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LARGE N LATTICE GAUGE THEORY
WITHOUT THIRD-ORDER PHASE TRANSITION

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ABSTRACT

Two-dimensional lattice QCD with Manton's action is solved in the $N = \infty$ limit. Contrary to the case when Wilson's action is used, no phase transition appears. Possible implications for higher dimensions are briefly discussed.

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1. INTRODUCTION

The lattice formulation [1] of QCD provides a viable approach to its large-scale structure. The physically relevant small (bare) coupling constant régime may be probed from the strong coupling régime; there one has strong coupling expansions indicating confinement and convergence for a non-vanishing region. One then tries to watch the result with what one expects from conventional perturbation theory. The success of this programme depends on the absence of phase transitions (or other singularities) between the strong and the weak coupling domain.

One such possible phase transition has been suggested recently by Gross and Witten [2,3]; the authors solve $U(N)$ gauge theory with Wilson's action on a two-dimensional lattice in the limit $N = \infty$ and find a third-order phase transition. Accepting the independent link approximation [4], this phase transition would persist in four dimensions [5]. It is argued in ref. [2] that the sharp cross-over between strong and weak coupling behaviour of the string tension which has been observed in Monte Carlo calculations of $SU(2)$ and $SU(3)$ models in four dimensions [6] may be a trace of such a phase transition at $N = \infty$.

In the present paper we solve two-dimensional $U(N)$ lattice gauge theory in the limit $N = \infty$ with an alternative action suggested by Manton [7]. We find that no phase transition occurs. Thus the obstacle of the third-order phase transition can be avoided, at least in two dimensions, by using a suitable action. Finally, we speculate on what might happen in four dimensions if Manton's action is used. In the independent link approximation it appears possible that new types of phase transitions might occur.

2. THE LARGE N LIMIT

In Euclidean lattice gauge theory the gauge field variables live on the links and may be represented by unitary $N \times N$ matrices for the gauge group $U(N)$.

Wilson's [1] action for a general configuration of variables is

$$S_W = \sum_p S_{W,p} = -\frac{1}{g^2} \sum_p \text{Tr}(U_p + U_p^\dagger - 2I), \quad (2.1)$$

where the sum is over all plaquettes and U_p is the oriented product of the link variables of the plaquette. The term $-2I$ (where I denotes the unit element in the group) normalizes the action such that its minimal value is zero. The action of a particular plaquette can be expressed in terms of the eigenvalues $e^{i\alpha_\nu}$ of U_p as

$$S_{W,p} = \frac{2}{g^2} \sum_{\nu=1}^N (1 - \cos \alpha_\nu). \quad (2.2)$$

Manton [7] has recently suggested an alternative form for the action, where the trace is replaced by the square of the shortest distance from U_p to the unit element in group space, i.e.

$$S_M = \sum_p S_{M,p} = \frac{1}{g^2} \sum_p (d(U_p, I))^2. \quad (2.3)$$

This means a replacement of (2.2) by

$$S_{M,p} = \frac{1}{g^2} \sum_{\nu=1}^N \alpha_\nu^2, \quad (2.4)$$

where the angle variables α_ν must be chosen in the interval $(-\pi, \pi)$, corresponding to the first Brillouin zone of the magnetic weight lattice [7]. In the continuum limit, where the unitary matrices are close to unity, the two expressions agree, and consequently one expects the two actions to describe the same theory.

In two dimensions the partition function factorizes into a product of plaquette integrals. This is due to the freedom to choose a gauge in which the variables of all links pointing in one direction are unity. Then each plaquette variable reduces to a product of two link variables and one of them may be removed by redefinition of the integration variables. The functional integral then reads

$$\begin{aligned} Z &= z^L, \\ z &= \int dW e^{-\frac{1}{g^2} (d(W, I))^2}. \end{aligned} \quad (2.5)$$

