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THE PHASE STRUCTURE OF A NON-ABELIAN
GAUGE HIGGS FIELD SYSTEM

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ABSTRACT

The phase structure of a non-Abelian gauge Higgs field system is studied by Monte-Carlo (MC) simulations. The analysis is done on a 4^4 lattice and approximating both the gauge group and the Higgs manifold by the icosahedral subgroup of $SU(2)$. Results are presented for a Higgs field in the fundamental representation, in which case the symmetry is completely broken for sufficiently strong coupling, and for a Higgs field in the adjoint representation, in which case a $U(1)$ symmetry (approximated by Z_{10}) survives. No evidence for a "Coulomb pocket" is found.

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1. INTRODUCTION

MC simulations have proven most valuable for investigating the phase structure of pure gauge field theories¹⁾. A step towards the final aim of understanding a coupled fermion gauge system consists of the introduction of scalar matter fields and the analysis of the resulting phase diagrams. For some Abelian discrete gauge groups of low order (Z_6) this has been done by Creutz²⁾; to our knowledge no MC calculations for non-Abelian gauge groups have been made. There are, however, distinct theoretical expectations³⁾⁻⁵⁾, mainly based on arguments concerning the convergence of the cluster expansion for strong and/or weak couplings. As an independent check and in order to localize the critical lines, we have studied the phase structure of a coupled gauge Higgs system with gauge group $SU(2)$ by MC methods.

To make this analysis feasible we had to comply with a few simplifications which we believe do not distort the physical results. We fix (compactify) the Higgs field to unit norm, i.e., we formally give the potential infinite weight. In the continuum limit, which will eventually be obtained at a critical point where the correlation length grows to infinity, the radial mode should not be relevant.

We work on a 4^4 lattice with periodic boundary conditions for both the gauge and the Higgs variables. Finite size effects are small for bulk quantities, such as the average contribution to the action per plaquette or per link (internal energies). One MC iteration consists of an update of all gauge field variables without changing the Higgs variables and then vice versa. Most information has been derived from thermal cycles, where the coupling is changed by a small amount, five MC iterations are performed to get closer to equilibrium, then another ten MC iterations are used to obtain average values of the internal energies before proceeding to the next coupling. In this way the system is cooled to the highest values of the coupling constant and then heated again to starting value. The shape of the surfaces giving the internal energies as a function of the couplings, the presence of hysteresis loops or large fluctuations have all been used to locate the probable critical lines. As the thermal cycles are performed with "constant speed", a comparison of the shape of different hysteresis loops also allows us to draw inferences on the type of the phase transitions. Beyond the internal energies, we have also measured the expectation value of the covariant product of Higgs fields at different points. This quantity, together with its fluctuation, may be used to discriminate between the symmetric and Higgs phases of the theory.

Another simplification consisted of approximating the $SU(2)$ gauge group with its 120 element icosahedral subgroup $\tilde{Y}$. The saving in computer time is very substantial and a previous analysis⁶⁾ has demonstrated that the icosahedral group gives, for the pure gauge system, results undistinguishable from those with gauge group $SU(2)$ for values of the gauge coupling below $\beta_c = 6.05$. The latter value is the position of the phase transition specific to the discreteness of the group and (more or less) determined by the action gap. At this value, however, the correlation length in the pure gauge theory is already considerably larger than the lattice size and there will be no difference from a spin wave approximation. Any important structure (which is noticeable on lattices of this size) should have evolved already at smaller values of β and there should be no basic difference between the situation at $\beta = 6$ and at $\beta = \infty$ for the full group $SU(2)$.

The action contains two terms:

$$S(u, \phi) = \beta \sum_P S_P(u_P) + \gamma \sum_{(i,j)} S_L(\phi_i, \phi_j, u_{ij}) \quad (1.1)$$

where the first sum is over all plaquettes and the second over all links of the lattice. The first part gives the pure gauge field action and we have used Wilson's⁷⁾ form

$$S_P(u_P) = \frac{1}{2} \text{Tr} (1 - u_P) \quad (1.2)$$

normalized so that $0 \leq \langle S_P \rangle \leq 1$. The second sum gives the gauge invariant Higgs action and will have the general form

$$S_L(\phi_i, \phi_j, u_{ij}) = \text{const} - (R(\phi_i)^\dagger R(u_{ij}) R(\phi_j) + \text{h.c.}) \quad (1.3)$$

where $R(\phi)$ and $R(U)$ denote vector and matrix representations of the Higgs and the gauge field elements of the group. We investigated the particularly interesting cases where the Higgs field is in (a) the fundamental or (b) the adjoint representation.

