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THE ACTION OF SU(N) LATTICE GAUGE THEORY
IN TERMS OF THE HEAT KERNEL ON THE
GROUP MANIFOLD

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ABSTRACT

We consider the heat kernel on the group manifold as an alternative to the Wilson action in lattice gauge theory, and we exhibit its strict analogy with the well-known Berezinski-Villain action. With the heat kernel action the Gross-Witten singularity is rigorously absent in two dimensions. The similarity of the heat kernel action to the Hamiltonian approach should provide a better convergence of the Lagrangian strong coupling expansion, while its behaviour at weak coupling should simplify the analysis of the weak coupling perturbative expansion.

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1. INTRODUCTION

There has been a great deal of effort recently to understand the transition from strong to weak coupling in lattice gauge theories. Numerical experiments^{1),2)} on small lattices with Wilson action show a rather sharp turnover from the strong coupling behaviour of the string tension to a weak coupling behaviour compatible with asymptotic freedom. In the leading $N \rightarrow \infty$ approximation of the $U(N)$ gauge theory in two dimensions, Gross and Witten³⁾ have shown that the string tension σ is non-analytic in $\beta = (g^2 N)^{-1}$ at some critical point β_c . Higher order corrections⁴⁾ show that a breakdown of the $1/N$ expansion occurs in some interval around β_c , where it should be necessary to take into account terms of order e^{-cN} (this is clear in the Hamiltonian approach where the $N = \infty$ critical point corresponds to an aperiodic instanton-like classical trajectory in the equivalent Fermi gas⁵⁾). It was conjectured that the $N = \infty$ singularity could manifest itself at finite N by a sharp bending of the function $\sigma(\beta)$ (which is what is actually observed) provided one can extend the result to four dimensions. On the other hand, the singularity would forbid a simple extrapolation from strong to weak coupling in the $1/N$ expansion. The question then arises: is this singularity unavoidable, i.e., is it a property of any gauge theory on the lattice, or rather is it a peculiarity of Wilson's action? A partial answer was found recently⁶⁾ by studying the alternative version of the action introduced by Manton⁷⁾. With this choice^{*)}, $A_M(U_p) = g^{-2} \text{dist}(U_p, I)^2$, the leading $N \rightarrow \infty$ contribution to the string tension in two dimensions is analytic in β (higher order corrections, however, are not known).

In our opinion, another much simpler action, which leads to a similar (actually stronger) result, i.e., no Gross-Witten singularity in two dimensions to all orders in $1/N$, was already encountered in various contexts^{**) 8-11)} a couple of years ago, but perhaps overlooked from this point of view⁸⁻¹¹⁾. We want to consider it again here, because it appears to be the most natural non-Abelian generalization of the so-called "Villain model"^{***)}. This too was noticed in Refs. 8-11), but we are able to go one step further in making the analogy with Villain's model more explicit. The action we want to introduce is given in terms

*) $\text{dist}(U, I)$ is the geodesic distance of U from the identity with respect to the invariant metric on the group.

**) We are grateful to G. Veneziano for drawing our attention to Ref. 9).

***) Manton's action, though completely acceptable from the point of view of gauge invariance and compactness, is singular at "conjugate" points. The heat kernel provides the most natural generalization to any Riemannian manifolds of the usual Gaussians in Euclidean space.

of the "heat kernel" on the gauge group, i.e., in terms of the matrix elements of $\exp\{g^2\Delta/2\}$, where Δ is the Laplace-Beltrami operator (with respect to the invariant metric on the group). The precise form of the action will be introduced in Section 2. For the moment, let us anticipate that the heat kernel on a compact Lie group can be given explicitly in terms of "periodic Gaussians" in invariant angles (co-ordinates of the maximal Abelian subgroup). This makes the connection with the $U(1)$ Villain model more explicit, even if the non-Abelian nature of the group does not lead to a direct interpretation of the theory in terms of vortices as happens in the Abelian case¹²⁾; we cannot exclude that this may be feasible, at least in some approximate sense.

