Ref.TH.3107-CERN

FUNCTIONAL EQUATIONS FOR PLANAR GRAPHS

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ABSTRACT

The closed equation for the generating functional for the planar Green function in scalar field theory is constructed.

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In field theory we usually have to deal with essentially non-linear systems. It is very important to choose the adequate perturbation expansion in order to have the possibility of describing the essential feature of such systems. For example, for chiral field theory such an important property as renormalizability depends on the choice of perturbation expansion¹⁾. For the gauge theory^{*)} the short distance dynamics is well described by the usual perturbation theory in the coupling constant, but this expansion is unacceptable for long distances. As was pointed out by 't Hooft⁴⁾ the adequate choice for the last goal might be the $1/N$ expansion where N is the number of colours. The reason for this hope is that actually the leading order of the $1/N$ expansion seems able to describe the features of strong interactions. Namely, the leading order corresponds to some dual resonance theory⁵⁾ and some phenomenological rules can be explained in the framework of the $1/N$ expansion⁶⁾. The leading order of the $1/N$ expansion for $U(N)$ or $SU(N)$ gauge theory is the sum of an infinite number of planar Feynman graphs. The contribution to the next-to-leading orders gives the graphs characterized by the topology of the two-dimensional surface on which one can draw these graphs. This topological expansion is also correct for all Lagrangians of the form

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}, \quad (1)$$

$$\mathcal{L}_0 = \frac{1}{2} \text{Tr} (\partial_\mu \varphi)^2, \quad \mathcal{L}_{int} = \sum_{n=3}^{\infty} c_n \left(\frac{N}{g^2} \right)^{1-\frac{n}{2}} \text{Tr} (\varphi^n),$$

where $(\varphi_i^j) \in U(N)$.

We do not doubt that the quantitative analysis of the sum of all planar graphs for QCD as well as for scalar theories (1) is a very important problem, but it is also an extremely difficult one.

This problem in the $D = 1$ dimensional scalar theory was solved by Brézin et al.⁷⁾. However, there is not direct method of summing the relevant planar graphs for $D > 1$. Apparently, one of the possibilities of attacking this problem is to write down the closed equations, if possible, for the generating functional for the planar Green function and then try to solve these equations. The closed system of equation for the planar loop Green function for gauge theory was obtained by Mareenko and Migdal⁸⁾ and an attempt to get the solution of this equation was made in Ref. 9). However, the equation suggested in Ref. 8) is rather specific to gauge theory.

*) We do not discuss here the rather extreme point of view that more interesting physical systems must have exact integrability^{2),3)}.

The main result of this paper is the construction of a closed equation for the generating functional for the planar Green function. Let me clarify why this problem is a non-trivial one. For simplicity I shall study only the case when $L = 3$ in the Lagrangian (1), $c_3 = 1/3$. The Schwinger-Dyson equations take the form:

$$\begin{aligned} \partial_{x_1}^2 G_n(x_1, x_2, \dots, x_n) + \frac{g}{\sqrt{N}} G_{n+1}(x_1, x_1, x_2, \dots, x_n) = \\ = \sum_{i=2}^n \delta(x_1 - x_i) G_{n-2}(x_2, \dots, \hat{x}_i, \dots, x_n), \end{aligned} \quad (2)$$

where

$$G_n(x_1, \dots, x_n) = \langle \varphi(x_1) \dots \varphi(x_n) \rangle = N_0 \int \varphi(x_1) \dots \varphi(x_n) e^{-\int \mathcal{L} d^D x} \prod_x d\varphi(x).$$

It is obvious that in each term of this equation we can take into account only planar graphs, so we get:

$$\begin{aligned} \partial_{x_1}^2 G_n^{pl}(x_1, \dots, x_n) + \frac{g}{\sqrt{N}} G_{n+1}^{pl}(x_1, x_1, \dots, x_n) = \\ = \sum_{\substack{k=0, \dots, n-1 \\ i_j=2, \dots, n}} \delta(x_1 - x_{i_k}) G_k^{pl}(x_1, x_{i_1}, \dots, x_{i_{k-1}}) G_{n-2-k}^{pl}(x_{i_{k+1}}, \dots, x_{i_{n-2}}). \end{aligned} \quad (3)$$

