when adding up all formulae that are obtained by cutting one bubble, with its $-\lambda^2$ insertions, in all possible ways. A bubble is here the type of diagram which is obtained after squaring the amplitude. In our case it is sufficient to consider only those cuts that lead to the diagrams of Figs. 1-4, as we are not interested in the two-jet region with the $q\bar{q}$ final state.

From this point on we can proceed with calculating in four instead of n dimensions. This greatly simplifies the phase space integrals of the final states containing four particles, i.e., the radiative graphs. The full benefits of this are explained in Chapter 5.

The only ambiguities left at this point are with respect to the various integrals that may still go through a divergence if we are not careful. Therefore we will now discuss these. The first one concerns the four-quark final state. If the gluon that "decays" into a $q\bar{q}$ pair has a propagator with a denominator that

looks like $W^2 - \lambda^2$ and the $q\bar{q}$ mass W can go down to zero because the quark mass is assumed to be zero, one has to integrate through a pole. To avoid this we replace the denominator of the gluon propagator by $W^2 + \lambda^2$ in all bubble structures that can contribute to the four-quark final state via their different cuts. This can be done because λ is just a formal parameter to regularize an amplitude, not a real mass. After having performed the integrals we choose the analytic continuation for $\lambda^2 \rightarrow -\lambda^2$ as we would like to have all our formulae for a positive λ^2 . This causes differences in terms with an odd power of λ^2 (which do not contribute anyway to the limit $\lambda \rightarrow 0$), and in terms with a $\ln \lambda^2$ as now $\ln \lambda^2 \rightarrow (\ln \lambda^2 \pm i\pi)$. As in taking $|\mathcal{M}|^2$ only a real part survives there is no difference for terms with one power of $\ln \lambda^2$, but there is a difference for terms with higher powers of $\ln \lambda^2$ as $\ln^2 \lambda^2$ will go to $(\ln^2 \lambda^2 - \lambda^2)$ under the replacement $\lambda^2 \rightarrow -\lambda^2$. Luckily we do not have to worry about this here as in the three-jet limit, the four-quark final state contains only one power of $\ln \lambda^2$. For the $q\bar{q}gg$ final state the above problem is not relevant due to the mass of the external gluons.

The other difficulties that are left concern the proper regularization of the diagrams contributing to the virtual amplitude. Let us take for example the second diagrams of Fig. 2. Its Feynman parameter integral can be written as

$$\int_0^1 \int_0^1 \int_0^1 \frac{z(1-z) dx dy dz}{(T)^2} \quad (1)$$

with

$$T = (x\gamma s + x(1-\gamma)s_7 + \gamma(1-x)s_6)z(1-z) - \lambda^2(1-x)(1-z) + \mu^2\gamma(1-\gamma)z^2$$

Here $s_7 = (p_q + p_{\bar{q}})^2$, $s_6 = (p_{\bar{q}} + p_g)^2$, λ is the mass term on the internal gluon and μ is the mass term of the external gluon, which we keep different from λ during the analytic continuations. In the end we will take $\mu = \lambda$ but if we do this right away it becomes impossible to avoid T having a zero in the integration region. The way this is avoided is that s , s_1 and s_2 have to be taken space-like temporarily and we also take μ^2 negative for the moment. Now T is negative in the whole integration region and the integral can be done, giving a real answer. Finally we turn s , s_1 and s_2 back to the timelike region and replace μ^2 by $-\lambda^2$. In principle we should have also taken an imaginary mass for one of the gluons in the corresponding radiative graph but as we only study the three-jet limit we can keep this gluon from approaching any poles, so its

exact mass is not important. Also the regulator mass of the external gluon is not really necessary for the whole amplitude of the diagram but only for its scalar loop. In the whole diagram all terms in $\ln \mu^2$ cancel, so whatever we did with μ^2 is not really important. The treatment of the other virtual graph involves a similar analytic continuation to obtain their final answers.

4. - VIRTUAL DIAGRAMS

Using the regularization-renormalization scheme described in the previous chapter we must calculate the diagrams of Fig. 2. The loop momentum integrals are not just scalar integrals (i.e., integrals that would occur in a theory like ϕ^3 or ϕ^4) but may have powers of the loop momentum in the numerator. To simplify the calculation we have used the reduction technique described in Ref. 18) in order to rewrite all occurring loop integrals in terms of scalar two-point, three-point and four-point functions. As in Ref. 18) one has to invert several matrices containing kinematical quantities. In principle one should take for these matrices the kinematics where $p_{\text{gluon}}^2 = \lambda^2$ but it can easily be seen that if the coefficients are expanded around $\lambda^2 = 0$, the resulting terms of $O(\lambda^2)$ will not contribute to the limit $\lambda^2 \rightarrow 0$. Once this is realized it is rather straightforward to write a SCHOONSCHIP program for this reduction to scalar loop integrals.

The necessary scalar loop integrals were done by hand and are given in Appendix A. To obtain a cross check on the ever dangerous π^2 terms we also did the Feynman parameter integral numerically for some points in phase space, using the analytical continuation technique explained in the previous chapter.

