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SUPERCURRENTS

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A B S T R A C T

Supercurrents that provide irreducible off-shell representations of N -extended supersymmetry with maximum spin $\geq N/2$ are constructed as bilinears of massless on-shell superfields. The corresponding prepotentials are derived and used to construct maximally gauge invariant field theories with maximum spin $\geq N/2$. The construction of non-conformal theories using these irreducible field multiplets is discussed; in particular the superfield form of massive spin 1 is given for $N = 2$ and compensating fields are proposed for $N = 4$ Poincaré supergravity. We also show how the components of supercurrents may be simply obtained with the aid of the "tableaux calculus" and we work out in detail several $N = 4$ and $N = 8$ examples.

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1. INTRODUCTION

Every rigid symmetry of a field theory has a corresponding Noether current that is conserved as a consequence of the field equations. In supersymmetric theories these currents are related to each other by supersymmetry transformations, that is they form supermultiplets. The simplest example is the massless Wess-Zumino model,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\psi}\not{\partial}\psi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \quad (1.1)$$

The (improved) stress tensor $T_{\mu\nu}$, for which $T^\mu_\mu = 0$, the (improved) spinor current S_μ , for which $\gamma \cdot S = 0$, and a conserved axial current $j_\mu^5 = i/4\bar{\lambda}\gamma_5\gamma_\mu\lambda + A\overset{\leftrightarrow}{\partial}_\mu B$, together form a supermultiplet [1],

$$(T_{\mu\nu}, S_\mu, j_\mu^5) \quad (1.2)$$

provided the field equations of the model (1.1) are imposed. The interesting point is that (1.2) then furnishes an off-shell representation of supersymmetry on fields; it determines directly a multiplet of gauge fields via the Noether coupling

$$\mathcal{L}_{int} = h \cdot J \quad (1.3)$$

The gauge transformations of h are determined by the conservation conditions on J . In the particular case above we have the correspondence

$$\begin{aligned} T_{\mu\nu} &\leftrightarrow h_{\mu\nu} \\ S_\mu &\leftrightarrow \psi_\mu \\ j_\mu^5 &\leftrightarrow A_\mu \end{aligned} \quad (1.4)$$

from which we deduce the field multiplet $(h_{\mu\nu}, \psi_\mu, A_\mu)$ of conformal supergravity, together with the obvious gauge transformations.

It is of interest to note that had we chosen a different conserved axial current $j_\mu^{5'} = -i/2\bar{\lambda}\gamma_5\gamma_\mu\lambda + A\overset{\leftrightarrow}{\partial}_\mu B$; i.e. with a different relative combination of spinor and scalar contributions, then we would have found a different multiplet of currents, the "axial charge multiplet"

$$(j_\mu^{5'}, \chi, s) \quad (1.5)$$

where $S = A^2 + B^2$ and $\chi = (A - i\gamma_5 B)\psi$. From this "supercurrent" we can deduce the field multiplet (A_μ, λ, D) . In the interacting theory, however, only j_μ^5 can remain conserved. This will indeed be a general feature; interacting theories have many fewer conserved currents than free theories. Nevertheless we can use any current multiplet*) of the free theory to deduce an off-shell representation of supersymmetry. Which of these field representations can be used to construct *interacting* theories is of course another question; although one knows that if the representation contains gauge fields with spins > 2 it cannot be done straightforwardly.

The above procedure for finding off-shell representations of supersymmetry is not new, although it has not yet been systematically exploited. It was used recently [2] to discover the multiplet of fields of $N = 4$ conformal supergravity [2,3] by constructing the stress tensor multiplet for $N = 4$ Maxwell theory. One can also deduce that the prepotential of $N = 2$ Maxwell theory is the $SU(2)$ triplet scalar superfield v^{ij} that has been found previously by other methods [4]. A new application, however, is to the spin 4 "Bel-Robinson multiplets", whose square can be used to construct three-loop counterterms [5,6,7] for supergravity theories.

There are various other new off-shell representations that can be constructed by this method, which are also of physical interest. Firstly, there are the various possible higher spin multiplets, e.g. spin 3 and spin 4, for $N = 4$, that have been conjectured to be relevant to the Poincaré, i.e. non-conformal, stress tensor multiplet of $N = 4$ supersymmetry. Secondly, the composite $SU(8)$ gauge fields of $N = 8$ supergravity [8] that are important in phenomenological applications [9,10] are expected to appear in multiplets of fields that have maximum spins 6 and 8. That the $N = 8$ spin 6 and spin 8 multiplets contain spin 1 fields in the $\mathbf{63}$ representation of $SU(8)$ is known by reference to the on-shell *massive* representations of $N = 8$ supersymmetry whose spin content is the same as the irreducible off-shell representations.

*) By current multiplet, or supercurrent, we do not mean to exclude higher spin conserved quantities.

The spin and $SU(N)$ content of the massive multiplets of states may be obtained by analysing the representations of the rest frame Clifford algebra [11]. This procedure has been carried out for the above $N = 8$ massive multiplets in the phenomenological investigations of Ref. [10]. However, this analysis of the Clifford algebra does not give the $SL(2, \mathbb{C})$ representations for the off-shell multiplets of currents. The techniques presented here allow us to develop the current multiplets in full detail.

In order to exploit systematically the construction of off-shell representations of supersymmetry via supercurrents we need on-shell massless superfields for all N [12]. This topic is reviewed in the next section. Following this we give all the supercurrents that can be obtained as bilinears of on-shell massless superfields. The latter have maximum spin $\geq N/4$, so that the currents constructed from them have integral maximum spin $\geq N/2$. From such a supercurrent one can abstract an off-shell representation of supersymmetry of which the particular bilinear supercurrents are concrete realizations.

Supercurrents determine prepotentials via the Noether coupling, but there is another method of obtaining these prepotentials which is to relax the on-shell constraints of the superfields of Section 2. This is discussed in Section 4 and we show how to obtain the superspace actions for these prepotentials. These are the linearized maximally gauge invariant theories for any N .

In Section 5 we discuss ways in which irreducible field representations may be combined to form field representations appropriate for theories without maximal gauge invariance, e.g. Poincaré supergravities. In particular, we discuss massive $N = 2$ spin 1 and $N = 4$ Poincaré supergravity.

In Section 6 we summarize our results, and in an Appendix we show how the component structure of supercurrents may be obtained by the "tableaux-calculus" [7], and we work out several $N = 4$ and $N = 8$ examples in detail.