However, this equation is not the closed equation for the planar Green functions $G_n^{pl}(x_1, \dots, x_n)$ because the second term in (3) does not have an internal definition:

$$G_{n+1}^{pl}(x_1, x_1, \dots, x_n) \neq \lim_{x_1' \rightarrow x_1} G_{n+1}^{pl}(x_1, x_1', \dots, x_n)$$

Indeed, if we join one leg of the planar Green function to another leg, we do not, in general, get only planar graphs. However, from Fig. 1 one can see what we have to do to describe the arguments of the Green function, which can be made equal without changing the topology of the graph under consideration.

For this purpose we have to put the legs of the Green function on the ring so that the ring does not cross the legs. The arguments of the legs nearest on the ring

can then be made equal without loss of planarity. One can reach this kind of ordering on the ring more simply by considering the leading order of the invariant Green function:

$$\frac{1}{N^{1+n/2}} \langle \text{Tr} (\varphi(x_1) \dots \varphi(x_n)) \rangle = w_n(x_1, \dots, x_n) + O(1/N). \quad (4)$$

It is obvious that the following equality takes place

$$w_{n+1}(x_1, x_1', \dots, x_{n+1}) = \lim_{x \rightarrow x'} w_{n+1}(x_1, x_1', \dots, x_{n+1}) \quad (5)$$

and one can get the closed equations for $w_n(x_1, \dots, x_n)$ ¹⁰⁾. However, the functions w_n do not symmetrize in their arguments so one cannot get, for these functions a generating functional which depends on one commuting source [cf. Ref. 11)]^{*}.

We shall describe the functional depending on more than one commuting source which satisfies the closed equation and generates the planar Green functions. As we just saw, one can get the closed equation for these functions only with the help of some kind of ordering of their legs. To perform this ordering we introduce some additional non-relativistic fermion field, ψ , which is the vector of the fundamental representation of the group $U(N)$

$$\mathcal{L}' = \mathcal{L} + \mathcal{L}_\psi + \mathcal{L}'_{int}, \quad (6)$$

$$\mathcal{L}_\psi = \bar{\psi}_i \mathcal{D} \psi^i, \quad \mathcal{L}'_{int}(x) = N^{-1/2} f(x) \bar{\psi}_i(x) \varphi_j^i(x) \psi^j(x),$$

where $f(x)$ is the scalar function and $\mathcal{D}$ is the non-relativistic propagator such that $(\mathcal{D}^{-1})(x, y) = K(x, y) = \theta(y_0 - x_0) \chi(x - y)$, $[\theta(x) = \begin{matrix} 1, & x \geq 0 \\ 0, & x < 0 \end{matrix}]$. For example, one can consider the following operator $\mathcal{D}$: $\mathcal{D} = \partial_{x_0} + v(x)$.

Let us consider the functional

$$\mathcal{Z}(f, \bar{\eta}, \eta) = \int \exp \left\{ \int [\mathcal{L}' + \bar{\psi} \eta + \bar{\eta} \psi] d^D x \right\} \prod_x d\psi d\bar{\psi} d\varphi, \quad (7)$$

^{*}) In Ref. 11) the generating functional depending on a non-commuting source was described.

which depends on two scalar functions and two vectors. For the connected part $W(\chi, f, \bar{\eta}, \eta)$ of $Z(\chi, f, \bar{\eta}, \eta)$ we have the usual representation:

$$\mathcal{Z}(\chi, f, \bar{\eta}, \eta) = \exp W(\chi, f, \bar{\eta}, \eta).$$

The generating functional (7) connects with the usual generating functional

$$\mathcal{Z}(y_i^j) = \int \exp \left\{ \int [\mathcal{L}(\varphi) + y_i^j \varphi_j^i] dx \right\} \prod_x d\varphi(x) \quad (8)$$

by the following formula:

$$\mathcal{Z}(\chi, f, \bar{\eta}, \eta) = \mathcal{Z}(N^{-1/2} f \frac{\delta^2}{\delta \bar{\eta} \delta \eta}) \exp \{ \bar{\eta} K \eta \}, \quad (9)$$

where the normalization constants are omitted and $\bar{\eta} K \eta = \int \bar{\eta}_i(x) K(x-y) \eta^i(y) dx dy$.