After we obtained the full expression for the amplitude of all diagrams of Fig. 2, leaving out all terms that vanish in the limit $\lambda \rightarrow 0$, we added them up and checked whether gauge invariance was still present. Each of the replacements of the polarization vector ϵ_γ by p_γ or ϵ_g by p_g gave zero indicating that any gauge breaking effects due to the introduction of the gluon mass stay confined to terms that do not contribute in the limit $\lambda \rightarrow 0$. After this we multiplied the diagrams of Fig. 2 with those of Fig. 1, performed the spin sums and took the traces.

We define the kinematical variables as follows. The quark and the antiquark four-momenta are given by p_1 and p_2 respectively, whereas the gluon momentum will be denoted by p_3 . The over-all four-momentum of the final state

is $p_4 = p_1 + p_2 + p_3$ so that $s = p_4^2$. We also define the invariants $s_5 = p_5^2 = (p_1 + p_3)^2$, $s_6 = p_6^2 = (p_2 + p_3)^2$ and $s_7 = p_7^2 = (p_1 + p_2)^2$. Using these invariants we can express the matrix element entirely in terms of s and the following scaled quantities :

$$\begin{aligned} x_5 &= s_5/s \\ x_6 &= s_6/s \\ x_7 &= s_7/s \\ \epsilon &= \lambda^2/s \end{aligned} \quad (2)$$

We also use the shorthand notation $x_{ij} = x_i + x_j$. In this way the order α_s matrix element is given by

$$T = 16 \frac{x_5^2 + x_6^2}{x_5 x_6} \quad (3)$$

up to kinematical factors and coupling constants. The entire contribution of the interference of the diagrams of Fig. 1 with those of Fig. 2 can now be given as

$$V = -T A - B(x_5, x_6, x_7) - B(x_6, x_5, x_7) \quad (4)$$

with

$$\begin{aligned} A &= \frac{1}{3} (19 - \frac{2}{3} n_f - \pi\sqrt{3} + 12\pi^2 + (33 - n_f)\Delta + 4 L_{56} - 16 L_{57} - 16 L_{67} \\ &+ \ln^2 \frac{\epsilon}{x_5 x_6 x_7} - 10 \ln^2 \frac{\epsilon}{x_5 x_6} - 8 \ln^2 \epsilon - (19 - n_f) \ln \epsilon) \end{aligned} \quad (5)$$

$$\begin{aligned} B(x_5, x_6, x_7) &= \frac{1}{3} \left\{ 160 \frac{x_5 x_7}{x_6} - 32 \frac{x_7}{x_5 x_6} + 128 \frac{x_7}{x_5 x_7} \right. \\ &+ \frac{32}{x_6 x_7} \left(4 \frac{x_6 x_7}{x_5 x_7} + 13 x_6 + 16 x_7 \right) \ln x_6 \\ &- 32 \left(\frac{x_7^2}{x_5 x_6^2} + 2 \frac{x_7}{x_5 x_6} \right) \ln x_7 \\ &\left. - 32 \left(\frac{x_6 + 2 x_7}{x_5} \right) \left(\ln x_6 \ln x_7 + L_{56} + L_{57} - \frac{\pi^2}{6} \right) \right\} \end{aligned} \quad (6)$$

The variable Δ is defined by $\Delta = -2/(n-4) - \gamma + \ln 4\pi$ but is set equal to zero once the $\overline{\text{MS}}$ scheme is used to renormalize the ultra-violet divergences. The notation L_{ij} stands for $\text{Li}_2(x_i+x_j)$. The number of flavours is represented by n_f . In practice we took the terms proportional to n_f separately as their divergence in $\ln \epsilon$ cancels the divergence of the radiative diagrams with four quarks in the final state. As these radiative diagrams have to be treated on their own (all quarks are taken massless while the gluons have an effective mass) we may as well have the extra check of the separate cancellation.

The final expression for the double differential cross-section of the interference terms becomes now :

$$\frac{d^2\sigma}{dE_q dE_{\bar{q}}} = \sigma_0 \frac{\alpha_s^2}{\pi^2} \frac{V}{3845} \quad (7)$$

σ_0 is here the lowest order cross-section for $e^+e^- \rightarrow q\bar{q}$. It is given by :

$$\sigma_0 = \frac{4\pi\alpha^2}{s} \sum_i Q_i^2 \quad (8)$$

The sum is over all contributing flavours.

5. - THE RADIATIVE DIAGRAMS

As explained in Chapter 3 we can evaluate all phase space integrals in four dimensions. This leads to great simplifications as we will explain now.

When looking at experimental results from PETRA it becomes clear that jet masses are of the order of 5 GeV at $s = 10^3$. In such systems it should make only very little difference if we give the gluon a mass of the order of 100 MeV to prevent the infra-red and collinear divergences. If we assume that individual jets have, at this value of s , a mass of up to 5 GeV then the effect of $\lambda = 100$ MeV is expected to be of the order of 2%, once the terms in $\ln\lambda^2$ have been cancelled. This means that we can allow for greater regulator masses than is usual in QED calculations. If for such values of λ the cancellations between the virtual and the radiative contributions do not cause too great a loss of numerical accuracy, it is entirely feasible to do the full radiative integral by Monte Carlo technique, rather than first splitting phase space up into pieces with soft,

collinear or hard radiation and doing the soft and collinear integrals analytically. It is indeed possible to argue that these cancellations should be relatively mild. The argument goes as follows.