Here dW denotes the normalized Haar measure on the group and L is the total number of plaquettes. Since the action depends only on the eigenvalues of W , all other degrees of freedom can be integrated over directly. One obtains

$$Z = \frac{1}{N!} \int \left(\prod_{\mu} \frac{d\alpha_{\mu}}{2\pi} \right) \left(\prod_{\mu \neq \nu} 2 \sin \left| \frac{\alpha_{\mu} - \alpha_{\nu}}{2} \right| \right) e^{-\frac{1}{g^2} \sum_{\nu} \alpha_{\nu}^2}$$

$$= \frac{1}{N!} \int \left(\prod_{\mu} \frac{d\alpha_{\mu}}{2\pi} \right) e^{-\frac{1}{g^2} \sum_{\nu} \alpha_{\nu}^2 + \sum_{\mu \neq \nu} \ln \left(2 \sin \left| \frac{\alpha_{\mu} - \alpha_{\nu}}{2} \right| \right)} \quad (2.6)$$

In the large N limit the eigenvalue distribution is approximated by a (positive) eigenvalue density function $\rho(\alpha)$ such that the number of eigenvalues in the interval $(\alpha, \alpha + d\alpha)$ is $N\rho(\alpha)d\alpha$ and $\int d\alpha \rho(\alpha) = 1$. The partition function is then written as

$$Z = \frac{1}{N!} \int \mathcal{D}\rho \, \delta\left(\int \rho d\alpha - 1\right) e^{-N^2 \tilde{S}}, \quad (2.7)$$

$$\tilde{S} = -\frac{1}{\lambda} \int d\alpha \rho(\alpha) \alpha^2 - \int d\alpha d\beta \rho(\alpha) \rho(\beta) \ln \left| 2 \sin \frac{\alpha - \beta}{2} \right|.$$

Here the temperature $\lambda = g^2 N$ has been introduced. In the stationary phase approximation [8] it is approximated by a single eigenvalue density function which minimizes $\tilde{S}$, that is if $\rho(\alpha) > 0$ for $|\alpha| < \alpha_c$ and $\rho(\alpha) = 0$ for $\pi \geq |\alpha| \geq \alpha_c$, then, in the interval $|\alpha| < \alpha_c$, $\rho(\alpha)$ satisfies the optimization equation

$$\frac{1}{\lambda} \alpha^2 - 2 \int_{-\alpha_c}^{\alpha_c} d\beta \rho(\beta) \ln \left| 2 \sin \frac{\alpha - \beta}{2} \right| = k(\lambda), \quad (2.8)$$

$k(\lambda)$ being a Lagrange multiplier corresponding to the normalization of ρ .

This equation is solved in appendix A [eqs. (A.9), (A.10)]. The result is

$$\rho(\alpha) = \begin{cases} \frac{1}{\pi\lambda} \ln \left(\frac{1 - a + 2 \cos \alpha + 2 \cos(\frac{1}{2}\alpha) \sqrt{2 \cos \alpha - 2a}}{1 + a} \right) & \text{for } \cos \alpha \geq a, \\ 0 & \text{for } \cos \alpha \leq a, \end{cases} \quad (2.9)$$

where $a = \cos \alpha_c = 2e^{-\lambda/2} - 1$. Since a ranges over the interval $[1, -1)$ when λ ranges over $[0, \infty)$, this solution is acceptable for all values of the coupling constant. Contrary to the situation when Wilson's action is used, there is no phase transition.

The internal energy per plaquette

$$E = \int d w \lambda S(w) e^{-S(w)} / \int d w e^{-S(w)}$$

may be expressed, in the stationary phase approximation for large N, as

$$E = N^2 \int d \alpha \rho(\alpha) \alpha^2 e^{-S(\rho)} / \int d \alpha \rho(\alpha) e^{-S(\rho)} = \\ = N^2 \int d \alpha \rho(\alpha) \alpha^2. \quad (2.10)$$

This integral is simplified in appendix B. The expression for the specific heat per degree of freedom assumes the following simple form

$$\frac{C}{N^2} = \frac{1}{N^2} \frac{\partial E}{\partial \lambda} = \frac{1}{\lambda^2} \int_0^\lambda \frac{dx}{2} \frac{x^2}{e^{x^2} - 1}. \quad (2.11)$$

In fig. 1 we compare this expression with the corresponding quantity when Wilson's action is used [2], i.e.

$$\frac{C}{N^2} = \begin{cases} \frac{2}{\lambda^2} & \lambda \geq 2, \\ \frac{1}{2} & \lambda \leq 2. \end{cases} \quad (2.12)$$

We notice that the two specific heats are quite different in nature and that there is no trace of any phase transition when Manton's action is used.

