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A NOTE ON THE "ONE-LINK" INTEGRAL OF
U(N) LATTICE GAUGE THEORY

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ABSTRACT

We calculate the "one-link" $U(N)$ integral in closed form by a direct method, i.e., polar decomposition and integration over angular variables. The result agrees with the known solution of the Brower-Nauenberg equation. A possible role of this result in understanding the weak coupling limit of lattice QCD is discussed.

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1. INTRODUCTION

The integral over the $U(N)$ group

$$Z(A) = \int_{U(N)} [dU] \exp \{ \text{Tr} (AU + U^\dagger A^\dagger) \} \quad (1)$$

where $[dU]$ is the Haar measure on $U(N)$ and A is any complex matrix, has recently been the object of intensive investigations ¹⁾⁻³⁾. Its interest has been stressed by many authors and we shall make some comments on this in the conclusions. It is sufficient here to recall that we may consider $A = g_0^{-2} \langle \tilde{U} \rangle$ as an average self-consistent plaquette; $Z(A)$ is an essential ingredient of a mean field type approximation of lattice gauge theory.

A closed form of $Z(A)$ has been proposed by Brower, Rossi and Tan ⁴⁾ who showed that

$$Z(\lambda) = \mathcal{N} \frac{\det \| \lambda_j^{\nu-1} I_{\nu-1}(2\lambda_j) \|}{\det \| \lambda_j^{2(\nu-1)} \|} \quad (2)$$

is a solution of the Brower-Nauenberg equation, $(\lambda_1, \dots, \lambda_N)$ being the eigenvalues of $\sqrt{AA^\dagger}$. It is not clear, however, whether the only boundary condition $Z(0)=1$ is sufficient to give a unique solution. It is true that the asymptotic growth at infinity ($\lambda \rightarrow \infty$, or equivalently $g_0 \rightarrow 0$) is correct, as it can be obtained by a Gaussian approximation directly from Eq. (1). However, no rigorous proof of uniqueness under these conditions was given, to our knowledge. On the other hand, the system of partial differential equations introduced in Ref. 3) leads to a unique solution, independently of the growth at infinity, but it is difficult to check whether $Z(\lambda)$ is in fact a solution (it was a painful experience to check it for $N=3$ - see also Appendix B).

A more direct derivation of Eq. (2) then seems desirable, and this note is devoted to this aim. The result depends on an integral formula of Itzykson and Zuber ⁵⁾ and Mehta ⁶⁾ which allows the integration over angular variables in the polar decomposition. The final form of Eq. (2) is obtained by applying a new identity satisfied by Bessel functions.

It is hoped that this direct method will also apply to $SU(N)$, with minor modifications.

2. THE INTEGRAL OVER $U(N)$ USING THE POLAR DECOMPOSITION

We let $A \equiv \Lambda$ be diagonal with non-negative entries $\lambda_1, \lambda_2, \dots, \lambda_N$.
Let us introduce polar co-ordinates in $U(N)$:

$$U = S \Phi S^\dagger \quad (3)$$

where $\Phi = \text{diag}[e^{i\varphi_1}, \dots, e^{i\varphi_N}]$ and S runs over some coset space. It is well known that the integral Eq. (1) becomes

$$Z(\lambda) = \frac{2^{N(N-1)}}{N! (2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} d\varphi_1 \dots d\varphi_N [dS] \prod_{i>j} \left(\sin \frac{\varphi_i - \varphi_j}{2} \right)^2 \exp \{ 2\pi i (\Lambda S dS^\dagger) \} \quad (4)$$

where $C = \text{diag}[\cos \varphi_1, \dots, \cos \varphi_N]$. A result of Ref. 5) allows us to integrate over S to obtain

$$Z(\lambda) = \frac{2^{\frac{1}{2}N(N-1)} \prod_{i=1}^{N-1} \nu!}{N! (2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} d\varphi_1 \dots d\varphi_N \prod_{i>j} \left[\frac{\sin^2 \frac{\varphi_i - \varphi_j}{2}}{(\lambda_i - \lambda_j)(\cos \varphi_i - \cos \varphi_j)} \right] \det \| e^{2\lambda_j \cos \varphi_k} \| \quad (5)$$