The total cross-section in order α_s^2 is rather small in the $\overline{MS}$ scheme ³⁾ : $(1.98 - 0.115 n_f) \cdot \sigma_0 (\alpha_s/\pi)^2$ or $1.98 - 0.115 n_f$ in units of $\sigma_0 (\alpha_s/\pi)^2$. For a rather reasonable cut to separate four-jet events from two- and three-jet events one obtains that the four-jet events can easily give cross-sections as large as 10 in units of $\sigma_0 (\alpha_s/\pi)^2$ ¹²⁾. This means that the contributions of the two- and three-jet configurations have to cancel this number almost entirely and we expect the three-jet and two-jet parts to be of order 10 each, ignoring for the moment the extra cancellations between them. For values of the regulator mass λ of about 0.1 to 1.0 GeV and s of about 10^3 , $\ln^2(\lambda^2/s)$ is of the order 100 and so a first expectation is that the cancellation between the virtual diagrams and the radiative diagrams in the three-jet region will make us lose only one order of magnitude in accuracy. As we do not need our final cross-sections to be more accurate than about 1% we can do the radiative integral entirely by Monte Carlo if it is possible to obtain an accuracy of the order of 0.1% for them in the three-jet region. That this is certainly possible and that indeed the cancellation takes place according to the above estimates will be shown in the next chapter. For the rest of this chapter we will concentrate on technical and kinematical aspects of the calculation.

The evaluation of the various matrix elements was done with the help of SCHOONSCHIP and is rather similar to the matrix elements of Refs. 11), 12) except for the mass terms in the denominator of the gluon propagators and the external gluon masses. The phase space integration was rewritten such that the three invariants that involve one of the gluons (say gluon 1 with momentum p_3) with the quark, the antiquark and the other gluon, respectively, are among the integration variables. Calling these variables y_1 , y_2 and y_3 , respectively, we can compensate for the fact that $d\sigma/dy_1$ peaks sharply for small y_1 by a change of variables. For example for y_1 this change is to use $\ln(y_1 - \lambda^2)$ as an integration variable instead of y_1 , as the peak at small y_1 is effectively proportional to $1/y_1$ with a lower boundary on the integration of λ^2 . To avoid extra peaks in other regions of phase space, connected to the other gluon becoming soft or collinear with one of the quarks we calculated for each Monte Carlo event the four variables $(p_{g1}+p_q)^2$, $(p_{g1}+p_{\bar{q}})^2$, $(p_{g2}+p_q)^2$ and $(p_{g2}+p_{\bar{q}})^2$. If the smallest of these four numbers belongs to gluon 1 (i.e., one of the first two numbers) we accept the point, otherwise we reject it. This will yield exactly half the cross-section, and due to the fact that gluons are indistinguishable, all differential distributions will be correct up to this over-all factor $\frac{1}{2}$.

The two-jet region with the double infra-red problems also has to be cut out. This can be done in a variety of ways but some caution is needed. One way is to implement a cut on the thrust by rejecting any configuration with a thrust larger than a given number $t^{\max}$. One can of course use any other variable that can distinguish two jets from three jets in an infra-red safe way, like the variable C in Ref. 7). Another method is to count the number of jets. This can be done using the scaling cuts of Ref. 12). The idea is that an event with n final state particles qualifies as an n jet event if all jet-jet invariant masses divided by $\sqrt{s}$ are larger than some dimensionless parameter ξ with $0 < \xi < \sqrt{2/n(n-1)}$. This way it can first be checked whether an event is a clear four-jet event (these could be treated separately). If an event is not a four-jet event we look which of the variables y_1, y_2 and y_3 is smallest. If this is, for instance, y_1 we consider the quark and gluon 1 together as a one-massive-particle. As we have now only three particles left in the final state we can check whether the event can be classified as a three-jet event using the same variable ξ . If this is not the case the event has to be a two-jet event and should be cut out.

Of course, the various methods like cuts in the thrust, in the variable C , in ξ or in the Sterman-Weinberg¹⁹⁾ variables ϵ and δ which are used in Ref. 9) are all related to some extent. For instance, the statement that a fraction ϵ of the energy is deposited in a cone with angle δ is closely related to putting a limit on the fraction of the total invariant mass that is projected into this cone, assuming that the masses of the observed physical particles (pions, etc.) can be ignored. We believe, however, that the use of a single cut parameter on variables that are directly proportional to the denominators of the propagators that cause the divergences is most useful for a theoretical analysis.