The expectation value of the Wilson loop operator is

$$W_L(\lambda) = \frac{1}{N} \langle \text{Tr} \prod_L U \rangle, \quad (2.13)$$

where $\prod_L U$ is an ordered product of U's on the links belonging to a closed loop L.

Since the action only depends on the eigenvalues of the matrices $W_{\underline{n}\underline{i}_2}$, it is in-

variant under separate unitary transformations $W_{\underline{n}\underline{i}_2} \rightarrow U_{\underline{n}} W_{\underline{n}\underline{i}_2} U_{\underline{n}}^+$ of the matrices

$W_{\underline{n}\underline{i}_2}$. In two dimensions W_L therefore factorizes into a product of single plaquette

Wilson loops

$$\int d w_{\underline{n}} e^{-\frac{1}{2}(\text{tr}(w_{\underline{n}}, I))^2} \text{Tr}(A w_{\underline{n}} B) \\ = \int d U \int d w_{\underline{n}} e^{-\frac{1}{2}(\text{tr}(w_{\underline{n}}, I))^2} \text{Tr}(w_{\underline{n}} U^+ B A U) =$$

$$\begin{aligned}
 &= \int dW_n e^{-\frac{1}{2}(\sum (W_n, I))^2} (W_n)_{da} (BA)_{bc} \frac{1}{N} \delta_{ad} \delta_{bc} \\
 &= \frac{1}{N} \text{Tr} (BA) \int dW_n \text{Tr}(W_n) e^{-\frac{1}{2}(\sum (W_n, I))^2}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 W_L(\lambda) &= (\omega(\lambda))^L \\
 \omega(\lambda) &= \frac{1}{N} \int dW \text{Tr} W e^{-\frac{1}{2}(\sum (W, I))^2} / \int dW e^{-\frac{1}{2}(\sum (W, I))^2}^{(2.14)}
 \end{aligned}$$

where L denotes the number of plaquettes enclosed by the loop. In the eigenvalue density function approximation this is

$$\omega(\lambda) = \int D\rho \int d\alpha e^{i\alpha} \rho(\alpha) e^{-N^2 \tilde{S}} / \int D(\rho) e^{-N^2 \tilde{S}},$$

and in the stationary phase approximation it becomes simply

$$\omega(\lambda) = \int e^{i\alpha} \rho(\alpha) d\alpha. \quad (2.15)$$

The integral is determined in appendix A [eq. (A.14)]

$$\omega_M(\lambda) = \frac{1}{\lambda} (1 - \alpha) = \frac{2}{\lambda} (1 - e^{-\lambda/2}). \quad (2.16)$$

This expression can be compared with the corresponding quantity when Wilson's action is used [2]

$$\omega_W(\lambda) = \begin{cases} \frac{1}{\lambda} & \lambda \geq 2, \\ 1 - \frac{\lambda}{4} & \lambda \leq 2. \end{cases} \quad (2.17)$$

In the weak coupling limit they agree up to $O(\lambda^2)$ terms. In the strong coupling limit Manton's action gives a result larger by a factor of two, reflecting the fact that Manton's action is larger than Wilson's. The string tension is

$$\sigma = -\frac{1}{a^2} \ln \omega(\lambda), \quad (2.18)$$

where a is the lattice spacing. If we change the lattice spacing and require that the coupling constant λ changes correspondingly such that the string tension remains constant, we obtain the β function

$$\beta(\lambda) = -a \frac{d\lambda}{d\ell} = -2 \frac{\ln \omega(\lambda)}{\omega'/\omega}. \quad (2.19)$$

The two β functions are compared in fig. 2. Evidently, with Manton's action the β function does not show any trace of the third-order phase transition.

3. FINAL REMARKS

It would be interesting to determine what difference the use of different actions makes in four dimensions. This question can be investigated by Monte Carlo calculations. Here we limit ourselves to some considerations based on the independent link approximation [4,5]. The integral over one link is then performed, while the other link variables are held fixed. The integral is of the form

$$\int dU e^{-\sum_p S(U U_p)} \quad (3.1)$$

where the sum is over the six plaquettes containing the link under consideration, and $U U_v$ is the product of the group elements of the plaquette v .