Another virtue of the heat kernel action is that it leads to a straightforward computation of the renormalization transformation of the Migdal-Kadanoff type for all groups in the weak coupling limit (see Appendix B). In two dimensions, the action is exactly self-reproducing, thus providing the exact renormalized two-dimensional action. In four dimensions, there are indications that the heat kernel could be very near to the stable distribution [notice that the Villain periodic Gaussian used in Ref. 13) is not exactly the heat kernel as it differs from it by a factor $\theta/\sin\theta$ for $SU(2)$ - this factor favours our suggested action against the simple periodic Gaussian].

The paper is organized as follows: in Section 2 we derive the new form for the action, starting from the Hamiltonian formulation of Kogut and Susskind, we discuss the Gross-Witten singularity and we give the Villain form of the action. Section 3 is devoted to a general discussion of the advantages of the heat kernel action, also keeping in mind the strong coupling expansion^{14),15)} and the very recent numerical computations¹⁶⁾. Since the mathematics of heat diffusion on Riemannian manifolds is unfamiliar to most physicists, we give a self-contained account of it in Appendix A where we derive the heat kernel on $U(N)$ and $SU(N)$ in the most elementary way.

2. THE ACTION DEFINED BY THE HEAT KERNEL

The most natural way of introducing the "heat kernel" action is to start from the Kogut-Susskind¹⁷⁾ Hamiltonian formalism and to try to recover a space-time (Euclidean) formulation by the customary Feynman-Kac-Nelson formula. We have

$$H = \frac{g^2}{2a} \sum_{\ell} E_{\ell}^a E_{\ell}^a + \sum_P V(U_P) \quad (1)$$

where the summation extends over all links ℓ and over all plaquettes P contained in some finite volume $\Lambda \subset \mathbb{R}^3$; usually, $V(U_P) = 1/2g^2a \operatorname{tr}(2 - U_P - U_P^{\dagger})$, but

we are going to make a different choice very soon. The Casimir operator $\sum_{\alpha} E^{\alpha} E^{\alpha}$ coincides with the Laplace-Beltrami operator $-\Delta$ on the group with respect to the invariant metric¹⁸⁾ (see Appendix A for details).

To calculate $\exp(-TH)$ one can apply Trotter's formula to obtain:

$$\langle U^{(0)} | e^{-TH} | U \rangle = \lim_{\substack{n \rightarrow \infty \\ a_0 = T/n}} \int [dU^{(1)}] \dots \int [dU^{(n-1)}] \quad (2)$$

$$\prod_{k=0}^{n-1} \langle U^{(k)} | \exp\left\{ \frac{g^2 a_0}{2a} \sum_{\ell} \Delta_{\ell} \right\} | U^{(k+1)} \rangle \exp\left\{ -a_0 \sum_p V(U_p^{(k+1)}) \right\}$$

with $U^{(n)} \equiv U$. We have now to approximate this exact formula. First of all we take a large, but finite, n in such a way that $a_0 = T/n = a$; secondly, one may approximate the matrix elements of $\exp\{(g^2/2)\sum_{\ell} \Delta_{\ell}\}$ in such a way as to recover Wilson's action (in the temporal gauge). The choice we want to make instead is to keep the kernel $\langle U | \exp \Delta | U' \rangle$ exactly and to choose the potential $V(U_p)$ so as to have an isotropic interaction in the four dimensional lattice. This means that we have to set

$$\begin{aligned} e^{-aV(U_p)} &= \langle I | \exp\left\{ \frac{1}{2} g^2 \Delta \right\} | U_p \rangle \\ &\equiv K(U_p, \frac{1}{2} g^2) \end{aligned} \quad (3)$$

(I is the identity element of the group). In this way we have constructed an approximate solution for the dynamics of H in terms of a space-time lattice action defined by

$$e^A = \prod_p \frac{K(U_p, \frac{1}{2} g^2)}{K(I, \frac{1}{2} g^2)} \quad (4)$$