For the kinematics we used the formula :

$$\frac{1}{(2\pi)^8} \frac{d^3 p_q}{2E_q} \frac{d^3 p_{\bar{q}}}{2E_{\bar{q}}} \frac{d^3 p_{g_1}}{2E_{g_1}} \frac{d^3 p_{g_2}}{2E_{g_2}} \delta^4(p_q + p_{\bar{q}} + p_{g_1} + p_{g_2} - P_{in}) =$$

$$\frac{1}{2048\pi^6 s} \int \frac{dy_1 dy_2 dy_3 dy_4 dy_5}{\sqrt{-\Delta_4(p_q, p_{\bar{q}}, p_{g_1}, p_{g_2})}} \quad (9)$$

with $y_4 = (P_{in} - p_q)^2$ and $y_5 = (P_{in} - p_{\bar{q}})^2$. The boundaries of the integrals can be derived from the condition that the Gram determinant Δ_4 has to be negative. They are given in Appendix B.

The Monte Carlo integration was done with the help of the routine VEGAS ²⁰⁾ in a slightly modified form such that it is possible to simultaneously obtain a total cross-section and a number of differential cross-sections in histogram form. The main advantage of VEGAS lies in the fact that it is very good at handling the discontinuities introduced by the cuts.

The calculation of the four-quark final state can be simplified considerably by applying some symmetry arguments. First we split the diagrams of Fig. 4 up in four groups A, B, C, and D, being the first pair of diagrams, the second pair, etc. These pairs are related to each other by exchanges of external quark lines. As we cannot distinguish the various quarks and antiquarks in any observable distributions we can use this exchange symmetry to simplify simultaneously the matrix element and its divergent structure. It is, for example, only necessary to take $4|A|^2$ instead of $|A|^2 + |B|^2 + |C|^2 + |D|^2$. Similarly $AB^\dagger + BA^\dagger$ gives the same results as $CD^\dagger + DC^\dagger$. Finally the quantities $AC^\dagger + CA^\dagger$, $AD^\dagger + DA^\dagger$, $BC^\dagger + CB^\dagger$ and $BD^\dagger + DB^\dagger$ lead to exactly the same contributions to all distributions we can construct of observable quantities. Also $AB^\dagger + BA^\dagger$ always gives a zero result if one cannot measure charge asymmetries so that $4(|A|^2 + AC^\dagger + CA^\dagger)$ is sufficient instead of $|A+B+C+D|^2$. Apart from having a much shorter formula to integrate over we have now also confined the strongest divergences to small values of the invariant mass of quarks 3 and 4 together.

6. - RESULTS

The first thing to be done is, of course, to check the cancellation of the infra-red and collinear divergences as this will tell us whether the method of Monte Carlo integration over the whole of phase space - excluding, of course, the two-jet region - is feasible. Although all our calculations are only sensitive to scaled variables like λ^2/s we decided to fix s at 900 GeV^2 , a representative value for both PETRA and PEP. This way the various values we take for λ can be easier compared with what we know about jets from e^+e^- physics. It should, however, be noted that all our results are completely scale-independent as we always calculate $(1/\sigma_0)(\pi^2/\alpha_s^2)\sigma(\xi, \lambda^2/s)$ for $O(\alpha_s^2)$ terms and $(1/\sigma_0)(\pi/\alpha_s)\sigma(\xi)$ for $O(\alpha_s)$ terms.

From Table 1 and Fig. 5, it is clear that the approach to the limit $\lambda \rightarrow 0$ is linear with λ for the $q\bar{q}gg$ final state with its corresponding virtual contributions. The limit value is 94.0 ± 0.3 . The cancellations are clearly not very severe as even for $\lambda = 30 \text{ MeV}$ we lose less than one order of magnitude in

accuracy. To stay away from the two-jet region we took here a cut of $\xi = 0.25$ which corresponds to an upper limit of 7.5 GeV on single jet masses at $\sqrt{s} = 30$ GeV. For the same value of ξ , Table 2 and Fig. 6 show that the four-quark final state with its virtual contributions approaches its limit roughly like λ^2 and the limit value is -19.5 ± 0.1 . This assumed that the number of contributing flavours is 5. One could, of course, argue that the number of contributing flavours should be more like 4 as it is rather hard to make a massive pair of bottom quarks from a virtual gluon but this would only diminish the absolute value of the four-quark result and give a total $O(\alpha_s^2)^2$, which is now already 74.5 ± 0.3 , that is up to a few per cent bigger. The real effective number of flavours lies, of course, somewhere between 4 and 5 but since the exact value of this number hardly affects our results we will ignore this and work with five flavours. The total $O(\alpha_s)$ contribution for $\xi = 0.25$ is found to be 4.55 [in units of $\sigma_0(\alpha_s/\pi)$], so the contributions of the first and the second order corrections to $e^+e^- \rightarrow \text{hadrons}$, excluding the two-jet region and ignoring fragmentation effects, are equal for a value of $\alpha_s = 0.192 \pm 0.001$. Clearly we are not talking about small corrections here as this is believed to be a reasonable value for α_s at PETRA and PEP energies.