When Wilson's action is used

$$\begin{aligned} \sum_p S(U U_p) &= \frac{1}{g^2} \sum_p \text{Tr}(2I - U U_p - U_p^\dagger U) \\ &= \frac{1}{g^2} 2 \sum_p \text{Tr}(I) - \frac{1}{g^2} \text{Tr}(U A + A^\dagger U^\dagger), \\ A &= \sum_p U_p. \end{aligned} \quad (3.2)$$

The idea is that the fluctuations should be small enough such that it is reasonable to replace them with their mean value. The stationary phase approximation to the integral (3.1) with (3.2) depends only on the single parameter [5]

$$Q = g^2 \text{Tr}(A A^\dagger)^{-1/2} \quad (3.3)$$

The solution in the large N limit again exhibits a third-order phase transition [5], now as a function of Q .

If Manton's action is used instead, (3.2) is replaced by

$$\sum_p S_M(U U_p) = \frac{1}{g^2} \sum_p \left(d(U, U_p^\dagger) \right)^2. \quad (3.4)$$

This does not reduce to such a simple form as (3.2), for the six distances in (3.4) cannot be combined to a single distance. For each p_0 we can transform U_{p_0} away and write the integral as

$$\int (\prod_v d\alpha_v) e^{-\frac{1}{g^2} \sum_v \alpha_v^2} \int dW e^{-\frac{1}{g^2} \sum_{p \neq p_0} (d(U, U_p^+))^2}, \quad (3.5)$$

where W is the unitary matrix which diagonalizes U . If the last exponential were absent, this would reduce to eq. (2.6).

In general it is hard to solve this integral, but there may be a possibility of additional apices due to the presence of the second integral in (3.5). In that case the analysis in the first part of appendix A could be applied. Then there arises the possibility of phase transitions occurring, for example, when the eigenvalues start to fill the region between two apices.

We have shown that in two dimensions the third-order phase transition discussed by Gross and Witten is absent when one makes use of the action suggested by Manton. It was argued in ref. [2] that if the third-order phase transition also occurs in higher dimensions, then, for small couplings, one cannot interchange the large N limit with the strong coupling expansion. The results of the present paper show that such a conclusion depends on the way in which one regularizes the action on the lattice. Unless a new type of phase transition of the type discussed above and specific to dimensions higher than two appears, the use of Manton's action may provide for a lattice version of QCD, where the interchange of the large N limit and the strong coupling expansion is also possible. If such a situation prevails in higher dimensions the approximation procedure of Bars and Green [9] may, in fact, be a sensible way of deriving a string picture from lattice QCD. In this context it is interesting to notice that the use of Manton's action leads to a very rapid convergence of the $1/N$ expansion, at least in two dimensions. In fact, by deriving [10] the specific heat per degree of freedom and the β function for the groups $SU(2)$ and $SU(3)$ with Manton's action, we find agreement with the $U(\infty)$ results (2.11) and (2.19) to such an accuracy that one sees no difference when comparing with the curves presented in figs. 1 and 2.

Finally, we remark that a way to study the problems of a possible phase transition when using Manton's action in dimensions higher than two is to perform a Monte Carlo calculation. Such a study is at present in progress [11].

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APPENDIX A

The eigenvalue distribution is determined by eq. (2.8) or, equivalently, its derivative with respect to α . Let us for the moment replace $(1/\lambda)\alpha^2$ by a more general action $(1/\lambda)f(\alpha)$. We then have

$$\frac{1}{\lambda} f'(\alpha_0) = \int d\alpha \rho(\alpha) \cot\left(\frac{\alpha_0 - \alpha}{2}\right), \quad (A.1)$$

where the integral is over some part of $[-\pi, \pi]$ and f denotes the principal value integral. Let us first discuss the case when the eigenvalues fill the whole group space. We assume that $f'(\alpha)$ is piecewise continuous and integrable. Since f is periodic we have

$$\int_{-\pi}^{\pi} f'(\alpha) d\alpha = 0.$$

Therefore eq. (A.1) can be solved by Hilbert's inversion formula [2], which gives

$$-\rho(\alpha_0) + C = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \frac{1}{\lambda} f'(\alpha) \cot\left(\frac{\alpha_0 - \alpha}{2}\right) d\alpha \quad (A.2)$$

The normalization condition on ρ determines the constant to $C = 1/2\pi$.