(The normalization factor is chosen in order to have $A = 0$ for constant field configurations.) This definition is consistent, because i) the heat kernel $K(U, g^2/2)$ is differentiable and strictly positive everywhere; ii) the small g behaviour is the same as Wilson's (and Manton's) action after a suitable identification of the coupling constant and iii) the strong coupling confining properties are preserved⁹⁾ [as is expected from the character expansion Eq. (6) which in the strong coupling limit $g \rightarrow \infty$ converges to Wilson's action]. We now observe that in (1+1) dimensions, no approximation whatsoever is involved in truncating the Trotter formula at finite n , since there is no potential V . As a consequence, the space-time lattice will give the same string tension as the Hamiltonian formalism, i.e.,

$$\sigma^2 = \frac{1}{2} g^2 C_f = \frac{1}{4} (g^2 N) \left(1 - \frac{1}{N^2}\right) \quad (5)$$

where C_f is the quadratic Casimir of the fundamental representation of $SU(N)$. This result can be easily checked from the character expansion⁹⁾, namely

$$K(U, \frac{1}{2} g^2) = \sum_{\mu} d_{\mu} \chi_{\mu}(U) e^{-\frac{1}{2} g^2 C_{\mu}} \quad (6)$$

where μ runs over all irreducible unitary representations $\mathcal{D}^{(\mu)}$ of $SU(N)$, χ_{μ} is the character of $\mathcal{D}^{(\mu)}$, $d_{\mu} = \chi_{\mu}(I)$ its dimension and C_{μ} its quadratic Casimir invariant which represents the eigenvalue of $-\Delta$ in each invariant subspace of $L_2[SU(N)]$.

The action we have defined was already considered from several points of view: as a weak coupling approximation to Wilson's action⁹⁾, as a strong coupling effective action in the Hamiltonian approach^{8),10)}, and as a generalized Villain model for CP^n models¹¹⁾. The main point we want to raise here is that $K(U, g^2/2)$ admits, in general, a representation in terms of periodic Gaussians in the invariant angles. Let $U = S\Phi S^{\dagger}$ where $\Phi = \text{diag}[e^{i\phi_1}, \dots, e^{i\phi_N}]$ [$\sum \phi_i = 0$ for $SU(N)$]; K is a function of $(\phi_1, \dots, \phi_N)$ only and it is given explicitly by

$$K(\phi, \frac{1}{2} g^2) = N \sum_{\{\ell\}} \prod_{i < j} \frac{e^{i(\phi_i - \phi_j + 2\pi(\ell_i - \ell_j))}}{2 \sin \frac{1}{2} |\phi_i - \phi_j + 2\pi(\ell_i - \ell_j)|} \exp \left\{ -\frac{1}{g^2} \sum_j (\phi_j + 2\pi \ell_j)^2 \right\} \quad (7)$$

where N does not depend on ϕ [and factors out in Eq. (4)] and the sum ranges over all integers $(\ell_1, \dots, \ell_N)$ [subject to the constraint $\sum \ell_j = 0$ for $SU(N)$]. Equation (7) is proved in Appendix A.

In the weak coupling limit, all terms in the sum are exponentially small with respect to the dominant $\ell_k = 0$ term (in a neighbourhood of the identity $U \sim I$); the factors $\phi/\sin\phi$ appearing in Eq. (7) have a convergent expansion near $\phi = 0$ and contribute to the second derivative of the action near $\phi = 0$. In fact we have

$$K \underset{\phi \rightarrow 0}{\sim} N \exp \left\{ -\left(\frac{1}{g^2} - \frac{N}{24}\right) \sum_j \phi_j^2 \right\} \quad (8)$$

where we have expanded $\phi/\sin\phi \sim \exp\{\phi^2/6 + O(\phi^4)\}$, and we have taken into account the constraint $\sum \phi_k = 0$ which gives $\sum_{i < j} (\phi_i - \phi_j)^2 \equiv N \sum \phi_i^2$. As a simple application

of Eq. 8 we shall give in Appendix B a straightforward derivation of the Migdal renormalization transformation. For $SU(2)$ we have

$$K(\varphi, \frac{1}{2}g^2) = \mathcal{N} \sum_{-\infty}^{+\infty} \frac{\varphi + 2\pi n}{\sin \varphi} \exp \left\{ -2g^{-2}(\varphi + 2\pi n)^2 \right\} \quad (9)$$

a formula which goes back to the fifties¹⁹⁾. Equation (7) is strongly reminiscent of the Berezinski-Villain action for Abelian groups, where, through an exact transformation¹²⁾, the curl of the discrete field ℓ_μ has been given the interpretation of vortex excitations, while the gradient part is used to decompactify the integration range. Here, the meaning of the integers $\{\ell_k\}$ is less clear due to the non-linear relationship between the plaquette invariant angles $\{\phi_j\}$ and the group integration variables U_ℓ .