2. ON-SHELL SUPERFIELDS

A free massless particle of spin s is described by a field strength tensor $F_{\alpha_1 \dots \alpha_{2s}}$ totally symmetric in its $2s$ (two-component) spinor indices and satisfying

$$\begin{aligned} \partial_{\alpha}^{\alpha'} F_{\alpha_1 \dots \alpha_{2s}} &= 0 \\ \square F_{\alpha_1 \dots \alpha_{2s}} &= 0 \end{aligned} \quad (2.1)$$

The second of Eqs. (2.1) is independent only for $s = 0$. For $s \geq 1$ the first of Eqs. (2.1) implies both the Bianchi identity and the equation of motion for the underlying gauge field. For N -extended supersymmetry we need field strength superfields, W , whose constraints ensure that the only independent components are F 's satisfying (2.1). Such superfields have been given in Ref. [12] for all N ; here we will present the result in a form suitable for our purposes. If $J_{\max}$ is the maximum spin of a supermultiplet, then we define

$$n = 2 J_{\max} \quad (2.2)$$

The parameter n serves to distinguish the three cases $n > N$, $n = N$, $n < N$, as follows:

i) $n > N$: in this case W has $m \equiv n - N$ spinor indices. It is totally symmetric in these indices and satisfies

$$\bar{D}_{\alpha i} W_{\beta_1 \dots \beta_m} = 0 \quad ; \quad D_{\alpha}^i W_{\beta_1 \dots \beta_m} = D_{(\alpha}^i W_{\beta_1 \dots \beta_m)} \quad (2.3)$$

ii) $n = N$: in this case W is a Lorentz and $SU(N)$ scalar satisfying

$$\bar{D}_{\alpha i} W = 0 \quad ; \quad D_{\alpha}^i D_{\beta}^j W = D_{(\alpha}^i D_{\beta}^j W \quad (2.4)$$

iii) $n < N$: in this case W is a Lorentz scalar with n antisymmetrized $SU(N)$ indices satisfying

$$\begin{aligned} D_{\alpha}^i W_{j_1 \dots j_n} &= \frac{(-1)^{n-1} n}{N-n+1} \delta_{[j_1}^i D_{\alpha}^{j_2} W_{j_2 \dots j_n]} \\ \bar{D}_{\alpha i} W_{j_1 \dots j_n} &= \bar{D}_{\alpha} [i W_{j_1 \dots j_n]} \end{aligned} \quad (2.5)$$

A special subcase is when $n = N/2 = \text{even integer}$, when a self-duality condition is imposed on W

$$\bar{W}^{i_1 \dots i_n} = \frac{1}{n!} \epsilon^{i_1 \dots i_n j_1 \dots j_n} W_{j_1 \dots j_n} \quad (2.6)$$

In this case the two Eqs. (2.5) are not independent.

In cases (i) and (ii) it is straightforward to show that the only components in W are the F 's satisfying (2.1). For case (iii) it is a little harder to prove because W is no longer chiral. By use of the identity

$$\delta_{[k_1}^i \bar{D}_{\rho j]} W_{k_2 \dots k_n] \lambda} = (-1)^{n-1} \bar{D}_{\rho \lambda} \delta_{[k_1}^i W_{k_2 \dots k_n] j} \quad (2.7)$$

one can prove that

$$D_{\alpha}^i \bar{D}_{\rho j} W_{k_1 \dots k_n} = i(n+1) \delta_{[k_1}^i \partial_{\alpha \rho]} W_{k_2 \dots k_n] j} \quad (2.8)$$

from which the on-shell condition follows on multiplication by $D^{\alpha j}$.

The highest spin in these on-shell superfields is an $SU(N)$ singlet. One can of course obtain representations with non-singlet highest spin by multiplying a given arbitrary $SU(N)$ representation into every component of one of the multiplets given above.

3. SUPERCURRENTS

From the on-shell superfields, W , of the preceding section we now construct irreducible supercurrents as bilinears of the form $W\bar{W}$ *). We distinguish two cases:

- i) $n > N$: in this case the lowest spin supercurrent is ($m = n - N$)

$$J_{\alpha_1 \dots \alpha_m; \dot{\alpha}_1 \dots \dot{\alpha}_m} = \bar{W}^{\dot{\alpha}_1 \dots \dot{\alpha}_m} W_{\alpha_1 \dots \alpha_m} \quad (3.1)$$

which, as a consequence of Eq. (2.3), satisfies the conservation condition

$$D^{\alpha i} J_{\alpha_1 \dots \alpha_m; \dot{\alpha}_1 \dots \dot{\alpha}_m} = 0 \quad (3.2)$$

- ii) $n \leq N$; in this case we can form the following irreducible supercurrents

$$J = (W\bar{W})_{\text{irrep of } SU(N)} \quad (3.3)$$

*) We consider here only real currents which have integral maximum spin. Currents with half-integral maximum spin may be obtained as bilinears of two *different* superfields or simply by truncation of higher N integral spin currents.

In terms of Young tableaux we can represent W and $\bar{W}$ as

$$W \sim \left\{ \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right\}^{N-n}, \quad \bar{W} \sim \left\{ \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right\}^n \quad (3.4)$$

so that the product $W\bar{W}$ decomposes into the following $SU(N)$ representations:

$$W\bar{W} \sim n \left\{ \begin{array}{|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right\}^{N-n} + \dots + p \left\{ \begin{array}{|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right\}^{N-p} + \dots + N \left\{ \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right\}^N \quad (3.5)$$

The irreducible representation whose first column has p boxes is an irreducible supercurrent whose maximum spin is p . The explicit expression for this supercurrent is

$$J_{i_1 \dots i_{N-p}}^{j_1 \dots j_{N-p}} = \left(W_{i_1 \dots i_{N-p}} R_{N-p+1} \dots R_N \bar{W}^{j_1 \dots j_{N-p}} R_{N-p+1} \dots R_N - \text{traces} \right) \quad (3.6)$$

so that J is totally antisymmetric in lower and upper indices and totally traceless. The constraint satisfied by J , which follows from Eq. (2.5), is

$$D_\alpha^q J_{i_1 \dots i_{N-p}}^{j_1 \dots j_{N-p}} = \delta_{[i_1}^\alpha \lambda_{\alpha i_2 \dots i_{N-p}}^{j_1 \dots j_{N-p}}] + \chi_{\alpha i_1 \dots i_{N-p}}^{q j_1 \dots j_{N-p}} - \frac{N-p}{(2p-N+2)} \delta_{[i_1}^{[j_1} \lambda_{\alpha i_2 \dots i_{N-p}}^{j_2 \dots j_{N-p}]} \quad (3.7)$$

where λ and χ are both totally antisymmetric in lower and upper indices and totally traceless. The explicit expression for λ is

$$\lambda_{\alpha i_2 \dots i_{N-p}}^{j_1 \dots j_{N-p}} = \frac{(N-p)(2p-N+2)}{(p+2)(2p-N+1)} D_\alpha^{i_1} J_{i_1 \dots i_{N-p}}^{j_1 \dots j_{N-p}} \quad (3.8)$$

In terms of Young tableaux this constraint is written as

$$DJ \sim p \left\{ \begin{array}{|c|c|} \hline & \\ \hline \end{array} \right\}^{N-p+1} + \left\{ \begin{array}{|c|c|} \hline & \\ \hline \end{array} \right\}^{N-p} \quad (3.9)$$

i.e. the largest representation in DJ with three boxes in the top row is constrained to vanish. In the case that J is a singlet (i.e. $p = N$), there is no first derivative constraint, but instead there is the second derivative constraint,

$$D^{ij} J = 0, \quad D^{ij} \equiv D^{\alpha i} D_{\alpha}^j \quad (3.10)$$

In addition, when the underlying field strength multiplet W is self conjugate [Eq. (2.6)], ($n = N/2 = \text{even integer}$), not all representations in the product (3.5) occur because $W\bar{W} \sim WW$ is symmetric under interchange of the sets of indices on the two W 's. For example, for $N = 4$, $n = 2$, we have only the representations,

$$\left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right)_{\text{Sym.}} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \quad (3.11)$$

We now establish that these supercurrents are irreducible. This guarantees that the component currents of Lorentz form $(a/2, b/2)$ describe only spin $(a + b)/2$ and therefore satisfy all possible conservation conditions. Case (i) is straightforward; the minimum spin is $n - N$ and the maximum spin is n (using 2.3), so that the range of spin is N . Since both minimum and maximum spins are $SU(N)$ singlets, the multiplet is irreducible with superspin $n - N$.