When this formula gives an acceptable, i.e. positive, eigenvalue density $\rho(\alpha)$ it is the solution to eq. (A.1). We are then in the strong coupling phase. This will always be the case for sufficiently large λ if f' is continuous, for then the right-hand side is bounded, and will be smaller than C for sufficiently large λ . In such a case there is a phase transition at the λ for which the solution $\rho(\alpha)$ to eq. (A.2) fails to be positive definite.

But if we relax the condition that $f'(\alpha)$ be continuous, and consider an $f'(\alpha)$ which is discontinuous for certain values of α_p , i.e. $f'(\alpha_p - 0) > f'(\alpha_p + 0)$, then we find that $\rho(\alpha)$ tends logarithmically to $-\infty$ at α_p and is therefore unacceptable for all finite values of λ . The correct solution to eq. (A.1) in such cases will always leave some portion of group space. The existence or absence of the strong coupling phase, in which the eigenvalues fill the whole group space, is therefore determined by the absence or presence of apices in the action, i.e. points α_p in which $f'(\alpha_p - 0) - f'(\alpha_p + 0) > 0$.

To solve (A.1) in the case when the eigenvalues do not fill the whole group space, we follow the method used in ref. [3]. To be explicit let us assume that the integration interval is $(\alpha) < \alpha_c < \pi$. We change variables as follows

$$z = e^{i\alpha}, \quad A = e^{i\alpha_c}, \quad \bar{A} = e^{-i\alpha_c},$$

$$\frac{dz}{z - z_0} = \frac{1}{2} d\alpha \left[i + \cot \frac{1}{2}(\alpha - \alpha_c) \right].$$

Then eq. (A.1) becomes

$$\int_{\bar{A}}^A \frac{dz}{z - z_0} \rho(z) = -\frac{1}{2\lambda} f'(z_0) + \frac{i}{2}. \quad (\text{A.3})$$

The integration is along an arc of the unit circle; we assume that $f(z_0)$ is an analytic function in a neighbourhood of this arc. The following functions, analytic outside the arc, are defined:

$$\Phi(z_0) = \int_{\bar{A}}^A \frac{dz}{z - z_0} \rho(z), \quad (\text{A.4})$$

$$F(z_0) = \frac{\Phi(z_0)}{\sqrt{(z_0 - A)(z_0 - \bar{A})}}. \quad (\text{A.5})$$

The branch of the square root is defined such that for real z it is > 0 for $z > 1$ and < 0 for $z < 1$. On the arc of integration these functions assume the following values ($\pm$ refers to the branch outside and inside the unit circle, respectively).

$$\Phi_{\pm}(z_0) = -\frac{1}{2\lambda} f'(z_0) + \frac{i}{2} \pm i\pi \rho(z_0), \quad (\text{A.6})$$

$$F_{\pm}(z_0) = \frac{\pm \left(-\frac{1}{2\lambda} f'(z_0) + \frac{i}{2} \right)}{\left(\sqrt{(z_0 - A)(z_0 - \bar{A})} \right)_{\pm}} - \frac{i\pi \rho(z_0)}{\left(\sqrt{(z_0 - A)(z_0 - \bar{A})} \right)_{+}}. \quad (\text{A.7})$$

Since $F(z_0)$ is analytic outside the arc and tends to zero as z_0^{-2} when $z_0 \rightarrow \infty$ it must be given by the contour integral around the arc

$$F(z_0) = -\frac{1}{2\pi i} \oint \frac{dz}{z - z_0} \left(-\frac{1}{2\lambda} f'(z) + \frac{i}{2} \right) \frac{1}{\sqrt{(z - A)(z - \bar{A})}}. \quad (\text{A.8})$$

To get the solution to eq. (2.8) we choose

$$f'(z) = -2i \ln(z)$$

with a branch cut along the negative real axis. The integral (A.8) can then be deformed to an integral along the negative real axis plus a residue at $z = z_0$.