3. DISCUSSION

It is important to examine the potential implications of the heat kernel action which we have displayed in the preceding Section. If one accepts universality, the actual continuum limit of the theory should be independent of the details of the interaction Lagrangian; thus, in principle, Wilson's action is as good as any other action which preserves local gauge invariance and exhibits the compactness of the gauge group integration. On the other hand, from a practical point of view, the situation can be very different in the sense that one action may give rise to more quickly convergent algorithms (like strong coupling expansions or Monte Carlo calculations). The favourable points we want to stress about the heat kernel Lagrangian are the following. i) In two dimensions, the Gross-Witten singularity is rigorously absent to all orders in $1/N$; thus, in two dimensions such a singularity is an artefact of the Wilson's action which could reflect itself in a failure of the $1/N$ expansion in four dimensions. It is obvious that it is better to start from an interaction which is free from this disease, at least in two dimensions. ii) In the weak coupling limit, Wilson's action contains [for $SU(2)$] terms of the type ϕ^4/g^2 which are washed away in crude weak coupling approximations and replaced by the smaller terms of order $O(1)\phi^4$ which are very well represented by the heat kernel up to non-negligible values ($\beta/2=2/g^2 \gtrsim 0.6$) of the coupling constant¹³⁾. A smoother approach to the weak coupling region is also shown by the recent Monte Carlo calculation by Lang et al.¹⁶⁾ with the heat kernel action [Eq. (9)]. The reduction of one order in $1/g^2$ of the ϕ^4 terms in the Lagrangian, i.e., from $1/g^2\phi^4 = O(g^2)$ for Wilson's action to $\phi^4 = O(g^4)$ for the heat kernel action should also simplify the analytic comparison of lattice theory with the continuum theory²⁰⁾. iii) At present, Hamiltonian strong coupling expansions appear to be in better shape, as

far as the radius of convergence is concerned, than Lagrangian strong coupling expansions. In fact, if we define as the boundary β_w of the weak coupling region the point where the $\ln\sigma(\beta)$ curve is tangent to the asymptotic freedom line, then according to the analysis of Lüscher et al.¹⁴⁾ of the Lagrangian series expansion, the string roughening point lies well below $\beta_w(\beta_R^{\text{Lag}} \sim 1.9, \beta_w^{\text{Lag}} \sim 2)$. According to the work of Kogut et al.¹⁴⁾, we have instead $\beta_R^{\text{Ham}} > \beta_w^{\text{Ham}}$, giving more confidence in the Hamiltonian strong coupling expansion in the weak coupling region. If one were to perform Monte Carlo calculations in the Hamiltonian formulation by an approximation of the evolution equation through Trotter's formula, preserving discrete rotational invariance (isotropy), one is led naturally to the heat kernel interaction as we have discussed in Section 2.

APPENDIX A - THE HEAT KERNEL ON $U(N)$ AND $SU(N)$ ²¹⁾⁻²⁴⁾

1. The heat kernel, i.e., the fundamental solution of the diffusion equation

$$\frac{\partial K}{\partial t} = \frac{D}{2} \Delta K \quad (A.1)$$

on a Lie group G , or more generally, on a Riemannian manifold, has a long history in the mathematical literature [see Refs. 20) and 21) and references therein]. In the physics literature, the diffusion equation was studied in detail in the case of the rotation group, in the context of the rotational Brownian motion of molecules¹⁹⁾. More recently^{23),24)}, attention was focused on the free Schroedinger equation on Lie groups ($D=i\hbar/m$), the aim being to write a Feynman path integral for particles on Riemannian manifolds. In this Appendix, we calculate the heat kernel for $U(N)$ and $SU(N)$. Let us first show that the Kogut-Susskind operator $\sum E^\alpha E^\alpha$ is identical to the Laplace-Beltrami operator on the group. E^α is defined by the commutation property^{*)}

$$[E^\alpha, \hat{U}] = \lambda^\alpha \hat{U} \quad (A.2)$$

where λ^α are the generators of the group in the fundamental representation and are normalized to $\text{tr} \lambda^\alpha \lambda^\beta = (1/2) \delta^{\alpha\beta}$. Let us introduce canonical co-ordinates x_α , i.e., $U = e^{ix_\alpha \lambda^\alpha}$; then we find

$$E^\alpha = -i \frac{\partial}{\partial x_\alpha} - \frac{1}{2} f_{\alpha\beta\gamma} x_\beta \frac{\partial}{\partial x_\gamma} + O(x^2) \quad (A.3)$$

and

$$\sum_\alpha E^\alpha E^\alpha = -\sum_\alpha \left(\frac{\partial}{\partial x_\alpha} \right)^2 + O(x^2) \quad (A.4)$$