It is convenient to introduce complex variables $z_k = e^{i\varphi_k}$, in terms of which we find

$$Z(\lambda) = \frac{\prod_{i=1}^{N-1} \nu!}{N! \Delta(\lambda)} \oint \dots \oint \prod_{k=1}^N \frac{dz_k}{2\pi i z_k} \frac{\Delta(z)}{\prod_{i>j} (1 - z_i \bar{z}_j)} \det \| e^{\lambda_j (z_k + \bar{z}_k^{-1})} \| \quad (6)$$

where $\Delta(\cdot)$ denotes the Vandermonde determinant. We can now simplify this expression by writing the determinants explicitly :

$$\begin{aligned}
 \Delta(z) \det \left\| e^{\lambda_j(z_k + \bar{z}_k^{-1})} \right\| &= \\
 &= \sum_P \delta_P \prod_{j=1}^N z_{P_j}^{j-1} \sum_Q \delta_Q \prod_{k=1}^N e^{\lambda_k(z_{Q_k} + \bar{z}_{Q_k}^{-1})} \\
 &= \sum_P \sum_R \delta_P \delta_{R^{-1}P} \prod_{j=1}^N z_{P_j}^{j-1} \prod_{k=1}^N e^{\lambda_{R_k}(z_{P_k} + \bar{z}_{P_k}^{-1})}
 \end{aligned} \tag{7}$$

where P, Q, R run over the permutation group, $Q = PR^{-1}$, and δ_P is the character of P in the alternate representation. Since the z 's are dummy integration variables, the result is independent of P ; $\sum_P$ gives a factor $N!$ and we obtain

$$Z(\lambda) = \frac{\pi^N \nu!}{\Delta(\lambda)} \oint \dots \oint \prod_{k=1}^N \frac{dz_k}{2\pi i} z_k^{k-2} \frac{\det \left\| e^{\lambda_j(z_i + \bar{z}_i^{-1})} \right\|}{\prod_{i>j} (1 - z_i z_j)} \tag{8}$$

Notice that the factor $\prod (1 - z_i z_j)^{-1}$ has singularities which are cancelled by the zeros of the determinant; this allows us to deform the contours to $|z_k| = \rho < 1$. We are left with a problem of calculus of residues. Let us expand

$$(1 - z_i z_j)^{-1} = \sum_{h_{ij}}^{\infty} (z_i z_j)^{h_{ij}}$$

and insert the generating function of the modified Bessel functions : we obtain

$$Z(\lambda) = \frac{\pi^N \nu!}{\Delta(\lambda)} \sum_{\{h\}} \det \left\| I_{k-1 + \sum_{j \neq k} h_{kj}}(2\lambda_n) \right\|_{k,n=1}^N \tag{9}$$

This is hard to accept as a closed expression of the integral, since it involves an infinite multiple series over the $N(N-1)/2$ indices h_{ij} ($\equiv h_{ji}$). The final step is to prove that Eq. (9) is in fact identical to Eq. (2). This is shown in Appendix A for $N \leq 4$ where we also give an argument (not yet 100% proof) based on mathematical induction.

3. CONCLUDING REMARKS

We have found the closed form of the "one-link" $U(N)$ integral by a direct integration. This overcomes the question of uniqueness in the solution of the Brower-Nauenberg equation ^{*}). Having this exact result at hand, what can we do now? In our opinion, the closed form of the one-link integral may be useful in giving a highly accurate asymptotic expansion in the weak coupling. Actually, the large N asymptotic formula

$$Z(\lambda) \sim \frac{e^{2\sum \lambda_j}}{\prod_{i,j} (\lambda_i + \lambda_j)^{1/2}} \quad (10)$$

is merely the outcome of a Gaussian approximation $U \sim 1 - iX - \frac{1}{2}X^2$. Equation (2), however, can give us all the higher corrections including exponentially small terms which would normally be overlooked in a weak coupling expansion. In fact, the asymptotic expansion of $I_v(z)$ contains terms of order $\exp(-z)$ [i.e., $\exp(-1/g_0^2)$] besides the dominant $\exp(z)$ terms. It may happen that if the dominant term gives a vanishing result (say for the string tension), the exponentially small terms would come into play. This could be the mechanism for understanding non-perturbative effects on the lattice and, hopefully, for reproducing the Monte Carlo results.