Although the discrete approximation could be restricted to the gauge variables, the major saving in computer time is achieved when the continuous Higgs manifold is also replaced by a finite collection of points. Moreover, we found it more elegant to treat both fields in the same way. When the Higgs field is in the fundamental representation, the isomorphism between the gauge and Higgs manifolds ($\approx S^3$) immediately suggests that the discretization used for the $SU(2)$ group be applied also to the Higgs variables. Indeed, setting

$$V_i = \begin{pmatrix} \phi_i^1 & -\phi_i^{2*} \\ \phi_i^2 & \phi_i^{1*} \end{pmatrix} \quad (1.4)$$

one establishes a one-to-one correspondence between the values of the Higgs field, normalized to one, and the unitary, unimodular matrices V_i . With a normalization such that $0 \leq \langle S_L \rangle \leq 1$, the Higgs action per link then takes the form

$$S_L^{(F)}(\phi_i, \phi_j, u_{ij}) = \frac{1}{2} \text{Tr} (1 - V_i u_{ij} V_j^\dagger) \quad (1.5)$$

The discrete approximation is obtained by restricting the matrices V_i , as well as the u_{ij} , to those belonging to the icosahedral subgroup $\tilde{Y}$.

The case of the adjoint representation requires a little more consideration. In the continuous case the values of the Higgs field (belonging to S^2) may be represented by matrices $\vec{\sigma} \cdot \vec{\phi}_i$, $\vec{\phi}_i^2 = 1$, or equivalently, may again be represented by $SU(2)$ matrices

$$V_i = \cos \Theta + i \vec{\sigma} \cdot \vec{\phi}_i \sin \Theta \quad (1.6)$$

but where the angle Θ ($\neq 0, \pi$) is now kept fixed. This realizes the isomorphism

$$S^2 \approx S^3/S^1 \approx SU(2)/U(1) \quad (1.7)$$

Replacing V_i by the matrices of $\tilde{Y}$ will thus produce a discretization of the Higgs manifold; the V_i cannot, however, range over the whole group, but must belong to one of the equivalence classes. Leaving aside the trivial one-element classes $\{I\}$ and $\{-I\}$, the icosahedral subgroup has four equivalence classes

$n = 1 \div 4$ of 12 elements [with $\theta = \pi/5$ in Eq. (1.6)], two classes of 20 elements (with $\theta = \pi/3$) and one class of 30 elements ($\theta = \pi/2$). The rotations belonging to these various classes can be put into correspondence with the vertices, faces or sides of the icosahedron. Clearly the elements of larger class give the denser finite representation of S^2 . However, they do not provide the discretization of the Higgs variables most useful for our purposes. Indeed, whereas the stability group of any Higgs vector in the continuous case is $U(1)$, the stability group of the rotations belonging to the 12, 20 or 30 element classes are, respectively, Z_{10} , Z_6 and Z_4 . These will be the surviving gauge groups when, for a sufficiently large Higgs coupling, the original gauge symmetry is maximally broken. Since we want this remnant gauge freedom to approximate $U(1)$ as closely as possible, we have used one of the 12 element classes as a discrete version of the Higgs manifold. Specifically, we have represented the Higgs variables by the 12 matrices, $\epsilon \tilde{Y}$, of the form of Eq (1.6) with $\theta = \pi/5$. The Higgs action (normalized again so that its expectation value lies between 0 and 1) is then given by

$$S_L^{(\Lambda)} = \frac{1}{2 \sin^2 \frac{\pi}{5}} \text{Tr} (1 - V_i U_{ij} V_j^\dagger U_{ij}^\dagger) \quad (1.8)$$

2. EXPECTATIONS AND RESULTS

2.1 The Higgs field in the fundamental representation

a) Expectations

For $\gamma = 0$ the Lagrangian reduces to that of the pure $\tilde{Y}$ gauge field theory, which has been investigated in detail⁶⁾. To simplify the discussion for the cases $\gamma = \infty$ or $\beta = 0$ note that by a gauge transformation

$$\begin{aligned} V_i &\rightarrow V'_i = g_i V_i \\ U_{ij} &\rightarrow U'_{ij} = g_i U_{ij} g_j^\dagger \end{aligned}$$

one may transform the Higgs field away by choosing $g_i = V_i^\dagger$ ("U gauge"). The Lagrangian becomes one of a gauge system coupled to a non gauge invariant field with strength γ . Thus for $\gamma = \infty$ all U's will be frozen to unity in order to minimize $\text{Tr}(1-U)$; $\langle S_p \rangle$ and $\langle S_L \rangle$ vanish identically. For $\beta = 0$, only the link term contributes to the action and the integral factorizes. It may be

solved explicitly and the MC results are in agreement with the analytical expression. For $\beta = \infty$ the U configuration becomes a gauge transform of the trivial (completely ordered) one; this leaves a spin system with global symmetry SU(2) (actually its icosahedral subgroup) and nearest neighbour coupling. Such a system will develop one phase transition of (presumably) second order. (As a matter of fact, some non-trivial features are associated with the finite size of the system; see later discussion.) From the clustering properties (strong/weak coupling expansion) one finds analyticity domains $\{\gamma, \beta\}$ both γ, β small or γ/β large enough³⁾. For $\gamma > 0$ there will be perimeter decay for both the Wilson loop and the 't Hooft loop⁸⁾ due to the screening of static charges through the Higgs field. For our discrete group there is also analyticity in $\{\gamma, \beta\}$ γ small and β large [cf. Seiler⁴⁾].