Now observe that $\sum_\alpha E^\alpha E^\alpha$ is invariant under group translations, formally

$$(L_V \psi)(U) = \psi(V^{-1}U) \quad (A.5)$$

$$L_V E^\alpha E^\alpha = E^\alpha E^\alpha L_V \quad (A.6)$$

*) $\hat{U}$ is the Hilbert space operator, while U denotes its formal eigenvalues in a continuous basis $|U\rangle$.

Hence,

$$\begin{aligned}
 (E^\alpha E^\alpha \psi)(U) &= (L_V E^\alpha E^\alpha L_V^{-1} \psi)(U) \\
 &= (E^\alpha E^\alpha L_U^{-1} \psi)(I) \\
 &= \lim_{x \rightarrow 0} (E^\alpha E^\alpha L_U^{-1} \psi)(\exp\{i x_\beta \lambda_\beta\}) \\
 &= -\sum_\alpha \left(\frac{\partial}{\partial x_\alpha}\right)^2 (L_U^{-1} \psi)(\exp\{i x_\beta \lambda_\beta\}) \Big|_{x=0}
 \end{aligned} \tag{A.7}$$

Now we define the metric $ds^2 = 2\text{tr}(\bar{d}U dU^\dagger)$ which is invariant under left and right translations. The Laplace-Beltrami operator Δ associated to ds^2 is also invariant and its action can be calculated as in Eq.(A.7), namely

$$(\Delta \psi)(U) = \lim_{x \rightarrow 0} (\Delta L_U^{-1} \psi)(\exp\{i x_\beta \lambda_\beta\}) \tag{A.8}$$

Since $ds^2 = 2\text{tr}(de^{ix_\alpha \lambda_\alpha} de^{-ix_\beta \lambda_\beta}) = \sum_\alpha dx^\alpha dx^\alpha + O(x^2, dx^2)$, we have $\Delta = -\sum_\alpha (\partial/\partial x_\alpha)^2 + O(x^2)$ near the identity. Hence $\sum_\alpha E^\alpha E^\alpha = -\Delta$ everywhere. It follows that the heat kernel has the well-known character expansion [Eq. (6)].

2. To find the fundamental solution of the heat equation, let us now introduce polar co-ordinates $U = S\Phi S^\dagger$, $\Phi = \text{diag}[e^{i\phi_1}, \dots, e^{i\phi_N}]$, in terms of which we have

$$ds^2 = 2 \sum_i d\phi_i^2 + 2 \sum_{i \neq j} |e^{i\phi_i} - e^{i\phi_j}|^2 |(s^\dagger ds)_{ij}|^2 \tag{A.9}$$

Due to the invariance of Δ , the fundamental solution of Eq. (A.1) (with $D = 2$)

$$\kappa(U, t) = \langle I | e^{t\Delta} | U \rangle \tag{A.10}$$

is a class function, i.e., it is a function of $(\phi_1, \dots, \phi_N)$ only. It is sufficient then to consider the "radial" part of Δ , namely

$$\dot{\Delta} = \frac{1}{2} \sum_i \frac{1}{J^2} \frac{\partial}{\partial \phi_i} J^2 \frac{\partial}{\partial \phi_i} \tag{A.11}$$

where $J(\phi) = \prod_{i < j} [2\sin(\phi_i - \phi_j)/2]$ is the fourth root of the determinant of the metric tensor. A more convenient form for $\dot{\Delta}$ is the following

$$\dot{\Delta} = \frac{1}{2J} \sum_i \left(\frac{\partial}{\partial \phi_i}\right)^2 J + \frac{1}{2} R_N \tag{A.12}$$

with $R_N = N(N^2-1)/12$.