The different approach to the limit $\lambda \rightarrow 0$ for the final states $q\bar{q}g\bar{g}$ resp. $q\bar{q}q\bar{q}$ can be traced back to the kinematics of their four-body final states. The difference arises from the fact that we take all quarks to be massless while all gluons behave as if they have a mass λ . This mass introduces terms in the phase space Jacobian that are effectively linear in λ in the region where the matrix element is large.

To see how cut-dependent this α_s^2 contribution is we calculated the observable three- and four-jet cross-sections in the limit $\lambda^2 \rightarrow 0$ as a function of the jet separation parameter ξ . The results are given in Fig. 7 together with the three-jet cross-section in order α_s . From Fig. 7 and the differential thrust distribution in Fig. 8, it is clear that the ratio of α_s^2 contributions to α_s cross-sections for three- (and four-) jet events is rather cut-off independent and is mainly a function of α_s itself.

One could now, of course, wonder whether the event shapes will undergo great changes too. From the ξ dependence and the thrust distribution it is already clear that at least some distributions are left relatively unchanged except for their over-all normalization. However, the ultimate test is, of course, a Dalitz plot for three-jet events. We know already that four-jet events are shaped differently so we will exclude them from the Dalitz plot via a $\xi = 0.25$ cut and only consider three-jet events taking again the limit $\lambda^2 \rightarrow 0$. The

Dalitz plot is given in Fig. 9 in terms of a histogram. Both for the α_s (top numbers in each square) as for the α_s^2 (bottom numbers) contributions we normalized the numbers to 1000 generated events (up to round off errors). The jets were ordered according to energy as up to now it is not reliably possible to tell on an event-by-event basis which jet belongs to the gluon, the quark or the anti-quark.

The main difference between the two distributions lies in the fact that the α_s^2 distribution is slightly shifted to the right. This is due to the fact that when the four-particle final state of the radiative diagrams behaves like a three-jet event one of its jets is massive (up to $\xi\sqrt{s}$). It is, however, expected that the addition of a fragmentation process will reduce this already quite small effect even more.

7. - CONCLUSIONS

The first conclusion we can draw from our results is that up to order α_s^2 perturbative QCD in the $\overline{MS}$ scheme does not show any signs of convergence when studying differential distributions. This is in agreement with the paper by Ellis, Ross and Terrano ⁷⁾ who reached the same conclusion in a calculation that involved n dimensional regularization to remove all divergences simultaneously. We also calculated the differential distributions of their variables C and D and agree with them. On the other hand we do not agree with the paper by Fabricius, Schmitt, Schierholz and Kramer ⁹⁾ who concluded that there exists a reasonable convergence for the perturbative expansion in the event shape.

The fact that the event shape is hardly altered by the large corrections means that the observable three-jet cross-section looks like the lowest order three-jet cross-section but is proportional to some complicated function of α_s . The precise form of this function depends, of course, on the three-jet criterion. For $\xi = 0.25$ it is up to α_s^2 : $(\alpha_s/\pi)(1+(\alpha_s/0.192))$. We expect a similar thing to happen to four-jet cross-sections when studied in higher order so comparison of four-jet cross-sections with three-jet and two-jet cross-sections will only give results for the full proportionality functions of α_s . It is, of course, meaningless to apply such a comparison only up to order α_s^2 .

One could, of course, try to choose a subtraction scheme that is different from $\overline{MS}$ such that the three-jet correction becomes small by construction. It is, however, not clear how this should be done without giving a large number

for the full cross-section in α_s^2 . If the full cross-section in $O(\alpha_s^2)$ is large and the three- and four-jet cross-sections are relatively small this would still leave a large correction to the two-jet cross-section, so there would not be a fundamental improvement.

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$\lambda(\text{GeV})$	Radiative	Virtual	Total
0.03	+723.3 $\pm$ 0.8	-629.72 $\pm$ 0.05	+93.6 $\pm$ 0.8
0.05	+592.9 $\pm$ 0.5	-499.82 $\pm$ 0.04	+93.1 $\pm$ 0.5
0.10	+437.7 $\pm$ 0.4	-345.08 $\pm$ 0.03	+92.6 $\pm$ 0.4
0.15	+357.7 $\pm$ 0.3	-266.06 $\pm$ 0.02	+91.6 $\pm$ 0.3
0.20	+306.2 $\pm$ 0.6	-215.14 $\pm$ 0.02	+91.1 $\pm$ 0.6
0.30	+239.9 $\pm$ 0.5	-150.63 $\pm$ 0.01	+89.3 $\pm$ 0.5
0.50	+166.6 $\pm$ 0.6	- 81.43 $\pm$ 0.01	+85.2 $\pm$ 0.6
1.0	+ 85.3 $\pm$ 0.3	- 9.03 $\pm$ 0.01	+74.3 $\pm$ 0.3

Table 1 - Total cross-sections from the radiative diagrams contributing to the $q\bar{q}g\bar{g}$ final state and the corresponding virtual diagrams for various values of λ , in units of $(\alpha_s/\pi)^2\sigma_0$. The cut to stay away from the two-jet region was $\xi = 0.25$.