b) Results

The results of several thermal cycles done at fixed β or fixed γ are all put together in Fig. 1a. The upper graph illustrates the measured expectation value of the gauge action per plaquette $\langle S_p \rangle$, the lower graph the expectation value of the Higgs action per link $\langle S_L \rangle$. Hysteresis loops are evident in many of the cycles and are clear signs of the critical slowing down associated with phase transitions. A pronounced maximum in the slope of the curves (corresponding to a peak in the specific heat) can also be taken as a sign of a possible critical phenomenon. In the presence of a phase transition of higher order, one expects an increase in the fluctuations. This is clearly visible along some of the curves in Fig. 1a. In particular, the line with $\gamma = 1$ in the graph for $\langle S_L \rangle$ runs very close to a line of phase transitions (indicated by the hysteresis loops in the simulations at $\beta = \text{const}$) and one notices a marked increase in the fluctuations relative, for instance, to the other lines with $\gamma = \text{const}$.

From these measurements we deduce the phase diagram exhibited in Fig. 1b. The phase transition due to the discreteness of the group starts at $\beta = 6.05$, $\gamma = 0$ and moves to smaller β values with increasing γ . Above $\gamma \approx 5$ the hysteresis essentially disappears leaving a slope maximum; this indicates a change of the order of the phase transition. At $\gamma \gtrsim 6$ we were not able to detect a signal of this phase transition.

The other critical line is of much more interest - it should closely resemble the structure of the continuous group. From the comparison of the hysteresis we are led to attribute to that critical line a second order phase transition everywhere. As expected this critical line ends in the γ, β plane at a position close to $\beta = 1.2$, $\gamma = 1.5$. Concerning the shape of the curve towards larger values of β we find no evidence of a splitting into two lines (which would lead to a pocket of Coulomb phase). It seems that the line of phase transition approaches asymptotically the value of γ where one observes a phase transition of the spin system.

The thermal cycle at $\beta = 6$ shows a wide hysteresis loop in the results for $\langle S_L \rangle$. This interesting effect deserves some discussion. When β becomes very large the gauge variables are frozen into a configuration with vanishing gauge action. Locally this must be a gauge transform of the identity and, in particular, by suitably arranging the Higgs variables, the minimum of the Higgs action may also locally be set to zero. However, because of the finite size and the periodicity of the boundary conditions, the gauge action may be zero even if the configuration of the gauge variables is not a global gauge transform of the identity. In such a configuration the minimum of the Higgs action is larger than zero. In principle, as γ increases, the configurations of the gauge field which are locally, but not globally ordered should become energetically unfavourable; but because the MC upgrading is an essentially local process, the gauge field may remain locked into one of these configurations even at large γ . This is what is observed in the simulation at $\beta = 6$. The initial configuration taken from a run with $\gamma = 0$, still contains some disorder in the gauge variables because it is derived from the upper part of the hysteresis loop around the transition at $\beta_c \approx 6$ (see the diagram for $\langle S_p \rangle$). As the MC simulation proceeds and γ increases, both $\langle S_p \rangle$ and $\langle S_L \rangle$ decrease, but the Higgs action cannot get below some non-vanishing minimum value (although $\langle S_p \rangle$ is essentially zero), because of some global disorder frozen in. Eventually, for large γ (≈ 4), the ordering effect of the Higgs part of the action prevails and the whole system undergoes a transition to a more ordered configuration. It is interesting that this unfolding of the global disorder of the gauge field requires a temporary increase of $\langle S_p \rangle$, clearly evident as a small peak in the line with $\beta = 6$ in the upper part of the diagram. The simulation at $\beta = 8$ was done starting from the identity configuration of the gauge field and no effects attributable to global disorder are observed.

