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QUANTUM GASES WITH INTERNAL SYMMETRY

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A B S T R A C T

A statistical thermodynamical description of quantum gases with internal symmetries is given. It appears that an internal symmetry leads to a substantial modification of the grand canonical partition function which should result in a change of many physical properties as compared to the no-symmetry case.

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The development in the theory of strong interactions in the last decade has established a concept of the asymptotic freedom. Strongly compressed hadronic matter behaves in a way typical of an almost ideal quantum gas.

Discussions of critical phenomena, condensation problems, the structure of a vacuum, etc., are performed in the language of statistical thermodynamics.

An ideal gas approximation is useful in model considerations when the interaction can be treated as a small perturbation. However, this approximation would become too drastic if we neglect, without justification, such important properties of the matter as quantum statistics or internal symmetries.

The aim of this paper is to consider an ideal relativistic quantum gas where both an internal symmetry and quantum statistics are explicitly taken into account. With these two properties one can proceed to build up realistic thermodynamics of the hadronic matter. On the basis of the previous considerations ¹⁾, a grand canonical partition function is constructed and this will be a starting point for further research. In the present short paper only the main ideas are given. Exact and complete calculations of the properties of the system will be a subject of a future work.

To begin with I shall summarize the main results of the paper ¹⁾, where a basic formalism for the thermodynamical description of systems with internal symmetries was introduced. More details and examples are also given in Ref. 2).

Let H be the Hamiltonian of our system and G the internal symmetry group. We assume here that G is a compact semi-simple n parameter Lie group and that the symmetry is an exact one. This means that

$$[H, Q_k] = 0 \quad k = 1, \dots, n \quad (1)$$

where Q_k are generators of the group G .

At relativistic energies the number of particles is no longer conserved so a concept of the canonical distribution should be modified. The canonical distribution will no longer be related to the fixed number of particles, but rather to the given irreducible representation of the symmetry group. Let us consider, for example, a gas consisting of particles all of which transform under the same representation $U^\alpha(g)$. Transformation properties of our gas will be

given by some reducible unitary representation $U(g)$ which is a multiple tensor product of the representation $U^\alpha(g)$. We shall obtain states transforming under different irreducible unitary representations of the symmetry group by decomposing the representation $U(g)$ into irreducible ones. We call

$$Z_\nu(\beta, V) := \text{Tr}_\nu e^{-\beta H} \quad (2)$$

a canonical partition function. The trace is taken here over all states which transform under the representation $U^\nu(g)$ and β is the inverse temperature.

The equation of state can be obtained from the canonical partition function (2) in the usual way

$$\beta P = \frac{\partial}{\partial V} \log Z_\nu(\beta, V) \quad (3)$$

$$\epsilon = -V^{-1} \frac{\partial}{\partial \beta} \log Z_\nu(\beta, V) \quad (4)$$

Here P is the pressure and ϵ the energy density. The system is also characterized by the eigenvalues of mutually commuting charges. These eigenvalues characterize the irreducible representation $U(g)$ under which the system transforms.

Let $\hat{\chi}^\nu(g)$ be the character of the representation $U^\nu(g)$. We define the function $\hat{Z}(\beta, g, V)$ as

$$\hat{Z}(\beta, g, V) = \sum_\nu \frac{\hat{\chi}^\nu(g)}{d(\nu)} Z_\nu(\beta, V) \quad (5)$$

where $d(\nu)$ is a dimension of the representation $U^\nu(g)$. With the function $\hat{Z}(\beta, g, V)$ one can calculate canonical partition functions. This can be done by using the orthogonality property of characters

$$\int d\mu(g) \overline{\hat{\chi}^\nu(g)} \hat{\chi}^{\nu'}(g) = \delta_{\nu\nu'} \quad (6)$$

which gives us

$$Z_\nu(\beta, V) = d(\nu) \int d\mu(g) \overline{\hat{\chi}^\nu(g)} \hat{Z}(\beta, g, V) \quad (7)$$

It was shown ^{1),2)} that the function $\hat{Z}(\beta, g, V)$ is closely related to the grand canonical partition function. The function $\hat{Z}(\beta, g, V)$, as a function on a group, depends only on characters and these are defined on classes of conjugated elements. So if the group G is of rank r , then the character $\hat{\chi}^V(g)$ can be considered as a function of the r variables $\gamma_1, \dots, \gamma_r$. These variables are parameters of the Cartan subgroup of the group G . To pass to the grand canonical partition function Z , one should perform the "Wick rotation" : $\gamma_k \rightarrow -i\beta\mu_k$. μ_k then become chemical potentials corresponding to the conserved charges Q_k which specify the system. We have

$$\begin{aligned}\hat{Z}(\beta, g, V) &= \hat{Z}(\beta, \gamma_1, \dots, \gamma_r, V) \\ Z(\beta, \mu_1, \dots, \mu_r, V) &= \hat{Z}(\beta, -i\beta\mu_1, \dots, -i\beta\mu_r, V)\end{aligned}\quad (8)$$

A relation of the function $\hat{Z}(\beta, g, V)$ to the grand canonical partition function will become obvious if one notices that the latter can be written as

$$Z(\beta, \mu_1, \dots, \mu_r, V) = \text{Tr} \exp \left[-\beta \left(H - \sum_{k=1}^r \mu_k Q_k \right) \right] \quad (9)$$

where Q_k is a set of conserved and mutually commuting charges corresponding to the internal symmetry group. The trace is taken over all physical states of the system.