$\lambda(\text{GeV})$	Radiative	Virtual	Sum
0.1	+26.33 $\pm$ 0.08	-45.81 $\pm$ 0.01	-19.48 $\pm$ 0.08
0.5	+14.24 $\pm$ 0.04	-33.62 $\pm$ 0.01	-19.38 $\pm$ 0.04
1.0	+ 9.30 $\pm$ 0.03	-28.42 $\pm$ 0.01	-19.12 $\pm$ 0.03
2.0	+ 4.99 $\pm$ 0.02	-23.29 $\pm$ 0.01	-18.30 $\pm$ 0.03
3.0	+ 3.02 $\pm$ 0.01	-18.31 $\pm$ 0.01	-15.29 $\pm$ 0.02

Table II - Total cross-sections from the radiative diagrams contributing to the four-quark final state and the corresponding virtual diagrams for various values of λ in units of $(\alpha_s/\pi)^2\sigma_0$. The cut to stay away from the two-jet region was $\xi = 0.25$.

APENDIX A

The various scalar loops that were needed to calculate all diagrams of Fig. 1 are presented here in the notation that is explained in the text. They are :

$$\begin{aligned}
 & \int \frac{d^4 q}{(q+p_1)^2 (q^2-\lambda^2) ((q-p_3)^2-\lambda^2) (q-p_2-p_3)^2} = \\
 & \quad \frac{i\pi^2}{s_5 s_6} \left[\ln^2 \left(\frac{\epsilon}{x_5 x_6} \right) + L_{57} + L_{67} - \frac{8\pi^2}{9} - 2 \ln(-1) \ln \frac{\epsilon}{x_5 x_6} \right] \\
 & \int \frac{d^4 q}{(q+p_1)^2 (q^2-\lambda^2) (q-p_2)^2 (q-p_2-p_3)^2} = \\
 & \quad \frac{i\pi^2}{s_6 s_7} \left[\frac{1}{2} \ln^2 \left(\frac{\epsilon}{x_6 x_7} \right) + \ln \frac{\epsilon}{x_6 x_7} \ln \frac{\epsilon}{x_6} + L_{57} - \frac{\pi^2}{3} - \ln(-1) \left(\ln \frac{\epsilon}{x_6 x_7} + \ln \frac{\epsilon}{x_6} \right) \right] \\
 & \int \frac{d^4 q}{(q+p_1+p_3)^2 (q+p_1)^2 (q^2-\lambda^2) (q-p_2)^2} = \\
 & \quad \frac{i\pi^2}{s_5 s_7} \left[\frac{1}{2} \ln^2 \left(\frac{\epsilon}{x_5 x_7} \right) + \ln \frac{\epsilon}{x_5 x_7} \ln \frac{\epsilon}{x_5} + L_{67} - \frac{\pi^2}{3} - \ln(-1) \left(\ln \frac{\epsilon}{x_5 x_7} + \ln \frac{\epsilon}{x_5} \right) \right] \\
 & \int \frac{d^4 q}{(q+p_1)^2 (q^2-\lambda^2) ((q-p_3)^2-\lambda^2)} = \frac{i\pi^2}{s_6} \left[\frac{1}{2} \ln^2 \frac{\epsilon}{x_6} - \frac{5\pi^2}{18} - \ln(-1) \ln \frac{\epsilon}{x_6} \right] \\
 & \int \frac{d^4 q}{(q+p_1)^2 (q^2-\lambda^2) (q-p_2-p_3)^2} = \frac{i\pi^2}{s_5 s_7} \left[\ln \epsilon \ln x_6 - \frac{1}{2} \ln^2 x_6 + L_{57} - \ln(-1) \ln x_6 \right] \\
 & \int \frac{d^4 q}{(q+p_1)^2 ((q-p_3)^2-\lambda^2) (q-p_2-p_3)^2} = \frac{i\pi^2}{s_6 s_7} \left[\ln \epsilon \ln x_5 - \frac{1}{2} \ln^2 x_5 + L_{67} - \ln(-1) \ln x_5 \right] \\
 & \int \frac{d^4 q}{(q^2-\lambda^2) ((q-p_3)^2-\lambda^2) (q-p_2-p_3)^2} = \frac{i\pi^2}{s_5 s_7} \left[\frac{1}{2} \ln^2 \frac{\epsilon}{x_5} - \frac{5\pi^2}{18} - \ln(-1) \ln \frac{\epsilon}{x_5} \right] \\
 & \int \frac{d^4 q}{(q+p_1)^2 (q^2-\lambda^2) (q-p_2)^2} = \frac{i\pi^2}{s_7} \left[\frac{1}{2} \ln^2 \frac{\epsilon}{x_7} - \frac{\pi^2}{6} - \ln(-1) \ln \frac{\epsilon}{x_7} \right] \\