Now we make use of the known solution of the corresponding problem for Hermitian matrices. Namely, let $U = \exp iX$; the same transformation S which diagonalizes U brings X to its diagonal form $X = S \text{diag}(\phi_1, \dots, \phi_N) S^\dagger$. The Laplace operator Δ_X on the space of Hermitian matrices acting on invariant functions is well known to be

$$\Delta_X = \frac{1}{D(\phi_1, \dots, \phi_N)} \sum_k \left(\frac{\partial}{\partial \phi_k} \right)^2 D(\phi_1, \dots, \phi_N) \quad (\text{A.13})$$

where

$$D(\phi) \equiv D(\phi_1, \dots, \phi_N) = \prod_{i < j} (\phi_i - \phi_j) \quad (\text{A.14})$$

Hence we have

$$2 \frac{\partial}{\partial t} = \frac{D(\phi)}{J(\phi)} \Delta_X \frac{J(\phi)}{D(\phi)} + R_N \quad (\text{A.15})$$

Since Δ_X is an ordinary Laplace operator in a Euclidean space, the fundamental solution to the corresponding diffusion equation is a simple Gaussian, namely

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{K}(X, t) &= \frac{1}{2} \Delta_X \mathcal{K} \\ \mathcal{K}(X, t) &= (2\pi t)^{-N^2/2} \exp \left\{ -\frac{1}{2t} \text{tr} X^2 \right\} \end{aligned} \quad (\text{A.16})$$

Because of Eq.

$$K_0(U, t) = (2\pi t)^{-N^2/2} \frac{D(\phi)}{J(\phi)} \exp \left\{ -\frac{1}{2t} \text{tr} X^2 + \frac{1}{2} R_N t \right\} \quad (\text{A.17})$$

is a solution to Eq. (A.1) which has the correct limit to $\delta(U)$ as $t \downarrow 0$ (up to a normalization); K_0 however is not a regular solution everywhere (it is not periodic in ϕ_j). To obtain the true fundamental solution, we have to sum the values taken by K_0 on all points X such that $\exp iX = U$, i.e., over all values $(\phi_j + 2\pi \ell_j)$ with integer $\{\ell_j\}$. This gives Eq. (7) of the text^{*)}. The restriction to $SU(N)$ is simply given by $\{\sum \phi_j = 0, \sum \ell_j = 0\}$. Notice that since $\mathcal{K}(X, t)$ is normalized with respect to the invariant volume $[dX] = D(\phi_1, \dots, \phi_N)^2 d\phi_1, \dots, d\phi_N [dS]$, $K(U, t)$ is not automatically normalized according to $[dU] = J(\phi)^2 d\phi_1, \dots, d\phi_N [dS]$; the proper normalization can be found in Ref. 23), but it is irrelevant for our purposes. Notice also that that naïve Villain form

^{*)} This is essentially the method of "images" of the classical potential theory.

adopted in Ref. 13) corresponds to the heat kernel of Δ_X summed over all inverse images $\exp^{-1}(U)$. There is nothing wrong with this choice, but the reproducing property of the heat kernel and the simple character expansion in terms of the Casimir invariant are lost.

3. We now want to show how the heat kernel on $U(N)$ is related to the heat kernel on its subgroup $U(1)$ and $SU(N)$. Let us introduce the variables

$$\begin{aligned}\theta_j &= \phi_j - \frac{1}{N} \sum_k \phi_k \\ \theta_0 &= \frac{1}{N} \sum_k \phi_k\end{aligned}\tag{A.18}$$

$(\theta_0, \theta_1, \dots, \theta_N)$ are invariant co-ordinates on $U(1) \otimes SU(N)$. The correspondence $U(N) \leftrightarrow U(1) \otimes SU(N)$ given by Eq. A.18 is multivalued, namely all the points

$$\begin{aligned}\theta_j^v &= \theta_j - \frac{2\pi v}{N} + 2\pi v \delta_{jN} \\ \theta_0^v &= \theta_0 + \frac{2\pi v}{N}\end{aligned}\quad (v = 0, \dots, N-1)\tag{A.19}$$

correspond to the same point in $U(N)$. This is written formally as $U(N) = SU(N) \otimes U(1)/Z_N$. It follows that

$$K_{U(N)}(\phi, t) = \sum_v K_{SU(N)}(\theta^v, t) K_{U(1)}(\theta_0^v, t/N)\tag{A.20}$$