To establish irreducibility in case (ii) we make use of the correspondence between the spin and $SU(N)$ content of irreducible multiplets of fields and of massive multiplets of states. When the Clifford vacuum of a massive multiplet of states is an $SU(N)$ singlet, the states of a given spin fall into irreducible $Sp(2N)$ representations [10]. In particular, the spin zero states in a multiplet

with maximum spin p (i.e. superspin $p - N/2$) fall into the same irreducible $Sp(2N)$ representation as the massive states of spin $p - N/2$ in the multiplet with the lowest possible maximum spin $N/2$ (i.e. superspin 0). This $Sp(2N)$ representation has the following decomposition in terms of $SU(N)$ representations [13]:

$$2p-N \left\{ \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\} + \dots + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} + \dots + p \left\{ \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right\} N-p + \dots + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} 2N-2p \quad (3.12)$$

On the other hand, the scalars in the supercurrents are obtained by differentiating J with D^{ij} and $\bar{D}_{ij}$. Since J has a two-column Young tableau satisfying the constraint (3.9) this differentiation produces a series of $SU(N)$ representations which is exactly of the form (3.12). It now remains to prove that the maximum spin contained in J is p . Consider the operator $D_{(\alpha_1}^{i_1} \dots D_{\alpha_p)}^{i_p}$, which is in the $(p/2, 0)$ $SL(2, C)$ representation and has a one-column $SU(N)$ tableau with p boxes; acting with this operator on J and using the constraints on J , we find the single representation with the tableau

$$N \left\{ \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right\} \begin{array}{l} N-p \\ 2p-N \end{array} \quad (3.13)$$

Acting again with $\bar{D}_{(\dot{\alpha}_1 i_1} \dots \bar{D}_{\dot{\alpha}_p i_p)}$, we then obtain a singlet of $SU(N)$ with the $SL(2, C)$ representation $(p/2, p/2)$. One cannot obtain a higher spin since the operator $D_{(\alpha_1}^{i_1} \dots D_{\alpha_{p+1})}^{i_{p+1}}$ annihilates J , so that p is the highest spin.

For a given on-shell superfield W the supercurrents discussed above are those of the lowest dimension. Others can be found by taking products of $D_{\alpha}^i W$, $\partial_{\alpha\dot{\alpha}} W$, etc., but the abstract structure of such supercurrents is the same as the lowest dimension supercurrent for another W . Hence no new abstract supercurrents are found this way.

We will now tabulate some currents of physical interest. For spin 1 (i.e. $J_{\max} = 1$) we can construct electric charge currents for $N = 1$ and $N = 2$

$$\begin{aligned} N=1: \quad J &= W \bar{W} \\ N=2: \quad J^i_j &= \bar{W}^i W_j - \frac{1}{2} \delta^i_j \bar{W}^k W_k, \quad 3 \text{ of } SU(2) \end{aligned} \quad (3.14)$$

For spin 2 we have the conformal stress-tensor multiplets:

$$\begin{aligned} N=1: \quad J_{\alpha\dot{\alpha}} &= W_{\alpha} \bar{W}_{\dot{\alpha}} \\ N=2: \quad J &= W \bar{W} \\ N=3: \quad J^i_j &= W^i \bar{W}_j - \frac{1}{3} \delta^i_j W^k \bar{W}_k, \quad 8 \text{ of } SU(3) \\ N=4: \quad J_{ijkl} &= W_{ij} W_{kl} - \frac{1}{12} \epsilon_{ijkl} \bar{W}^{mn} W_{mn}, \quad 20' \text{ of } SU(4) \end{aligned} \quad (3.15)$$

of which the $N = 1$ [1] and $N = 4$ [2] cases have appeared before. For spin 4 we have the Bel-Robinson multiplets as bilinears of the $N = 1$ to 8 supergravity superfields*):

$$\begin{aligned} N=1: \quad J_{\alpha\beta\gamma, \dot{\alpha}\dot{\beta}\dot{\gamma}} &= W_{\alpha\beta\gamma} \bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \\ N=2: \quad J_{\alpha\beta, \dot{\alpha}\dot{\beta}} &= W_{\alpha\beta} \bar{W}_{\dot{\alpha}\dot{\beta}} \\ N=3: \quad J_{\alpha\dot{\alpha}} &= W_{\alpha} \bar{W}_{\dot{\alpha}} \\ N=4: \quad J &= W \bar{W} \\ N \geq 5: \quad J^{ijkl}_{mnpq} &= (\bar{W}^{ijkl} W_{mnpq}) \text{ irrep. of } SU(N) \end{aligned} \quad (3.16)$$

of which the $N = 1$ case has been given previously [6]. The representations to be chosen for $N > 4$ are

*) Note that supercurrents of type (ii), which are singlets, may also be represented by a single column Young tableau with N boxes, and therefore have maximum spin N .

$$\begin{aligned}
 N = 5 : & \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \square & \\ \hline \square & \\ \hline \end{array} & = \underset{\sim}{24} \\
 N = 6 : & \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \square & \\ \hline \end{array} & = \underset{\sim}{189} \\
 N = 8 : & \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} & = \underset{\sim}{1,764}
 \end{aligned} \tag{3.17}$$

The spin 4 currents can be used to construct the supergravity linearized three-loop counterterms [5,6], in manifestly $SU(N)$ invariant form [7].

Other supercurrents of interest are obtained by choosing a lower dimensional $SU(N)$ representation for J , where this is possible. From $N = 4$ spin 1 one can construct the supercurrent

$$J = \bar{W}^{ij} W_{ij} \tag{3.18}$$

which has the maximum spin 4. From the on-shell $N = 4$ spin $3/2$ superfield W_{ijk} we can construct the spin 3 $N = 4$ supercurrent:

$$\bar{J}_i{}^j = W_{ihl} \bar{W}^{jhl} - \frac{1}{4} \delta_i{}^j W_{hlm} \bar{W}^{hlm} \tag{3.19}$$

which is a $15_{\sim}$ of $SU(4)$. From the $N = 8$ supergravity on-shell superfield W_{ijkl} , one can construct two supercurrents other than the one containing the Bel-Robinson tensor; these are

$$\begin{aligned}
 J &= (W^2)_{\sim 20} \\
 J &= (W^2)_{\sim 1}
 \end{aligned} \tag{3.20}$$

which have maximum spin 6 and 8, respectively. The complete component field expansions of the abstract multiplets for $N = 4$, spin 3 and 4, and $N = 8$, spin 4, 6, and 8, are given in the Appendix. For $N = 8$ the $SU(N)$ and spin content of these multiplets is already known [10] from the correspondence with massive on-shell state multiplets.