2.2 The Higgs field in the adjoint representation

a) Expectations

For $\beta = 0$ the integral again factorizes and may be solved explicitly (all ϕ_i may be transformed to some ϕ_0) confirming our MC results. For $\beta = \infty$ the system becomes a spin system with the icosahedral subgroup of $SO(3)$ as a symmetry group, i.e., the classical Heisenberg model for a subgroup of $O(3)$; one phase transition (presumably second order) is expected. For $\gamma = \infty$ the stability group is Z_{10} and we expect two phase transitions of second order at $\beta = 1$ and 4.08 ⁹⁾. Further results^{3),4)} are the evidence of a Higgs phase (perimeter decay of the Wilson loop, area decay of the 't Hooft loop) for both γ, β large and of a confinement phase (area decay for the Wilson loop, perimeter decay for the 't Hooft loop) for small β and γ .

b) Results

The thermal cycles in this case are shown in Fig. 2a. The analysis is done as for the previous case. The resulting phase diagram is shown in Fig. 2b. We observe that both critical lines now continue until $\gamma = \infty$. Actually $\gamma = 8$ already provides a good approximation to a Z_{10} theory with its two phase transitions. Using the full continuous group $SU(2)$ would lead to a $U(1)$ theory at $\gamma = \infty$. This would not change the position of the first phase transition at $\beta = 1$ but the second phase transition now at $\beta = 4.08$ would move to infinity. At the same time the phase transition at $\gamma = 0, \beta = 6.05$ would move to infinite values of β . From the hysteresis we expect that the critical line from $\gamma = \infty, \beta = 1$ to $\gamma \approx 0.75, \beta = \infty$ indicates a second order phase transition. The other one starts as a first order phase transition at $\gamma = 0, \beta = 6.05$ and appears to become second order above $\gamma \geq 2$.

3. LOOKING AT THE HIGGS CONFINEMENT TRANSITION

If one tries to characterize the transition from the Higgs region to the confinement region (with the matter field in the fundamental representation) one encounters the difficulty that there is no gauge invariant order parameter⁹⁾. In our model this is reflected in the phase diagram Fig. 1 which shows just one single phase to the left of the $\beta = 6$ line. On the other hand, the naive picture just assumes that the Higgs field itself develops a vacuum expectation value. We would like to understand the degree of validity of this assumption.

We have therefore looked at the behaviour of the following operator:

$$X = \sum_{\vec{n}_1} \frac{\text{Tr}}{2} (V_{\vec{n}_1, \vec{\Delta}}^+ W(\vec{n}_1, \vec{\Delta}) V_{\vec{n}_1}) \quad (3.1)$$

across the transition line at $\beta = 5.2$ ($\gamma = 0.6, 0.7, 0.8, 0.9$). Here $\vec{n}_1$ and $\vec{n}_1 + \vec{\Delta}$ are the two antipodal points in a fundamental hypercube (i.e., $\vec{n}_1 + \vec{\Delta} = (n_{1x} + 1, n_{1y} + 1, n_{1z} + 1, n_{1t} + 1)$) and W is the average over the 24 paths of minimum length from $\vec{n}_1$ to $\vec{n}_1 + \vec{\Delta}$ of the ordered products of the U field. In (3.1) the sum over $\vec{n}_1$ sweeps the whole lattice. This is the only reasonable choice at $\beta = 5.2$ where the correlation length is larger than the lattice. In general we would choose a volume whose size is a definite fraction of the correlation length. Then X will develop a large expectation value (compared to its fluctuations) if $\phi^\dagger \phi$ does so.

We have measured X , at each value of γ , over 1000 successive configurations of the lattice. The histograms are shown in Fig. 3. We observe that the average value is different from zero both below and above the transition at $\gamma = 0.75$. However, in the disordered phase the fluctuations and the average value are of the same order, while in the Higgs phase the average is much larger than the fluctuations. This is exactly what one expects to justify the semiclassical approximation around a new vacuum characterized (in a certain gauge) by $\phi \neq 0$.

4. CONCLUSIONS

MC methods appear to be well suited to discussing coupled Higgs gauge systems. In the cases considered we find no surprises, i.e., no deviations from the theoretical expectations. However, contrary to the approaches based on series expansions, we are capable of determining the field phase structure quite efficiently. For a closer investigation of order parameters one will have to move to larger lattices.

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FIGURE CAPTIONS

- Fig. 1 : 1a) Values of $\langle S_p \rangle$ and $\langle S_L \rangle$ as measured in simulations of thermal cycles for case (a): the Higgs field in the fundamental representation. The values of β or γ are 0, 1, 2, 3, 4, 5, 6, 8.
1b) The phase diagram deduced from the thermal cycles shown above.
- Fig. 2 : 2a) Values of $\langle S_p \rangle$ and $\langle S_L \rangle$ as measured in simulations of thermal cycles for case (b): the Higgs field in the adjoint representation. The values of β or γ are 0, 1, 2, 3, 4, 5, 6, 8.
2b) The phase diagram deduced from the thermal cycles shown above.
- Fig. 3 : Histograms showing the distribution of the instantaneous values of the operator X measured a total of 1000 times at $\gamma = 0.6, 0.7, 0.8, 0.9$.

FIG. 1a

FIG. 1b

FIG. 2a

Fig. 2b

FIG. 3