For example, the following representations go from Eq. (9)

$$\langle T \varphi(x_1) \dots \varphi(x_n) \rangle = \left(\prod_{i=1}^n \frac{\delta}{\delta f(x_i)} \right) \frac{\delta^2}{\delta \bar{\eta}_i(y) \delta \eta_i(z)} \mathcal{Z}(\chi, f, \bar{\eta}, \eta) \Big|_{\substack{\bar{\eta}=\eta=0 \\ \chi=1 \\ f=0}} \quad (10)$$

$$y_0 < x_{10} < \dots < x_{n0} < z_0$$

for the invariant Green function, and for the non-invariant one:

$$\begin{aligned} G_{i_1 i'_1, \dots, i_n i'_n}(x_1, \dots, x_n) &= \\ &= \prod_{j=1}^n \frac{\delta^3}{\delta f(x_j) \delta \bar{\eta}_{i_j}(y_j) \delta \eta_{i'_j}(z_j)} \mathcal{Z}(\chi, f, \bar{\eta}, \eta) \Big|_{\substack{\bar{\eta}=\eta=0 \\ \chi=1, f=0}} \end{aligned} \quad (11)$$

$$y_{10} < x_{10} < z_{10} < y_{20} < x_{20} < z_{20} < \dots < y_{n0} < x_{n0} < z_{n0}.$$

Equation (11) also conserves for planar graphs so we can put the index (p\ell) for Z and G in Eq. (11), and Eq. (10) changes to

$$W_n(x_1, \dots, x_n) =$$

$$= \left(\prod_{i=1}^n \frac{\delta}{\delta f(x_i)} \right) \frac{\delta^2}{\delta \bar{\eta}_i(y) \delta \eta_i(z)} \mathcal{L}^{pl}(\chi, f, \bar{\eta}, \eta) \Big|_{\substack{\eta = \bar{\eta} = 0 \\ \chi = 1 \\ f = 0}},$$

$$y_0 < x_{10} < \dots < x_{n0} < z_0.$$

Let me introduce the following operators:

$$\mathcal{K}^{(1)}(y, x, z) \mathcal{A}(\chi, f) = S_\delta(y-y', z-z') \left\{ \frac{\delta^3}{\delta \chi(x-y') \delta \chi(z'-x) \delta f(x)} \times \right.$$

$$\left. \left[f(z) \frac{\delta}{\delta f(z)} + \eta^i(z) \frac{\delta}{\delta \eta^i(z)} \right] \left[f(y) \frac{\delta}{\delta f(y)} + \bar{\eta}_i(y) \frac{\delta}{\delta \bar{\eta}_i(y)} \right] \mathcal{A}(\chi, f) \right\}; \quad (12)$$

$$\mathcal{K}^{(2)}(y, x, t, z) \mathcal{A}(\chi, f) = S_\delta(y-y', z-z', t-t') \left\{ \frac{\delta^5}{\delta f(x) \delta f(t) \delta \chi(x-y) \delta \chi(t'-x) \delta \chi(z'-t)} \times \right.$$

$$\left. \left[f(y) \frac{\delta}{\delta f(y)} + \bar{\eta}_i(y) \frac{\delta}{\delta \bar{\eta}_i(y)} \right] \left[f(z) \frac{\delta}{\delta f(z)} + \eta^i(z) \frac{\delta}{\delta \eta^i(z)} \right] \right\} \mathcal{A}(\chi, f); \quad (13)$$

$$\mathcal{M}(y, x) \mathcal{A}(\chi(v), f) = S_\delta(x-x') \left\{ \frac{\delta^2}{\delta \chi(x-y) \delta v(x')} \left[f(y) \frac{\delta}{\delta f(y)} + \bar{\eta}_i(y) \frac{\delta}{\delta \bar{\eta}_i(y)} \right] \mathcal{A}(\chi, f) \right\}$$