4. PREPOTENTIALS

The prepotentials for maximally gauge invariant theories with maximum spin $\geq N/2$ may be obtained directly from the supercurrents by Noether coupling. However, another approach is to relax the on-shell constraints on the field strength superfields of Section 2. We are able to do this straightforwardly for $n \geq N$ for which the field strength superfield is chiral. However, it no longer satisfies the second of constraints (2.3), or (2.4) if $n = N$, but rather the appropriate Bianchi identity which ensures that the components of W are field strengths of underlying potentials. The Bianchi identity may be imposed by a Lagrange multiplier superfield. The superfield W may then be eliminated to produce an action for the Lagrange multiplier superfield which therefore can be interpreted as the prepotential.

We again distinguish three cases^{*)}:

- i) $n > 2N$, n even; in this case the constraint to be imposed is the conformal Bianchi identity for the lowest spin field, e.g. $N = 1$ conformal supergravity, which is described by $W_{\alpha\beta\gamma}$ satisfying

$$\bar{D}_{\dot{\alpha}} W_{\alpha\beta\gamma} = 0, \quad \partial_{\alpha}^{\dot{\alpha}} \bar{D}^{\dot{\gamma}} \bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = \partial_{\alpha}^{\dot{\alpha}} D^{\gamma} W_{\alpha\beta\gamma} \quad (4.1)$$

- ii) $n \leq 2N$, n even; in this case the constraint is that the scalars of the same dimension as the highest spin field strength are real. In this category one has $N = 1, 2$ Yang-Mills and $N = 2, 3, 4$ conformal supergravity. For example, for $N = 2$ conformal supergravity the constraints are

^{*)} Here $n = 2J_{\max}$ refers to the maximum spin of the prepotential and of its corresponding supercurrent and is at least twice the value of n for the underlying on-shell fields of this supercurrent.

$$\bar{D}_{\dot{\alpha}\dot{\beta}} W_{\alpha\beta} = 0 \quad D^{\alpha\beta} W_{\alpha\beta} = \bar{D}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} \quad (4.2)$$

where $D_{\alpha\beta} = \frac{1}{2} \epsilon_{ij} D_{\alpha}^i D_{\beta}^j$.

iii) n odd; in this case one superfield is not enough to describe the system; one needs two. For example, if $N = 1$, $n = 3$, then the superfields are the expected one, $W_{\alpha\beta}$, plus an additional one, $W'_{\alpha\beta}$, both chiral and satisfying the constraint

$$D^{\alpha} W'_{\alpha\beta} = \partial_{\beta}^{\dot{\gamma}} \bar{D}^{\dot{\delta}} \bar{W}_{\dot{\gamma}\dot{\delta}} \quad (4.3)$$

An obvious exception is $N = 1$, $n = 1$. We shall not discuss further the case of n odd.

The action for the fields described by the, now off-shell, superfields W is, in cases (i) and (ii) above, the square of W integrated over the chiral subspace*) but it is convenient to regard W as merely chiral and to impose the remaining constraint by a Lagrange multiplier. For example, for the superfield $W_{\alpha\beta}$ in (4.2) we have the action

$$I = \frac{1}{2} \int d^4x d^4\theta W^{\alpha\beta} W_{\alpha\beta} + c.c. + i \int d^4x d^4\theta d^4\bar{\theta} V (D^{\alpha\beta} W_{\alpha\beta} - \bar{D}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}}) \quad (4.4)$$

When W is eliminated by its algebraic field equation, one finds an action for V alone,

$$I = \frac{1}{2} \int d^4x d^4\theta (\bar{D}^4 D^{\alpha\beta} V) (\bar{D}^4 D_{\alpha\beta} V) + c.c. \quad (4.5)$$

so that V is the prepotential. In terms of V the field strength $W_{\alpha\beta}$ is

$$W_{\alpha\beta} = i \bar{D}^4 D_{\alpha\beta} V \quad (4.6)$$

Similarly, for $N = 2$ Maxwell we have the action

$$I = \frac{1}{2} \int d^4x d^4\theta W^2 + c.c. + i \int d^4x d^4\theta d^4\bar{\theta} V^{ij} (D_{ij} W - \bar{D}_{\dot{i}\dot{j}} \bar{W}) \quad (4.7)$$

from which we deduce,

$$W = i \bar{D}^4 D^{ij} V_{ij} \quad (4.8)$$

*) Actions of this form were proposed in Ref. [12] but the off-shell constraints were not given.

Notice that this construction has the property that the solution of the W field equation gives the same prepotential as the solution of the original constraint.

We now list some prepotentials of interest that can be obtained in this way^{*)}, the first three of which are already known [15,16,17]

$$\begin{aligned}
 N=1, \quad n=2 \quad W_\alpha &= i \bar{D}^2 D_\alpha V \\
 n=4 \quad W_{\alpha\beta\gamma} &= \bar{D}^2 \partial_{(\alpha} \bar{D}^2 D_{\beta} V_{\gamma)} \\
 N=2, \quad n=2 \quad W &= i \bar{D}^4 D^{ij} V_{ij} \\
 n=4 \quad W_{\alpha\beta} &= i \bar{D}^4 D_{\alpha\beta} V \\
 N=3, \quad n=4 \quad W_\alpha &= i \bar{D}^6 D^3_{\alpha j} V^j_i \quad ; \quad D^3_{\alpha j} = \epsilon_{jkl} D^{ik} D^l_\alpha \\
 N=4, \quad n=4 \quad W &= i \bar{D}^8 D^{4ij,kl} V_{ij,kl} \\
 &\quad D^{4ij,kl} = D^{(i}_\alpha D^j_{\beta)} D^{kl} D^{\alpha\beta} \\
 N=8, \quad n=8 \quad W &= i \bar{D}^{16} D^{8ijkl,mnrs} V_{ij,kl,mn,rs} \\
 &\quad D^{8ijkl,mnrs} = D^{(i}_\alpha D^j_{\beta)} D^k_{\gamma} D^l_{\delta)} D^{mn} D^{pq} D^{rs} D^{\alpha\beta\gamma\delta}
 \end{aligned} \tag{4.9}$$

The gauge transformations for these prepotentials may be extracted directly from the constraints on the supercurrents to which they couple. As an example, let us work out the gauge transformation for $V^{ij,kl}$, the $N=4$ conformal supergravity prepotential in the $20'$ representation of $SU(4)$. The corresponding current satisfies the constraint

$$D^\rho J_{ij,kl} \sim (D^\rho J_{ij,kl})_{20} \tag{4.10}$$

and similarly for $\bar{D}J$. Hence the Noether coupling

$$\int d^4x d^4\theta d^4\bar{\theta} V^{ij,kl} J_{ij,kl} \tag{4.11}$$

^{*)} In a recent preprint [14], the conformal supergravity prepotentials have been obtained by a different method. These authors have also pointed out that for $N=2$ and 4 these prepotentials do not contain the antisymmetric part of the vierbein.

will be invariant under the gauge transformation

$$\delta V^{ijkl} = \bar{D}_{\dot{m}} \bar{\Lambda}^{\dot{m}ijkl} + c.c. \quad (4.12)$$

where Λ is in the $\underline{60}$ representation of $SU(4)$, i.e. the gauge parameter is in the representation excluded in the constraint on J .