 & \int \frac{d^4 q}{(q+p_1)^2 (q-p_2)^2 (q-p_2-p_3)^2} = \frac{i\pi^2}{s_6 s_6} \left[\ln \epsilon \ln x_7 + 2 L_{56} \right] \\
 & \int \frac{d^4 q}{(q^2-\lambda^2) (q-p_2)^2 (q-p_2-p_3)^2} = \frac{i\pi^2}{s_6} \left[\ln^2 \frac{\epsilon}{x_6} + \frac{\pi^2}{6} - \ln(-1) \ln \frac{\epsilon}{x_6} \right] \\
 & \int \frac{d^4 q}{(q+p_1+p_3)^2 (q+p_1)^2 (q^2-\lambda^2)} = \frac{i\pi^2}{s_5} \left[\ln^2 \frac{\epsilon}{x_5} + \frac{\pi^2}{6} - \ln(-1) \ln \frac{\epsilon}{x_5} \right] \\
 & \int \frac{d^4 q}{(q+p_1+p_3)^2 (q+p_1)^2 (q-p_2)^2} = \frac{i\pi^2}{s_5 s_6} \left[\ln \epsilon \ln x_7 + 2 L_{56} \right] \\
 & \int \frac{d^4 q}{(q+p_1+p_3)^2 (q^2-\lambda^2) (q-p_2)^2} = \frac{i\pi^2}{s_6 s_7} \left[\ln \epsilon \ln x_5 - \frac{1}{2} \ln^2 x_5 + L_{67} - \ln(-1) \ln x_5 \right]
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{d^n q}{(q+p_1)^2 (q^2-\lambda^2)} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' - \ln \epsilon + 1] \\
 \int \frac{d^n q}{(q+p_1)^2 ((q-p_3)^2-\lambda^2)} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' - \ln x_5 + 2 - \ln(-1)] \\
 \int \frac{d^n q}{(q+p_1)^2 (q-p_2-p_3)^2} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' + 2 - \ln(-1)] \\
 \int \frac{d^n q}{(q^2-\lambda^2)((q-p_3)^2-\lambda^2)} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' - \ln \epsilon + 2 - \frac{\pi}{\sqrt{3}}] \\
 \int \frac{d^n q}{(q^2-\lambda^2)(q-p_2-p_3)^2} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' - \ln x_6 + 2 - \ln(-1)] \\
 \int \frac{d^n q}{((q-p_3)^2-\lambda^2)(q-p_2-p_3)^2} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' - \ln \epsilon + 1] \\
 \int \frac{d^n q}{(q+p_1)^2 (q-p_2)^2} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' - \ln x_7 + 2 - \ln(-1)] \\
 \int \frac{d^n q}{(q-p_2)^2 (q^2-\lambda^2)} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' - \ln \epsilon + 1] \\
 \int \frac{d^n q}{(q-p_2)^2 (q-p_2-p_3)^2} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' - \ln \epsilon + 2 - \ln(-1)] \\
 \int \frac{d^n q}{(q+p_1+p_3)^2 (q+p_1)^2} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' - \ln \epsilon + 2 - \ln(-1)] \\
 \int \frac{d^n q}{(q+p_1+p_3)^2 (q^2-\lambda^2)} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' - \ln x_5 + 2 - \ln(-1)] \\
 \int \frac{d^n q}{(q+p_1+p_3)^2 (q-p_2)^2} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 [\Delta' + 2 - \ln(-1)] \\
 \int \frac{d^n q}{q^2-\lambda^2} \frac{1}{(2\pi)^{n-4}} &= i\pi^2 \lambda^2 [\Delta' - \ln \epsilon + 1]
 \end{aligned}$$