The residue gives the first term in (A.8) and there remains the result

$$\pi \lambda \rho(z_0) = \left(\sqrt{(z_0 - \bar{A})(z_0 - A)} \right)_+ + \int_0^\infty \frac{dx}{x + z_0} \frac{1}{\sqrt{(x+A)(x+\bar{A})}} .$$

It is straightforward to evaluate this integral by making first the substitution $x = -z, + 1/t$. The result is

$$\pi \lambda \rho(z) = \ln \left(\frac{\cos \alpha_c + (1 + \sqrt{z^2 - 2z \cos \alpha_c + 1})(1 + \sqrt{z^{-2} + 2z^{-1} \cos \alpha_c + 1})}{1 + \cos \alpha_c} \right) \quad (\text{A.9})'$$

which is equivalent to eq. (2.9).

To determine the relation between λ and α_c we compare the analytic continuation of (A.8) for large z_0 with (A.9). The constant term and the $1/z_0$ term give us, respectively,

$$\lambda = 2 \ln \left(\frac{2}{1 + \cos \alpha_c} \right) , \quad (\text{A.10})$$

$$\int_{-\alpha_c}^{\alpha_c} d\alpha e^{i\alpha} \rho(\alpha) = \frac{1}{\lambda} (1 - \cos \alpha_c) , \quad (\text{A.11})$$

while higher powers of $1/z_0$ give us higher Fourier components of $\rho(\alpha)$. The integral (A.11) is precisely the Wilson loop [eq. (2.15)].

APPENDIX B

The energy per plaquette is according to eqs. (2.9) and (2.10)

$$\frac{E}{N^2} = \int_{-\alpha_c}^{\alpha_c} \frac{d\alpha}{\pi} \frac{\alpha^2}{\lambda} \ln \left(\frac{1-a+2\cos\alpha+2\cos(\frac{1}{2}\alpha)\sqrt{2\cos\alpha-2a}}{1+a} \right), \quad (B.1)$$

$$a = \cos\alpha_c = 2e^{-\lambda/2} - 1.$$

We will now derive a simpler expression for the energy by repeated differentiations and partial integrations of the expression above. First we get by differentiation

$$\frac{d}{d\lambda} \left(\frac{\lambda E(\lambda)}{N^2} \right) = \frac{1}{\pi} \int_{-\alpha_c}^{\alpha_c} d\alpha \frac{\alpha^2 \cos(\frac{1}{2}\alpha)}{\sqrt{2\cos\alpha-2a}} \quad (B.2)$$

In terms of new variables $\gamma = \frac{1}{2}\alpha$, $\eta = \sqrt{(1-a)/2}$ this is

$$\frac{4}{\pi} \int d\gamma \gamma^2 \frac{\cos\gamma}{\sqrt{\eta^2 - \sin^2\gamma}}. \quad (B.3)$$

Next, the idea is to get rid of the factor γ^2 by two partial integrations. We obtain the following intermediary expressions ($\sin\gamma_{\max} = \eta$)

$$\begin{aligned} \frac{4}{\pi} \int_{-\gamma_{\max}}^{\gamma_{\max}} d\gamma \gamma^2 \frac{\cos\gamma}{\sqrt{\eta^2 - \sin^2\gamma}} &= 4\gamma_{\max}^2 - \frac{8}{\pi} \int_{-\gamma_{\max}}^{\gamma_{\max}} d\gamma \gamma \arcsin(\eta^{-1}\sin\gamma); \\ \frac{d}{d\eta} \frac{4}{\pi} \int_{-\gamma_{\max}}^{\gamma_{\max}} d\gamma \gamma^2 \frac{\cos\gamma}{\sqrt{\eta^2 - \sin^2\gamma}} &= \frac{8}{\eta\pi} \int_{-\gamma_{\max}}^{\gamma_{\max}} d\gamma \gamma \frac{\sin\gamma}{\sqrt{\eta^2 - 1 + \cos^2\gamma}} \\ &= \frac{8}{\pi\eta} \int_{-\gamma_{\max}}^{\gamma_{\max}} d\gamma \operatorname{arcsinh}\left(\frac{\cos\gamma}{\sqrt{\eta^2 - 1}}\right), \end{aligned} \quad (B.4)$$

and

$$\begin{aligned} \frac{d}{d\eta} \eta \frac{d}{d\eta} \frac{4}{\pi} \int_{-\gamma_{\max}}^{\gamma_{\max}} d\gamma \gamma^2 \frac{\cos\gamma}{\sqrt{\eta^2 - \sin^2\gamma}} &= \\ &= -\frac{8}{\pi} \frac{\eta}{\eta^2 - 1} \int_{-\gamma_{\max}}^{\gamma_{\max}} d\gamma \frac{\cos\gamma}{\sqrt{\eta^2 - \sin^2\gamma}} = -\frac{8\eta}{\eta^2 - 1}. \end{aligned} \quad (B.5)$$