Similarly, in the case that the supercurrent has spinor indices, the Noether coupling is

$$\int d^4x d^{2N}\theta d^{2N}\bar{\theta} V^{\alpha_1 \dots \alpha_m, \dot{\alpha}_1 \dots \dot{\alpha}_m} J_{\alpha_1 \dots \alpha_m, \dot{\alpha}_1 \dots \dot{\alpha}_m} \quad (4.13)$$

and from the conservation condition on J , Eq. (3.2), the gauge transformation of V that leaves this interaction term invariant is

$$\delta V_{\alpha_1 \dots \alpha_m, \dot{\alpha}_1 \dots \dot{\alpha}_m} = D_{(\alpha_1}^i \Lambda_{\alpha_2 \dots \alpha_m)} \dot{\alpha}_1 \dots \dot{\alpha}_m i + c.c. \quad (4.14)$$

5. NON-CONFORMAL THEORIES

So far we have considered only off-shell supersymmetric theories that are maximally gauge invariant, e.g. massless spin 1, conformal spin 2, etc. There are others of interest that are not of this type, in particular, Poincaré supergravity and massive spin 1, (the relevance of the latter will become apparent later). However, the multiplets of fields for such theories can often be obtained by combining irreducible multiplets. The explicit way in which this can be done for $N = 1$ [16] and $N = 2$ [17] supergravity is by means of "compensating multiplets", which is a generalization of the Stueckelberg mechanism. The simplest supersymmetric example is the construction of massive $N = 1$ spin 1 from a massless gauge potential V and a compensating chiral superfield ϕ :

$$I = \frac{1}{2} \left[\int d^4x d^2\theta W^\alpha(V) W_\alpha(V) + c.c. \right] + m^2 \int d^4x d^2\theta d^2\bar{\theta} \left(V - \frac{1}{m} \phi - \frac{1}{m} \bar{\phi} \right)^2 \quad (5.1)$$

This is invariant under the gauge transformation $\delta V = \Lambda + \bar{\Lambda}$, $\delta \phi = m\Lambda$. In the "unitary gauge", $\phi = 0$, the Lagrangian reduces to the usual massive $N = 1$ spin 1 theory.

There is another mechanism of combining multiplets that avoids compensating fields, sometimes called the gauge mixing mechanism [18], which we again illustrate for $N = 1$ spin 1:

$$I = \frac{1}{2} \int \eta^2 d^4x d^2\theta d^2\bar{\theta} + m \int \eta V d^4x d^2\theta d^2\bar{\theta} + \frac{1}{2} \left[\int W^\alpha(V) W_\alpha(V) d^4x d^2\theta + c.c. \right] \quad (5.2)$$

Here V is the $N = 1$ gauge potential, while $G = D^\alpha \psi_\alpha + \bar{D}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}$ is the field strength superfield for the chiral gauge spinor superfield ψ_α . The latter describes a dual form of the Wess-Zumino multiplet in which the physical pseudoscalar is replaced by a gauge antisymmetric tensor [19]. This Lagrangian has *two* gauge invariances, that of ψ_α and that of V , which are preserved by the coupling term GV by virtue of the Bianchi identity, $D^2 G = \bar{D}^2 G = 0$. This is not a "compensating" mechanism because neither gauge field transforms under the gauge transformation of the other. However, the Lagrangian (5.2) can be shown to be equivalent to that of (5.1) by means of a duality transformation. One first writes the kinetic term for ψ_α in first order form

$$\frac{1}{2} \eta^2(\psi_\alpha) \rightarrow -\frac{1}{2} G'^2 + G' (D^\alpha \psi_\alpha + \bar{D}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}) \quad (5.3)$$

where G' is unconstrained. The field ψ_α now appears as a Lagrange multiplier imposing the constraint $G' = mV + \phi + \bar{\phi}$ with ϕ chiral. This leads directly back to (5.1). On the other hand, we can express the kinetic term for V in first order form

$$\frac{1}{2} W^\alpha(V) W_\alpha(V) \rightarrow -\frac{1}{2} W'^\alpha W'_\alpha + W'^\alpha \bar{D}^2 D_\alpha V \quad (5.4)$$

where W'^α is chiral, but otherwise unconstrained. V now appears as a Lagrange multiplier imposing the constraint that $\psi_\alpha = 1/m W'_\alpha + \text{gauge transformation}$. This leads to the Lagrangian

$$\mathcal{L}(x) = \frac{1}{2} \int \eta^2(W') d^2\theta d^2\bar{\theta} + m^2 \left[\int W'^\alpha W'_\alpha d^2\theta + c.c. \right] \quad (5.5)$$

All three Lagrangians (5.1), (5.2), and (5.5) describe the same physical degrees of freedom, although in (5.5) the massive spin 1 particle is described by an antisymmetric tensor.

It is well known that a massive representation of N-supersymmetry on states is also, when complexified, a massive representation of 2N-supersymmetry with a central charge. Recently, it has been shown that this is also true for representations on fields [20], at least for N = 1. In particular, one can use the massive N = 1 spin 1 multiplet to obtain a massive N = 2 spin 1 multiplet with a central charge. Furthermore, one can then take the massless limit (in which case the complexification simply doubles the representation), and one obtains massless N = 2 spin 1 in either the Grimm, Sohnius, Wess [21] or the Sohnius, Stelle, West [22] form depending upon whether one starts from (5.1) or (5.2), respectively. In view of this connection, it is clearly of interest to study the massive N = 2 spin 1 theory. In this case it is not obvious how the compensating field method can work. The direct N = 2 analogue of (5.1) leads to a massive but higher derivative N = 2 theory, while an obvious trial compensating multiplet, the hypermultiplet, does not yet have an off-shell formulation without a central charge. However, one can construct a massive N = 2 model using the gauge-mixing mechanism. The direct analogue of (5.2) is

$$\begin{aligned} \mathcal{L}(x) = & \int d^2\theta^{(ij)} d^2\bar{\theta}^{(kl)} G_{ij}(C) G_{kl}(C) + m \int d^4\theta d^4\bar{\theta} G_{ij}(C) v^{ij} \\ & + \left[\int W^2(v) d^4\theta + \text{c.c.} \right] \end{aligned} \quad (5.6)$$

Here $G_{ij} = D_{ij}C + \bar{D}_{ij}\bar{C}$ is a field strength for the chiral potential C and satisfies the Bianchi identity

$$D_\mu^h G_{ij} = \frac{2}{3} \delta_{(i}^h D_\mu^l G_{j)l} \quad (5.7)$$

which ensures the invariance [22] of the kinetic term for C and the coupling to v^{ij} . In a Wess-Zumino gauge the potential C contains the fields of the "linear multiplet" (otherwise known as the "tensor multiplet"). The step that led from (5.2) to (5.5) can now be repeated with the following result:

$$\mathcal{L}(x) = \int d^2\theta^{(ij)} d^2\bar{\theta}^{(kl)} G_{ij}(C) G_{kl}(C) + m^2 \left[\int d^4\theta C^2 + \text{c.c.} \right] \quad (5.8)$$

in which the massive spin 1 particle is described by an antisymmetric tensor. On the other hand, the steps that led from (5.2) to (5.1) cannot be repeated, since

a term of the form $\int G'_{ij} G'_{kl} d^2\theta^{(ij)} d^2\theta^{(kl)}$ will not be supersymmetric; unlike the $N = 1$ case the constraint on G_{ij} , Eq. (5.7), *cannot* be relaxed. This suggests that the compensated version of massive $N = 2$ spin 1 may not exist.