In these formulae we use $\Delta' = \Delta - \ln s$. The $\ln s$ is to be absorbed by the definition of the coupling constant. The analytic continuations, leading to the imaginary parts and terms proportional to π^2 were always reduced to terms with $\ln(-s)$ after which we wrote $\ln(-s) = \ln s + \ln(-1)$. This defines the $\ln(-1)$ terms.

APPENDIX B

The starting point from where to derive kinematics for a $2 \rightarrow 3$ or a $1 \rightarrow 4$ reaction in terms of invariants exclusively is a relation like formula (9) in the text. The boundaries are then determined by expressing the Gram determinant in terms of masses and invariants and requiring it to be negative definite. This will lead to a polynomial that is quadratic in each of the variables so, taking one variable, one can find for which values it makes the Gram determinant zero. These values are functions of all other variables and define the integration boundaries for this first invariant. The condition that these boundaries have to exist (the discriminant of the quadratic equation has to be larger than zero) will determine the boundaries of the next integration variable. This method can be repeated until all boundaries are known. It should be noted that the discriminants are not always quadratic in all their variables and depending on the order in which one needs the integration variables one may need to solve quartic equations.

The order of integration has to be chosen carefully as it can influence the rate of convergence of the Monte Carlo calculation considerably. The order of integration that one uses in an analytical calculation is not necessarily the optimum order for a Monte Carlo integration. We will illustrate this here. In an analytical integration of the four-particle final states that are encountered in this paper, one would do the integrals over y_1 , y_2 and y_3 first and keep the variables y_4 and y_5 as the last variables. This means that the boundaries of y_1 have to be expressed in terms of y_4 and y_5 among others. In a Monte Carlo calculation, however, one puts heavy emphasis on getting good access to the regions where a variable causes rapid changes of the integrant. As y_1 , y_2 and y_3 can cause a near divergence (only avoided by the presence of a small λ) we would like to express their boundaries in as few variables as possible, so the optimum solution would be to integrate y_4 and y_5 out first. The presence of cuts (to stay away from the two-jet region) and the fact that we use $\ln(y_1 - \lambda^2)$, $\ln(y_2 - \lambda^2)$ and $\ln(y_3 - \lambda^2)$ instead of y_1 , y_2 , y_3 alters the situation again. Now the greatest contributions to the uncertainty in the integral is due to the discontinuities of the cuts which are most prominent in y_4 and y_5 . In the end it turned out that to integrate over $\ln(y_1 - \lambda^2)$, $\ln(y_2 - \lambda^2)$ and $\ln(y_3 - 4\lambda^2)$ first was slightly better due to these cuts, so we used this. The corresponding boundaries are presented below :

$$4\lambda^2 < Y_5 < S$$

$$\frac{4\lambda^2 S}{Y_5} < Y_4 < S$$

$$Y_3^{\max} = \frac{Y_4 Y_5}{S}$$

$$Y_3^{\min} = \max(4\lambda^2, Y_4 + Y_5 - S)$$

$$Y_1^{\max} = \lambda^2 + \frac{1}{2}(Y_5 - Y_3)(1 + \sqrt{1 - \frac{4\lambda^2}{Y_3}})$$

$$Y_1^{\min} = \frac{\lambda^2}{Y_1^{\max}} \left(\lambda^2 - Y_5 + \frac{Y_5^2}{Y_3} \right)$$

$$Y_2^{\max} = \frac{-b + \sqrt{D}}{2a}$$

$$Y_2^{\min} = \frac{c}{a Y_2^{\max}}$$

with

$$D = 16 S Y_3 (Y_1^{\max} - Y_1)(Y_1 - Y_1^{\min})(Y_3^{\max} - Y_3)(S + Y_3 - Y_4 - Y_5)$$

and

$$a = (Y_5 - Y_3)^2$$

$$b = 2(Y_5 - Y_3)((Y_5 - Y_1 - Y_3 + 2\lambda^2)(Y_3 + Y_4) - Y_4 Y_5 - (Y_4 + Y_5)\lambda^2 + Y_3 S) + 4 Y_3 (Y_4 - S)(Y_5 - Y_1 - Y_3 + \lambda^2)$$

$$c = ((Y_5 - Y_1 - Y_3 + 2\lambda^2)(Y_3 - Y_4) + Y_4 Y_5 - (Y_4 + Y_5)\lambda^2 - Y_3 S)^2 + 4\lambda^2(Y_3 S(Y_5 - Y_1 - Y_3 + \lambda^2 - S) + Y_4(Y_5 S + (Y_5 - Y_1 - Y_3 + 2\lambda^2)(Y_3 - Y_4 - Y_5)))$$

The Gram determinant becomes now :

$$-\Delta_4(p_{q_1}, p_{q_2}, p_{q_1}, p_{q_2}) = 16a(Y_2^{\max} - Y)(Y - Y_2^{\min})$$

These formulae are written in such a way that they minimize potential problems with numerical stabilities.

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FIGURE CAPTIONS

- Figure 1 Feynman diagrams for $e^+e^- \rightarrow q\bar{q}g$.
- Figure 2 Feynman diagrams for the virtual contributions in order α_s^2 .
- Figure 3 Feynman diagrams for the reaction $e^+e^- \rightarrow q\bar{q}gg$ in lowest order.
- Figure 4 Feynman diagrams for the reaction $e^+e^- \rightarrow q\bar{q}q\bar{q}$ in lowest order.
- Figure 5 The cross-section for the $q\bar{q}gg$ final state plus its corresponding virtual diagrams as a function of λ with $\xi = 0.25$.
- Figure 6 The cross-section for the $q\bar{q}q\bar{q}$ final state plus its corresponding virtual diagrams as a function of λ with $\xi = 0.25$.
- Figure 7 Cross-sections as a function of the cut-off parameter ξ . All curves have been expressed in terms of scaled variables such that the α_s contribution is $(1/\sigma_0)(\pi/\alpha_s) \cdot \sigma(O(\alpha_s))$ and the α_s^2 contributions are $(1/\sigma_0)(\pi^2/\alpha_s^2) \cdot \sigma(O(\alpha_s^2))$.
- Figure 8 Differential cross-section with respect to thrust. Again the scaling factors $(1/\sigma_0)(\pi/\alpha_s)$ and $(1/\sigma_0)(\pi^2/\alpha_s^2)$ have been used for the $O(\alpha_s)$ resp. $O(\alpha_s^2)$ histograms.
- Figure 9 Dalitz histograms of E_1 v. E_2 with E_1 the energy of the jet with the largest energy and E_2 the energy of the jet with the second largest energy. Normalization is to 1000 generated events and two- and four-jet events have been left out ($\xi = 0.25$). Top numbers represent the distribution for three-jet events in $O(\alpha_s)$ while the bottom numbers are for three-jet events in $O(\alpha_s^2)$.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9