where $K_{U(1)}$ is the usual Villain expression. Notice that since $U(N)$ is not simple, the metric (A.9), i.e., $ds^2 = 2\text{tr}(dU dU^\dagger)$ is not the only invariant metric. We are free to choose a relative weight between the $U(1)$ and $SU(N)$ content, i.e., the heat kernel contains two coupling constants, as should be expected from the theory on the continuum. The direct proof of Eq. (A.20) runs as follows. The identity^{*)}

$$\sum_i \phi_i^2 = \frac{1}{N} \sum_{i < j} (\phi_i - \phi_j)^2 + \frac{1}{N} (\sum_i \phi_i)^2\tag{A.21}$$

allows us to write Eq.(7) as

$$K_{U(N)}(\phi, t) = \sum_{\{\ell\}} \prod_{i < j} \theta_t(\theta_i - \theta_j + 2\pi(\ell_i - \ell_j)) e^{-\frac{N}{2t}(\theta_0 + \frac{2\pi}{N} \sum \ell_j)^2}\tag{A.22}$$

where

*) A special case of the identity $\sum_i (\vec{\alpha}_i \cdot \vec{\phi})^2 = C_2 (\sum_i \vec{\gamma}_i \cdot \vec{\phi})^2$ valid for any compact group, where $\vec{\alpha}_i$ are the positive roots, $\vec{\gamma}_i$ are simple roots and C_2 is the quadratic Casimir of the adjoint representation [see Ref. 24)].

$$\theta_t(x) \equiv \frac{x}{2 \sin \frac{x}{2}} \exp \left\{ -\frac{x^2}{2 N t} \right\} \quad (\text{A.23})$$

(Warning: we are ignoring numerical factors.)

Now let $\sum \ell_i = Nn + \nu$ ($0 \leq \nu \leq N-1$); we transform the sum over $\ell_1, \dots, \ell_N$ as follows:

$$\sum_{\{\ell\}} = \sum_{\nu=0}^{N-1} \sum_{n=-\infty}^{+\infty} \sum_{\{\ell\}} \delta \left(\sum_i \ell_i - nN - \nu \right) \quad (\text{A.24})$$

Since the arguments of θ_t are invariant under a uniform translation $\ell_i \rightarrow \ell_i - n$, we find

$$\begin{aligned} K_{U(N)}(\Phi, t) &= \sum_{\nu=0}^{N-1} \sum_{\{\ell\}} \delta \left(\sum_i \ell_i - \nu \right) \prod_{i < j} \theta_t(\theta_i - \theta_j + 2\pi(\ell_i - \ell_j)) \times \\ &\times \sum_{n=-\infty}^{+\infty} \exp \left\{ -\frac{N}{2t} \left(\theta_0 + \frac{2\pi\nu}{N} + 2n\pi \right)^2 \right\} \end{aligned} \quad (\text{A.25})$$

Now we set $\bar{\ell}_i = \ell_i - \nu \delta_{iN}$; the sum over $\{\ell\}$ in Eq. (A.25) becomes

$$\begin{aligned} &\sum_{\sum \bar{\ell}_i = 0} \prod_{i < j} \theta_t(\theta_i - \theta_j + 2\pi(\bar{\ell}_i - \bar{\ell}_j) + 2\pi\nu(\delta_{iN} - \delta_{jN})) \equiv \\ &\equiv \sum_{\sum \ell_i = 0} \prod_{i < j} \theta_t(\theta_i^\nu - \theta_j^\nu + 2\pi(\ell_i - \ell_j)) \equiv K_{SU(N)}(\theta^\nu, t) \end{aligned} \quad (\text{A.26})$$

where $\theta_i^\nu = \theta_i - \nu/N(2\pi) + 2\pi\nu\delta_{iN}$ is precisely the Z_N transformation [Eq. (A.19)].