If we take the fields of the massive $N = 2$ model of (5.6) and complexify them we should, according to the previous argument, have a representation of massive $N = 4$ supersymmetry with a central charge. Indeed this is true, but upon taking the massless limit one only recovers the known $N = 4$ representation with an off-shell central charge of Ref. [22]. This suggests that the difficulties encountered in the construction of an off-shell $N = 4$ spin 1 model without a central charge are connected to the difficulties described above for the compensated massive $N = 2$ spin 1 model.

For supergravity theories the gauge-mixing mechanism for combining multiplets would presumably require higher rank analogues [23] of antisymmetric tensors, with their attendant difficulties, so the compensating field mechanism seems the most viable one. We present here some arguments which suggest that a possible set of compensating field multiplets for $N = 4$ supergravity is six $N = 4$ Maxwell multiplets*). Given that this is so, the full off-shell structure of $N = 4$ Poincaré supergravity will have to await the resolution of the auxiliary field problem for $N = 4$ Maxwell theory which, however, may not exist in the absence of a central charge [24].

The field content of six (on-shell) $N = 4$ Maxwell multiplets is as follows:

$$\begin{aligned} \dim 1: & \quad A_{\mu}^{ij} (\underline{6}), \quad \phi (\underline{1}), \quad \psi_i^j (\underline{15}), \quad \psi^{ijk} (\underline{20}') \\ \dim \frac{3}{2}: & \quad \lambda_a^i (\underline{4}), \quad \psi_{\alpha ij} (\underline{20}) \end{aligned}$$

where we have taken the six to be a $\underline{6}$ of $SU(4)$. The vectors are the only physical fields of Poincaré supergravity not already contained in the Weyl multiplet; the scalar ϕ and the four spinors λ_{α}^i may be used to compensate for Weyl and S-supersymmetry,

*) This result has been obtained independently by S.J. Gates and W. Siegel (private communication).

while ϕ_1^j may be used to compensate for $SU(4)$. The remaining fields are auxiliary and are of the right dimension to act as Lagrange multipliers for the high dimension auxiliary fields in the Weyl multiplet. Hence, coupling the $N = 4$ Weyl multiplet to six (on-shell) Maxwell multiplets^{*)} and fixing the superconformal gauges by $\phi \sim \kappa$, $\phi_1^j = \lambda_\alpha^i = 0$, should give $N = 4$ Poincaré supergravity with the missing auxiliary fields being those of the compensating multiplets, as for $N = 1$ and 2 [16,17]. Presumably, if the six Maxwell multiplets are exchanged for an $O(4)$ Yang-Mills multiplet, then the gauged $O(4)$ de-Sitter supergravity will emerge.

6. COMMENTS

In this paper we have shown how irreducible off-shell representations of supersymmetry with maximum spin $\geq N/2$ may be systematically constructed. This leaves two classes of problems: firstly the construction of multiplets with maximum physical spin $< N/2$, and secondly the construction of multiplets of fields for non-maximally gauge invariant theories. In the former category, we have the $N = 2$ hypermultiplet without a central charge and $N = 4$ Yang-Mills, while in the latter category we have $N = 3$ and 4 Poincaré supergravity. Supergravity theories for $N \geq 5$ have both problems. The second problem has a solution in some cases by means of compensating fields or the gauge-mixing mechanism. However, for $N = 4$ Poincaré supergravity we have argued that six $N = 4$ Maxwell multiplets are the correct compensating multiplets, so that the two problems are interrelated.

The problem of constructing off-shell Poincaré supergravity theories is also related to the problem of finding the non-conformal stress-tensor multiplet. (The two problems are not identical since even if the stress-tensor multiplet is found it does not follow that a Lagrangian can be constructed from the corresponding field multiplet.) One expects that the non-conformal stress tensor should be a sum of irreducible current multiplets of which the conformal spin 2 current multiplet is one (for $N \leq 4$). In the Appendix we have tabulated the $N = 4$ spin 3 and spin 4 multiplets and have attempted to construct a non-conformal stress-tensor

^{*)} Note that such a coupling does not require the presence in the action of kinetic terms for the Weyl multiplet, because of the closure of the algebra on these fields.

by combining them with the spin 2 multiplet, but if one insists that the higher spin currents not be conserved, it is easy to establish that no simple combination succeeds. An alternative possibility is that the $N = 4$ conformal stress-tensor multiplet is its own trace. This is suggested by an analysis [25] of the $N = 4$ stress-tensor supercurrent in five dimensions, where the underlying on-shell Maxwell theory is not conformal invariant. This supercurrent has the same number of components as the conformal supercurrent, but with an internal symmetry reduced to $Sp(4)$. There may well be an $O(4)$ version of this supercurrent, but whether it could be concretely realized as a bilinear is not known. [For the $Sp(4)$ this identification is possible if the underlying four-dimensional multiplet has a central charge.]

In conclusion there is a nexus of problems which include the $N = 2$ hypermultiplet, the $N = 4$ Yang-Mills, and Poincaré supergravity for $N \geq 3$ that fall outside the scope of the systematic methods that have been developed both here and elsewhere. We have seen that these problems are related and it may be that there is no solution within the usual framework, as has been conjectured in Ref. [26]. The implications of this fact, if true, remain obscure.

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APPENDIX

COMPONENT EXPANSION OF SUPERCURRENTS

In the text we have discussed supercurrents as constrained superfields. Here we show how the component fields of the supercurrents of the form (3.3) (i.e. Lorentz scalars) may be easily obtained by means of the "tableaux calculus" introduced in a previous article [7]. The spinor derivative D_{α}^i is represented by a one-box tableau containing a dot, while $\bar{D}_{\dot{\alpha}i}$ is represented by a single column of $N - 1$ boxes each containing a cross:

$$D_{\alpha}^i \sim \boxed{\cdot} ; \bar{D}_{\dot{\alpha}i} \sim \left\{ \begin{array}{c} \boxed{\times} \\ \boxed{\times} \\ \vdots \\ \boxed{\times} \end{array} \right\}_{N-1} \quad (A.1)$$

The currents we consider have tableaux of the form

$$J \sim p \left\{ \begin{array}{c} \boxed{} \\ \boxed{} \\ \vdots \\ \boxed{} \end{array} \right\}_{N-p} \quad (A.2)$$

and satisfy the constraints

$$\boxed{\cdot} \times J \sim \begin{array}{c} \boxed{} \\ \boxed{} \\ \vdots \\ \boxed{\cdot} \end{array} + \begin{array}{c} \boxed{} \\ \boxed{} \\ \vdots \\ \boxed{} \end{array} \quad (A.3)$$

or, for $p = N$,

$$\boxed{\cdot \cdot} \times J = 0 \quad (A.4)$$

Since J is always self-conjugate, the $\bar{D}$ constraints follow by complex conjugation.

$$J \sim \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \quad (A.8)$$

It has the following component in its expansion:

$$D^2 \bar{D} J \supset \begin{array}{|c|c|c|} \hline & & X \\ \hline & X & \cdot \\ \hline & \cdot & \\ \hline X & & \\ \hline \end{array} \quad (A.9)$$