On the other hand, due to the exact symmetry, the function $\hat{Z}(\beta, g, V)$ can be given in the form ^{1),2)} :

$$\hat{Z}(\beta, g, V) = \text{Tr} \exp \left[-\beta H + i \sum_{k=1}^r \gamma_k Q_k \right] \quad (10)$$

So we see that the relation (8) is a consequence of (9) and (10).

The equations of states can be obtained as :

$$\beta P = \frac{\partial}{\partial V} \log Z(\beta, \bar{\Gamma}_1, \dots, \bar{\Gamma}_r, V) \quad (11)$$

$$\varepsilon = -V^{-1} \frac{\partial}{\partial \beta} \log Z(\beta, \bar{\Gamma}_1, \dots, \bar{\Gamma}_r, V) \quad (12)$$

$$v_i = V^{-1} \frac{\partial}{\partial \Gamma_i} \log Z(\beta, \Gamma_1, \dots, \Gamma_n, V) \quad (13)$$

where $\Gamma_i = \beta \mu_i$ and v_i are densities of conserved charges.

From the previous considerations it follows that all thermodynamical properties of a system are contained in the function $\hat{Z}(\beta, g, V)$. We shall now construct this function in the case of a relativistic quantum gas with an internal symmetry.

Let us consider a gas consisting of particles which transform under the irreducible unitary representation $U^\alpha(g)$ of the internal symmetry group. The single-particle wave function is given as $\psi_{E_i, \xi_\rho}(x)$ where (E_i, ξ_ρ) represents a complete set of single-particle quantum numbers. Here ξ_ρ denotes quantum numbers related to the internal symmetry and ρ runs over this set of eigenvalues. For the $d(\alpha)$ dimensional representation, $U^\alpha(g)$, $\rho = 1, \dots, d(\alpha)$. All other quantum numbers are denoted by E_i , $i = 1, \dots, \infty$. We introduce abstract state vectors $|n_{1,1} \dots n_{1,d(\alpha)} \dots n_{\infty,1} \dots n_{\infty,d(\alpha)}\rangle$ where the notation means that there are $n_{1,1}$ particles in the state (E_1, ξ_1) , $n_{1,2}$ particles in the state (E_1, ξ_2) , etc. Our vectors form an orthonormal basis in the Hilbert space, i.e. :

$$\begin{aligned} \langle n_{1,1} \dots n_{1,d(\alpha)} \dots n_{\infty,1} \dots n_{\infty,d(\alpha)} | n'_{1,1} \dots n'_{1,d(\alpha)} \dots n'_{\infty,1} \dots n'_{\infty,d(\alpha)} \rangle = \\ = \delta_{n_{1,1} n'_{1,1}} \dots \delta_{n_{1,d(\alpha)} n'_{1,d(\alpha)}} \dots \delta_{n_{\infty,d(\alpha)} n'_{\infty,d(\alpha)}} \end{aligned} \quad (14)$$

We also have

$$e^{-\beta H} |n_{j,s}\rangle = e^{-n_{j,s} \beta E_j} |n_{j,s}\rangle \quad (15)$$

and

$$e^{i \sum_{k=1}^r \gamma_k Q_k} |n_{j,s}\rangle = e^{i n_{j,s} \sum_{k=1}^r \gamma_k Q_k^{(s)}} |n_{j,s}\rangle \quad (16)$$

Here r is a rank of the internal symmetry group, and Q_k are generators of the Cartan subgroup.

If Eq. (10) is written out in detail we have

$$\hat{Z}(\beta, \gamma_1, \dots, \gamma_r, V) = \sum_{\{n_{j,s}\}} \langle \{n_{j,s}\} | e^{-\beta H} e^{i \sum_{k=1}^r \gamma_k Q_k} | \{n_{j,s}\} \rangle \quad (17)$$

where $\{n_{j,\rho}\}$ denotes the whole set of occupation numbers. The relations (14)-(16) give us

$$\hat{Z}(\beta, \gamma_1, \dots, \gamma_r, V) = \sum_{\{n_{j,s}\}} \prod_{j=1}^{\infty} \prod_{s=1}^{d(\alpha)} \exp[-n_{j,s} (\beta E_j - i \sum_{k=1}^r \gamma_k q_k^{(s)})]$$

The sum over occupation numbers factors into a product of sum, one referring to each one-particle state

$$\hat{Z}(\beta, \gamma_1, \dots, \gamma_r, V) = \prod_{j=1}^{\infty} \prod_{s=1}^{d(\alpha)} \sum_n \exp[-n (\beta E_j - i \sum_{k=1}^r \gamma_k q_k^{(s)})] \quad (18)$$