^{*}) Notice that in Ref. 7) a direct integration was given for $N=2$ and $N=3$. However, the identification with Eq. (2) was not given explicitly, but on the basis of the assumed uniqueness of the solution.

A P P E N D I X A

Let us consider the series [Eq. (9) for $N=2$]

$$\mathcal{J}^{(2)}(x,y) \equiv \sum_0^\infty \begin{vmatrix} I_k(x) & I_{k+1}(x) \\ I_k(y) & I_{k+1}(y) \end{vmatrix} \quad (\text{A.1})$$

Multiply by $(x+y)$ and apply the recursion relation to obtain

$$\begin{aligned} (x+y) \mathcal{J}^{(2)}(x,y) &= \sum_0^\infty \begin{vmatrix} y I_k(x) + x I_{k+2}(x) & I_{k+1}(x) \\ x I_k(y) + y I_{k+2}(y) & I_{k+1}(y) \end{vmatrix} \\ &= \left(\sum_{k=0}^\infty - \sum_{k=1}^\infty \right) \begin{vmatrix} I_k(x) & x I_{k+1}(x) \\ I_k(y) & y I_{k+1}(y) \end{vmatrix} = \begin{vmatrix} I_0(x) & x I_1(x) \\ I_0(y) & y I_1(y) \end{vmatrix} \end{aligned} \quad (\text{A.2})$$

In general, we have

$$\mathcal{J}_\mu^{(2)}(x,y) = \sum_{k=\mu}^\infty \begin{vmatrix} I_k(x) & I_{k+1}(x) \\ I_k(y) & I_{k+1}(y) \end{vmatrix} = (x+y)^{-1} \begin{vmatrix} I_\mu(x) & x I_{\mu+1}(x) \\ I_\mu(y) & y I_{\mu+1}(y) \end{vmatrix} \quad (\text{A.3})$$

To proceed to $N > 2$, let us introduce the notation

$$[\mu_1 | \mu_2 | \dots | \mu_N] \equiv \det \| I_{\mu_k}(x_j) \|_{k,j} \quad (\text{A.4})$$

For $N=3$ we have

$$\begin{aligned} \mathcal{J}^{(3)}(x,y,z) &= \sum_{i,j,k}^\infty [i+j | i+k+1 | j+k+2] \\ &= \sum_{j=1}^\infty \sum_{i,k}^\infty [i+j-1 | i+k+1 | j+k+1] \end{aligned} \quad (\text{A.5})$$

By exchanging i and j we see that only the $i=0$ terms survive :

$$J^{(3)} = \sum_{j,k}^{\infty} \llbracket j | k+1 | j+k+2 \rrbracket \quad (A.6)$$

By the same argument, only $j=0$ gives a contribution and we are left with

$$\begin{aligned} J^{(3)} &= \sum_k^{\infty} \llbracket 0 | k+1 | k+2 \rrbracket = \\ &= I_0(x) \frac{\begin{vmatrix} I_1(y) & y I_2(y) \\ I_1(z) & z I_2(z) \end{vmatrix}}{y+z} + (\text{cyclic perm.}) \\ &= \frac{\begin{vmatrix} x(x^2 + x(y+z) + yz) I_0(x) & x I_1(x) & x^2 I_2(x) \\ y(y^2 + y(x+z) + xz) I_0(y) & y I_1(y) & y^2 I_2(y) \\ z(z^2 + z(x+y) + xy) I_0(z) & z I_1(z) & z^2 I_2(z) \end{vmatrix}}{x y z (x+y)(y+z)(z+x)} \end{aligned} \quad (A.7)$$

The recursion relation is now sufficient to obtain the desired result. Obviously an initial shift u on the indices $\Sigma \llbracket i+j+u | i+k+u+1 | j+k+u+2 \rrbracket$ would give the same expression with all indices shifted by u .