This component is in the $SL(2,C)$ representation $(1, \frac{1}{2})$, i.e. there is no $(0, \frac{1}{2})$ component. This may be seen as follows: *a priori*, the $SU(4)$ tableau of (A.9) occurs in the product

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} \times \left(\begin{array}{|c|} \hline X \\ \hline X \\ \hline X \\ \hline \end{array} \times \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array} \right) \quad (A.10)$$

which has the unique $SL(2,C)$ representation $(0, \frac{1}{2})$. J is therefore multiplied by the following mixed tableaux

$$\begin{array}{|c|c|} \hline X & \cdot \\ \hline X & \\ \hline X & \\ \hline \cdot & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline X & \cdot & \cdot \\ \hline X & & \\ \hline X & & \\ \hline \end{array} \quad (A.11)$$

It is now straightforward to see that one cannot obtain the $SU(4)$ representation of (A.9) without violating the usual Young rules or the rule stated above for allowed tableaux. A similar argument shows that the tableau (A.9) can indeed occur in the $(1, \frac{1}{2})$ representation.

The tableaux calculus treats D and $\bar{D}$ as if they anticommuted, so that the components of J obtained by acting with D 's and $\bar{D}$'s are only defined modulo derivatives of lower dimension components. This is sufficient to identify the components, their spin, and $SL(2,C)$ content. The actual conserved currents will, of course, contain particular combinations of derivatives of lower components.

Note that the case $p = N$ is very simple; since J is a singlet its higher components can be represented as tableaux of dots and crosses alone, satisfying the rule that $D^{\bar{r}}\bar{D}^s J$ occurs only in the $(r/2, s/2)$ $SL(2, C)$ representation.

Using these methods, we have worked out the components for the $N = 4$ spin 3 and spin 4 multiplets and the $N = 8$ spin 4, 6, and 8 multiplets, and they are given in Tables 1 to 5, respectively^{*)}. Note that all components in $SL(2, C)$ representations $(a/2, b/2)$ with both a and b non-zero, satisfy conservation conditions. Because of this, higher spin multiplets cannot be straightforwardly obtained by multiplication of the lowest spin multiplet by an external spin (in contrast to the case of massive state multiplets).

^{*)} The $2352'$ representation in the $N = 8$ multiplets is real. The dimensions are normalized so that the supercurrent superfield has dimension zero.

REFERENCES

- [1] S. Ferrara and B. Zumino, Nucl. Phys. B87 (1975) 207.
- [2] E. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. B182 (1981) 173.
- [3] J.G. Taylor, *in* Superspace and supergravity (eds. S.W. Hawking and M. Roček) (Cambridge University Press) (1981).
- [4] L. Mezincescu, JINR preprint P2-12572 (June, 1979).
- [5] S. Deser, J. Kay and K.S. Stelle, Phys. Rev. Lett. 38 (1977) 527.
S. Ferrara and P. van Nieuwenhuizen, Phys. Lett. 78B (1978) 573.
S. Deser and J.H. Kay, Phys. Lett. 76B (1978) 400.
P. Howe and U. Lindström, Nucl. Phys. B181 (1981) 487, and in Superspace and supergravity (eds. S.W. Hawking and M. Roček) (Cambridge University Press) (1981).
R. Kallosh, Phys. Lett. 99B (1981) 122, and JETP Lett. 33 (1981) 292.
- [6] S. Ferrara and B. Zumino, Nucl. Phys. B134 (1978) 301.
- [7] P. Howe, K.S. Stelle and P.K. Townsend, CERN preprint TH.3065 (1981).
- [8] E. Cremmer and B. Julia, Nucl. Phys. B159 (1979) 141.
- [9] J. Ellis, M.K. Gaillard and B. Zumino, Phys. Lett. 94B (1980) 343.
- [10] J.-P. Derendinger, S. Ferrara and C.A. Savoy, CERN preprint TH.3052 (1981).
- [11] S. Ferrara, C.A. Savoy and B. Zumino, Phys. Lett. 100B (1981) 393.
- [12] W. Siegel, Nucl. Phys. B177 (1981) 325.
- [13] V. Rittenberg and E. Sokatchev, Bonn preprint HE-81-5 (1981).
- [14] W. Siegel and S.J. Gates, CalTech. preprint (1981).
- [15] S. Ferrara and B. Zumino, Nucl. Phys. B79 (1974) 413.
- [16] M. Kaku and P.K. Townsend, Phys. Lett. 76B (1978) 54.
W. Siegel, Harvard preprint HUTP-77/A077 (1977); Nucl. Phys. B142 (1978) 301.

- [17] E. Bergshoeff, M. de Roo, J.W. van Holten, B. de Wit and A. van Proyen, *in* "Superspace and supergravity (eds. S.W. Hawking and M. Roček) (Cambridge University Press) (1981).
- [18] A. Aurilia and Y. Takakashi, Alberta preprint Thy-4-81 (1981).
- [19] W. Siegel, Phys. Lett. 85B (1979) 333.
- [20] A.S. Galperin, E.A. Ivanov and V.I. Ogievetsky, Dubna preprint E2-80-750 (1980).
- [21] R. Grimm, M. Sohnius and J. Wess, Nucl. Phys. B133 (1978) 275.
- [22] M. Sohnius, K.S. Stelle and P.C. West, *in* Superspace and supergravity (eds. S.W. Hawking and M. Roček) (Cambridge University Press) (1981).
- [23] S. Deser, P.K. Townsend and W. Siegel, Nucl. Phys. B184 (1981) 333.
- [24] W. Siegel and M. Roček, CalTech. preprint (1981).
- [25] P. Howe and U. Lindström, CERN preprint TH.3075 (1981).
- [26] V. Rivelles and J.G. Taylor, King's College preprint (1981).

Table 1

N = 4, spin 3

Spin	SU(4)	SL(2,C)	Dimension
3	1	$(\frac{3}{2}, \frac{3}{2})$	3
$\frac{5}{2}$	$4 + 4^*$	$(\frac{3}{2}, 1) + (1, \frac{3}{2})$	$\frac{5}{2}$
2	1 + 15	(1, 1)	2
	6 + 6	$(\frac{3}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{3}{2})$	2
$\frac{3}{2}$	$4 + 4^*$	$(1, \frac{1}{2}) + (\frac{1}{2}, 1)$	$\frac{3}{2}$
	$20 + 20^*$		
	$4 + 4^*$	$(\frac{3}{2}, 0) + (0, \frac{3}{2})$	$\frac{3}{2}$
1	$1 + 15 + 20'$	$(\frac{1}{2}, \frac{1}{2})$	1
	$6 + 6 + 10 + 10^*$	$(1, 0) + (0, 1)$	1
	$1 + 1$	$(1, 0) + (0, 1)$	2
	$4 + 4^*$	$(\frac{1}{2}, 0) + (0, \frac{1}{2})$	$\frac{3}{2}$
	$4 + 4^*$	$(\frac{1}{2}, 0) + (0, \frac{1}{2})$	$\frac{1}{2}$
	$20 + 20^*$	$(\frac{1}{2}, 0) + (0, \frac{1}{2})$	$\frac{1}{2}$
0	$6 + 6$	(0, 0)	1
	15	(0, 0)	0