For bosons, the occupation number is unrestricted so we have

$$\begin{aligned} \hat{Z}_B(\beta, \gamma_1, \dots, \gamma_r, V) &= \prod_{j=1}^{\infty} \prod_{s=1}^{d(\alpha)} \sum_{n=0}^{\infty} \exp[-n (\beta E_j - i \sum_{k=1}^r \gamma_k q_k^{(s)})] = \\ &= \prod_{j=1}^{\infty} \prod_{s=1}^{d(\alpha)} (1 - \exp[-\beta E_j + i \sum_{k=1}^r \gamma_k q_k^{(s)}])^{-1} \end{aligned} \quad (19)$$

For fermions, the occupation numbers are either 0 or 1, and we obtain

$$Z_F(\beta, \gamma_1, \dots, \gamma_r, V) = \prod_{j=1}^{\infty} \prod_{s=1}^{d(\alpha)} (1 + \exp[-\beta E_j + i \sum_{k=1}^r \gamma_k q_k^{(s)}]) \quad (20)$$

Both formulae can be written in an explicit invariant form, under similarity transformations, if you notice that $\exp[i \sum_{k=1}^r \gamma_k q_k^{(\rho)}]$ is an element of the representation of the Cartan subgroup.

Let $D^C(g)$ be a representation of the Cartan subgroup. In our case it has a diagonal form

$$D_{ss'}^C(g) = \exp[i \sum_{k=1}^r \gamma_k q_k^{(s)}] \delta_{ss'} \quad s, s' = 1, \dots, d(\alpha) \quad (21)$$

We then have

$$\prod_{g=1}^{d(\omega)} (1 \pm e^{-\beta E} \mathcal{D}_{gg}^c(g)) = \det (1 \pm e^{-\beta E} \mathcal{D}^c(g)) \quad (22)$$

This gives us

$$\hat{Z}_{B,F}(\beta, g, V) = \exp \left[\mp \sum_{j=1}^{\infty} \text{Tr} \log (1 \mp e^{-\beta E_j} \mathcal{D}^c(g)) \right] \quad (23)$$

where the upper (lower) signs refer to bosons (fermions). To obtain Eq. (23) the identity

$$\det A = \exp [\text{Tr} \log A]$$

was used.

The expression (23) can even be written in a more compact form

$$\begin{aligned} \hat{Z}_{B,F}(\beta, g, V) &= \exp \left[\mp \text{Tr}_1 \log (1 \mp e^{-\beta H} \mathcal{D}^c(g)) \right] = \\ &= [\det_1 (1 \mp e^{-\beta H} \mathcal{D}^c(g))]^{\mp 1} \end{aligned} \quad (24)$$

where the lower index 1 means that all operations are performed in the one-particle subspace.

There is a possibility that the extension of Eq. (24) to many-particle subspaces would allow us to describe a gas with a clustering property. This problem, however, needs further consideration. We are now able to calculate, according to Eq. (7), contributions related to states transforming under the given representation $U^V(g)$ of the symmetry group.

The grand canonical partition function $Z(\beta, \mu_1, \dots, \mu_r, V)$ can be obtained by means of the "Wick rotation" as in Eq. (8).

Let us consider, as an example, the pions' gas with the isospin taken into account. The symmetry group is $SU(2)$, which is of rank 1, and pions transform under the $I = 1$ representation. We then have

$$\hat{Z}(\beta, \gamma, V) = \frac{1}{\prod_{j=1}^{\infty} (1 - e^{-\beta E_j} e^{-i\gamma})(1 - e^{-\beta E_j})(1 - e^{-\beta E_j} e^{i\gamma})} \quad (25)$$

This gives us

$$\log \hat{Z}(\beta, \gamma, V) = - \sum_{j=1}^{\infty} \log(1 - 2e^{-\beta E_j} \cos \gamma + e^{-2\beta E_j}) - \sum_{j=1}^{\infty} \log(1 - e^{-\beta E_j}) \quad (26)$$

and the grand canonical partition functions :

$$\log \tilde{Z}(\beta, \mu, V) = - \sum_{j=1}^{\infty} \log(1 - 2e^{-\beta E_j} \cosh \mu \beta + e^{-2\beta E_j}) - \sum_{j=1}^{\infty} \log(1 - e^{-\beta E_j}) \quad (27)$$

The canonical partition function $Z_I(\beta, V)$ is now given by

$$Z_I(\beta, V) = \frac{1}{\pi} \int_0^{2\pi} d\gamma \sin^2 \frac{\gamma}{2} \hat{\chi}^I(\gamma) \hat{Z}(\beta, \gamma, V) \quad (28)$$

where

$$\hat{\chi}^I(\gamma) = \frac{\sin(\bar{I} + \frac{1}{2})\gamma}{\sin \frac{\gamma}{2}}$$

We see that some important properties of the system, such as condensation phenomena or a transition to the classical limit, will be different from the no-symmetry case. This will be a subject of the next publication.

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