Now E is recovered by two integrations. First we obtain

$$\eta \frac{d}{d\eta} \frac{4}{\pi} \int_{-\gamma_{\max}}^{\gamma_{\max}} d\gamma \gamma^2 \frac{\cos\gamma}{\sqrt{\eta^2 - \sin^2\gamma}} = -4 \ln(1-\eta^2), \quad (B.6)$$

which by a change of variables leads to

$$\frac{d}{d\lambda} \frac{4}{\pi} \int d\gamma \gamma^2 \frac{\cos \gamma}{\sqrt{\eta^2 - \sin^2 \gamma}} = - \frac{1-\eta^2}{\eta^2} \ln(1-\eta^2) = \frac{\lambda/2}{e^{\lambda/2} - 1} \quad (B.7)$$

$$\frac{d}{d\lambda} \left(\lambda \frac{E(\lambda)}{N^2} \right) = \frac{4}{\pi} \int d\gamma \gamma^2 \frac{\cos \gamma}{\sqrt{\eta^2 - \sin^2 \gamma}} = \int_0^\lambda dx \frac{x/2}{e^{x/2} - 1} \quad (B.8)$$

The final results are

$$\frac{E(\lambda)}{N^2} = \frac{1}{2} \int_0^\lambda dx \left(x - \frac{x^2}{\lambda} \right) \frac{1}{e^{x/2} - 1}, \quad (B.9)$$

and

$$\frac{C}{N^2} = \frac{d}{d\lambda} \left(\frac{E(\lambda)}{N^2} \right) = \frac{1}{2\lambda^2} \int_0^\lambda \frac{x^2 dx}{e^{x/2} - 1} \quad (B.10)$$

REFERENCES

- [1] K.G. Wilson, Phys. Rev. D 10 (1974) 2445.
- [2] D.J. Gross and E. Witten, Phys. Rev. D 21 (1980) 446.
- [3] S. Wadia, University of Chicago, preprint EFI 79/44 (1979).
- [4] A.M. Polyakov, Phys. tett. B82 (1979) 247.
- [5] R. Brower and M. Nauenberg, Group integration for lattice gauge theory at large N and at small coupling, University of California, Santa Cruz, preprint (1980).

E. Brézin and D.J. Gross, University of California, Santa Barbara, preprint NSF-ITP-80/42 (1980).
- [6] M. Creutz, Phys. Rev. D 21 (1980) 2308.

M. Creutz, Phys. Rev. Lett. 45 (1980) 313.

C. Rebbi, Phys. Rev. D 21 (1980) 3351.

G. Bhanot and C. Rebbi, preprint CERN TH.2979 (1980).

B. Lautrup and M. Nauenberg, preprint CERN TH.2945 (1980).
- [7] N.S. Manton, MIT preprint CTP 859 (1980).
- [8] E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, Commun. Math. Phys. 59 (1978) 35.
- [9] I. Bars and F. Green, Phys. Rev. D 20 (1979) 3311.
- [10] C.B. Lang, P. Salomonson and B.S. Skagerstam, preprint CERN TH.2995 (1980).
- [11] C.B. Lang, C. Rebbi, P. Salomonson and B.S. Skagerstam, in preparation.
- [12] N.I. Muskhelishvili, Singular integral equations (P. Noordhof, N.V., Groningen, 1953).

Figure captions

- Fig. 1 : The specific heat per degree of freedom when making use of Wilson's action $[U_W^{(\infty)}]$ and Manton's action $[U_M^{(\infty)}]$ as functions of λ .
- Fig. 2 : The β functions when making use of Wilson's action $[U_W^{(\infty)}]$ and Manton's action $[U_M^{(\infty)}]$ as functions of λ .

Fig. 1

Fig.2