The case $N=4$ is similarly treated; by the antisymmetry property we can easily reduce the series

$$J^{(4)} = \sum_{\substack{i,j,k \\ p,q,s}}^{\infty} \llbracket i+j+k | i+p+q+1 | j+p+s+2 | k+q+s+3 \rrbracket \quad (A.8)$$

to the following

$$\begin{aligned} J^{(4)} &= \frac{1}{2} \sum_{p,q}^{\infty} \llbracket p | p+1 | q | q+1 \rrbracket = \\ &= (x+y)^{-1} (z+t)^{-1} \left| \begin{vmatrix} I_0(x) & x I_1(x) \\ I_0(y) & y I_1(y) \end{vmatrix} \right| \left| \begin{vmatrix} I_0(z) & z I_1(z) \\ I_0(t) & t I_1(t) \end{vmatrix} \right| + (\text{cycl. perm.}) \end{aligned} \quad (A.9)$$

and finally to the desired form by the recursion relation. It would be nice to have a general proof for any N of the identity $(h_{ij} \equiv h_{ji})$

$$\sum_{\substack{\{h_{ij}\} \\ i < j}} \llbracket \sum h_{1j} + \mu \mid \sum h_{2j} + \mu + 1 \mid \dots \mid \sum h_{Nj} + \mu + N - 1 \rrbracket \equiv$$

$$\equiv \frac{\det \left\| x_k^{\nu-1} I_{\mu+\nu-1}(x_k) \right\|}{\prod_{i < j} (x_i + x_j)} \quad (\text{A.10})$$

At this stage, all we can say is that having established the identity for some N one can induce it for $N+1$ by looking at the Brower-Nauenberg equation : Equations (2) and (9) are both solutions, they coincide on the whole hyperplanes $P_k = \{\lambda_k = 0\}$ and they have the same growth at ∞ . In these conditions, uniqueness should be easier to prove.

A P P E N D I X B → Brézin and Gross vs Brower and Nauenberg

We want to show how the system introduced in Ref. 3) admits a unique regular solution in the simplest case ($N=2$). We have

$$\begin{cases} x Z_{xx} + 2 Z_x + \frac{y}{y-x}(Z_y - Z_x) = 4 Z \\ y Z_{yy} + 2 Z_y + \frac{x}{y-x}(Z_y - Z_x) = 4 Z \end{cases} \quad (B.1)$$

By introducing new variables $t = \sqrt{x} + \sqrt{y}$, $s = \sqrt{xy}$ we transform Eq. (B.1) to the following

$$\begin{cases} t Z_{tt} + 4 s Z_{ts} + s t Z_{ss} + 4 Z_t + t Z_s - 16 t Z = 0 \end{cases} \quad (B.2a)$$

$$\begin{cases} t Z_{tt} - s t Z_{ss} + 2 Z_t - t Z_s - 16 t Z = 0 \end{cases} \quad (B.2b)$$

Equation (B.2a) alone is just the Brower-Nauenberg equation. Now let $Z = \sum_0^\infty a(m,n) t^m s^n$. We obtain two recursion relations for the coefficients which determine the solution completely. We find

$$\begin{aligned} Z(s,t) &= \sum_{n,k}^\infty \frac{(2n-1)!!}{n!^2 k!(k+n)!} (2t)^{2k} (-8s)^n \\ &= \frac{1}{8t} \frac{\partial}{\partial t} \sum_{n=0}^\infty \frac{(2n-1)!!}{n!^2} (-8s)^n (2t)^{-n} I_n(4t) \\ &= \frac{1}{8\pi t} \frac{\partial}{\partial t} \int_0^\pi e^{4t \cos \vartheta} J_0(8\sqrt{s} \sin \vartheta) d\vartheta \end{aligned} \quad (B.3)$$

The integral is known ⁸⁾ to give a product of Bessel functions, and finally

$$Z(s,t) = \frac{1}{8t} \frac{\partial}{\partial t} \left[I_0(2t + 2\sqrt{t^2 - 4s}) I_0(-2t + 2\sqrt{t^2 - 4s}) \right] \quad (\text{B.4})$$

which is easily identified with Eq. (2). By contrast, the Brower-Nauenberg equation, Eq. (B.2a), gives only one recursion relation which determines $Z(s,t)$ from its knowledge at $s=0$. Of course, it may happen that just one choice of $Z(0,t)$ is compatible with the asymptotic growth $Z \sim t^{-1} s^{-\frac{1}{2}} \exp\{4t\}$ but this seems hard to prove (or disprove).

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