APPENDIX B - MIGDAL-KADANOFF RECURSION RELATIONS

The mathematical structure of the heat kernel provides a straightforward technique for computing its power and its convolution in the weak coupling limit. Such operations appear for example in approximate renormalization transformations of the Migdal-Kadanoff type and can appear in other lattice calculations as well. We recall that one is confronted with two operations: the convolution $\mathcal{B}_\lambda$

$$\begin{aligned} K(U, \tfrac{1}{2}g^2) &\xrightarrow{\mathcal{B}_\lambda} \underbrace{K \otimes K \otimes \dots \otimes K}_{\lambda^2 \text{ times}} \equiv \\ &\equiv \int [dV_1] \dots \int [dV_{\lambda^2-1}] K(UV_1^\dagger, \tfrac{1}{2}g^2) K(V_1V_2^\dagger, \tfrac{1}{2}g^2) \dots K(V_{\lambda^2-1}, \tfrac{1}{2}g^2) \end{aligned} \quad (B.1)$$

and the superposition $\mathcal{S}_\lambda$

$$K \xrightarrow{\mathcal{S}_\lambda} K \lambda^{d-2} \quad (B.2)$$

where d is the space-time dimension. With the heat kernel, the first operation trivially gives

$$K(U, \tfrac{1}{2}g^2) \xrightarrow{\mathcal{B}_\lambda} K(U, \tfrac{1}{2}\lambda^2 g^2) \quad (B.3)$$

In the weak coupling region, neglecting exponentially small terms, expanding $\theta/\sin\theta \sim 1 + \theta^2/6 + O(\theta^4)$ and using the identity (A.21) we get

$$K(U, \tfrac{1}{2}g^2) = N \exp \left\{ - \left(\frac{1}{g^2} - \frac{N}{24} \right) \sum_i \phi_i^2 + O(\phi^4) \right\} \quad (B.4)$$

where the $O(\phi^4)$ term is independent of g . Thus, for $g^2 \ll 1$ it holds

$$K(U, \tfrac{1}{2}g^2) \xrightarrow{\mathcal{S}_\lambda} K(U, \tfrac{1}{2}g'^2) \quad (B.5)$$

where

$$\frac{1}{g'^2} - \frac{N}{24} = \lambda^{d-2} \left(\frac{1}{g^2} - \frac{N}{24} \right) \quad (B.6)$$

It follows

$$\frac{1}{g^2} \xrightarrow{\mathcal{S}_\lambda \mathcal{B}_\lambda} \frac{\lambda^{d-4}}{g^2} - \frac{N}{24} (\lambda^{d-2} - 1) \quad (B.7a)$$

$$\frac{1}{g^2} \xrightarrow{\mathcal{B}_\lambda \mathcal{S}_\lambda} \frac{\lambda^{d-4}}{g^2} - \frac{N}{24} (\lambda^{d-2} - 1) \lambda^{-2} \quad (B.7b)$$

Taking into account the identification

$$\frac{1}{g^2} - \frac{N}{24} \equiv \frac{1}{g_o^2} \quad (B.8)$$

where g_o is the continuum bare coupling constant, we obtain (neglecting $O(g^6)$)

$$g_o^2 \xrightarrow{S_\lambda \mathcal{B}_\lambda} \lambda^{4-d} g_o^2 \left[1 + \frac{N}{24} (\lambda^2 - 1) g_o^2 \right] \quad (B.9a)$$

$$g_o^2 \xrightarrow{\mathcal{B}_\lambda S_\lambda} \lambda^{4-d} g_o^2 \left[1 + \frac{N}{24} \lambda^{4-d} (1 - \lambda^{-2}) g_o^2 \right] \quad (B.9b)$$

which are the well-known recursion relations²⁵⁾.

A final comment about $U(N)$. Since the Lie algebra $\mathfrak{u}(N)$ of $U(N)$ is the direct sum $\mathfrak{su}(N) \oplus \mathfrak{u}(1)$, the generators of $\mathfrak{su}(N)$ and $\mathfrak{u}(1)$ must decouple in the continuum limit. With the heat kernel action, we have shown in Appendix A that $K_{U(N)} = \sum_{\gamma \in Z(N)} K_{U(1)}^\gamma K_{SU(N)}^\gamma$, but in the limit $g \rightarrow 0$ the terms corresponding to $\gamma \neq \text{identity}$ are exponentially small, i.e., the action decouples very early as opposed to Wilson's action which contains couplings of the order $1/g^2(\theta_0^2 \sum \theta_j^2)$. Under the approximate recursion relation $\mathcal{B}_\lambda S_\lambda$ and $S_\lambda \mathcal{B}_\lambda$ one finds that the $U(1)$ coupling is fixed, which means that under the renormalization transformation one has to modify the metric ds^2 in order to adjust the relative weight of $U(1)$ and $SU(N)$ actions.

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