(see Figs. 2 and 3), where $\mathcal{A}(\chi, f)$ is arbitrarily functional on two functions χ and f , and $S_\delta^{(2)}(y-y'; z-z')$, $S_\delta^{(3)}(y-y', z-z', t-t')$ are the operators which single out the δ singularity of the distributions. For example, if

$$\Phi(y, y', z, z') = \delta(y-y') \delta(z-z') \phi_1(y, z) + \delta(y-z') \delta(y'-z) \phi_2(y, z) +$$

$$+ \delta(y-z) \delta(y'-z') \phi_3(y, y') + \delta(z-z') \phi_4(z, y, y') + \dots + \phi(y, y', z, z'),$$

where ϕ_i, ϕ are bounded functions, then

$$S_\delta(y-y', z-z') \Phi(y, y', z, z') = \phi_1(y, z). \quad (14)$$

For equal arguments the operators $\mathcal{K}^{(n)}$ are understood in the weak sense, i.e.,

$$\mathcal{K}^{(2)}(y, x, x, z) \mathcal{A} = \lim_{x' \rightarrow x} (\mathcal{K}^{(2)}(y, x, x', z) \mathcal{A})$$

Let us consider the functional $W^{pl}(\chi, f, \eta, \bar{\eta})$, which describes only the planar graphs. It is easy to see that it satisfies the following equation

$$\begin{aligned} & \partial_x^2 [\mathcal{K}^{(1)}(y, x, z) W^{pl}(\chi, f, \bar{\eta}, \eta)] + \\ & + \mathcal{K}^{(4)}(y, x, x, z) W^{pl}(\chi, f, \bar{\eta}, \eta) = \\ & = \frac{f(x)}{N} [\mathcal{M}(y, x) W^{pl}(\chi, f, \bar{\eta}, \eta)] [\bar{\mathcal{M}}(x, z) W^{pl}(\chi, f, \bar{\eta}, \eta)]. \end{aligned} \quad (15)$$

The operator $\bar{\mathcal{M}}$ is defined by a formula similar to (11), but instead of η we must put $\bar{\eta}$. This equation is represented in Fig. 4.

Substituting the ansatz (7) in Eq. (15)

$$\begin{aligned} & \left\{ \mathcal{Z} \left(\frac{f}{N^{1/2}} \frac{\delta^2}{\delta \eta \delta \bar{\eta}} \right) \exp \bar{\eta} K \eta \right\} \cdot \left[\partial_x^2 \mathcal{K}^{(1)}(y, x, z) + \frac{g}{N^{1/2}} \mathcal{K}^{(2)}(y, x, x, z) \right] \cdot \\ & \mathcal{Z} \left(\frac{f}{N^{1/2}} \frac{\delta^2}{\delta \eta \delta \bar{\eta}} \right) \exp \bar{\eta} K \eta = \\ & = \frac{f(x)}{N} \left\{ \left[\frac{\delta \mathcal{Z}(y)}{\delta \eta_i(y)} \frac{\delta^2}{\delta \eta^i(y) \delta \bar{\eta}_k(x)} + \mathcal{Z}(y) \bar{\eta}_i(y) \frac{\delta}{\delta \bar{\eta}_i(x)} \right] e^{\bar{\eta} K \eta} \right\}_x \\ & \left\{ \left[\frac{\delta \mathcal{Z}(y)}{\delta \eta_m^{n(z)}} \frac{\delta^2}{\delta \eta^m(x) \delta \bar{\eta}_n(z)} + \mathcal{Z}(y) \eta^i(z) \frac{\delta}{\delta \eta^i(x)} \right] e^{\bar{\eta} K \eta} \right\}, \end{aligned}$$

one can explicitly verify that in the usual perturbative expansion the planar graphs for $Z(f, \chi, \bar{\eta}, \eta)$ are generated by this equation. The equation for the function $w_n(x_1, \dots, x_n)$ can also be obtained from Eq. (15).

ACKNOWLEDGMENTS

I would like to express my gratitude to P. Cvitanović and G. Veneziano for discussions and to the Theoretical Physics Division at CERN for their kind hospitality.

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planar graph

planar graph

nonplanar graph

fig.1

$$y \xrightarrow{\quad} x \quad \Theta(x_0 - y_0) \chi(x - y)$$

fig 2

fig.3

fig.4