Table 2

N = 4, spin 4

Spin	SU(4)	SL(2,C)	Dimension
4	1	(2,2)	4
$7/2$	$4 + 4^*$	$(2, \frac{3}{2}) + (\frac{3}{2}, 2)$	$7/2$
3	1 + 15	$(\frac{3}{2}, \frac{3}{2})$	3
	6 + 6	$(2, 1) + (1, 2)$	3
$3/2$	$4 + 20^* + 4^* + 20$	$(\frac{3}{2}, 1) + (1, \frac{3}{2})$	$5/2$
	$4^* + 4$	$(2, \frac{1}{2}) + (\frac{1}{2}, 2)$	$5/2$
2	$1 + 15 + 20'$	(1,1)	2
	$6 + 10^* + 6 + 10$	$(\frac{3}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{3}{2})$	2
	1 + 1	$(2, 0) + (0, 2)$	2
$3/2$	$4 + 20^* + 4^* + 20$	$(1, \frac{1}{2}) + (\frac{1}{2}, 1)$	$3/2$
	$4^* + 4$	$(\frac{3}{2}, 0) + (0, \frac{3}{2})$	$3/2$
1	1 + 15	$(\frac{1}{2}, \frac{1}{2})$	1
	6 + 6	$(1, 0) + (0, 1)$	1
$1/2$	$4 + 4^*$	$(\frac{1}{2}, 0) + (0, \frac{1}{2})$	$1/2$
0	1	(0,0)	0

Table 3

N = 8, spin 4

Spin	SU(8)	SL(2,C)	Dimension
4	1	(2,2)	4
$\frac{7}{2}$	$8 + 8^*$	$(2, \frac{3}{2}) + (\frac{3}{2}, 2)$	$\frac{7}{2}$
3	63 $28 + 28^*$	$(\frac{3}{2}, \frac{3}{2})$ $(2, 1) + (1, 2)$	3 3
$\frac{5}{2}$	$216^* + 216$ $56 + 56^*$	$(\frac{3}{2}, 1) + (1, \frac{3}{2})$ $(2, \frac{1}{2}) + (\frac{1}{2}, 2)$	$\frac{5}{2}$ $\frac{5}{2}$
2	720 $420^* + 420$ $70 + 70$	(1,1) $(\frac{3}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{3}{2})$ $(2, 0) + (0, 2)$	2 2 2
$\frac{3}{2}$	$1344 + 1344^*$ $56^* + 56$ $504^* + 504$	$(\frac{1}{2}, 1) + (1, \frac{1}{2})$ $(\frac{3}{2}, 0) + (0, \frac{3}{2})$ $(\frac{3}{2}, 0) + (0, \frac{3}{2})$	$\frac{3}{2}$ $\frac{5}{2}$ $\frac{3}{2}$
1	2352' $28^* + 28$ $378^* + 378$ $1512^* + 1512$	$(\frac{1}{2}, \frac{1}{2})$ $(1, 0) + (0, 1)$ $(1, 0) + (0, 1)$ $(1, 0) + (0, 1)$	1 3 2 1
$\frac{1}{2}$	$8^* + 8$ $168^* + 168$ $1008^* + 1008$ $2352^* + 2352$	$(\frac{1}{2}, 0) + (0, \frac{1}{2})$	$\frac{7}{2}$ $\frac{5}{2}$ $\frac{3}{2}$ $\frac{1}{2}$
0	1 + 1 $36 + 36^*$ $336 + 336^*$ $1176 + 1176^*$ 1764	(0,0)	4 3 2 1 0

Table 4
N = 8, spin 6

Spin	SU(8)	SL(2,C)	Dimension
6	1	(3,3)	6
$1\frac{1}{2}$	$8 + 8^*$	$(3, \frac{5}{2}) + (\frac{5}{2}, 3)$	$1\frac{1}{2}$
5	$1 + 63$	$(\frac{5}{2}, \frac{5}{2})$	5
	$28 + 28^*$	$(3, 2) + (2, 3)$	5
$\frac{9}{2}$	$8 + 216^* + 8^* + 216$	$(\frac{5}{2}, 2) + (2, \frac{5}{2})$	$\frac{9}{2}$
	$56 + 56^*$	$(3, \frac{3}{2}) + (\frac{3}{2}, 3)$	$\frac{9}{2}$
4	$1 + 63 + 720$	(2,2)	4
	$28 + 420^* + 28^* + 420$	$(\frac{5}{2}, \frac{3}{2}) + (\frac{3}{2}, \frac{5}{2})$	4
	$70 + 70$	$(3, 1) + (1, 3)$	4
$\frac{7}{2}$	$8 + 216^* + 1344^* + 8^* + 216 + 1344$	$(2, \frac{3}{2}) + (\frac{3}{2}, 2)$	$\frac{7}{2}$
	$56 + 504^* + 56^* + 504$	$(\frac{5}{2}, 1) + (1, \frac{5}{2})$	$\frac{7}{2}$
	$56^* + 56$	$(3, \frac{1}{2}) + (\frac{1}{2}, 3)$	$\frac{7}{2}$
3	$1 + 63 + 720 + 2352'$	$(\frac{3}{2}, \frac{3}{2})$	3
	$28 + 420^* + 1512^* + 28^* + 420 + 1512$	$(2, 1) + (1, 2)$	3
	$70 + 378^* + 70 + 378$	$(\frac{5}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{5}{2})$	3
	$28^* + 28$	$(3, 0) + (0, 3)$	3
$\frac{5}{2}$	$8 + 216^* + 1344^* + 2352^* + 8^* + 216 + 1344 + 2352$	$(\frac{3}{2}, 1) + (1, \frac{3}{2})$	$\frac{5}{2}$
	$56 + 504^* + 1008^* + 56^* + 504 + 1008$	$(2, \frac{1}{2}) + (\frac{1}{2}, 2)$	$\frac{5}{2}$
	$8^* + 8$	$(\frac{5}{2}, 0) + (0, \frac{5}{2})$	$\frac{7}{2}$
	$56^* + 168^* + 56 + 168$	$(\frac{5}{2}, 0) + (0, \frac{5}{2})$	$\frac{5}{2}$
2	$1 + 63 + 720 + 2352' + 1764$	(1,1)	2
	$28 + 420^* + 1512^* + 1176^* + 28^* + 420 + 1512 + 1176$	$(\frac{3}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{3}{2})$	2
	$1 + 1$	$(2, 0) + (0, 2)$	4
	$28^* + 36^* + 28 + 36$	$(2, 0) + (0, 2)$	3
	$70 + 378^* + 336^* + 70 + 378 + 336$	$(2, 0) + (0, 2)$	2
$\frac{3}{2}$	$8 + 216^* + 1344^* + 2352^* + 8^* + 216 + 1344 + 2352$	$(1, \frac{1}{2}) + (\frac{1}{2}, 1)$	$\frac{3}{2}$
	$8^* + 8$	$(\frac{3}{2}, 0) + (0, \frac{3}{2})$	$\frac{7}{2}$
	$56^* + 168^* + 56 + 168$	$(\frac{3}{2}, 0) + (0, \frac{3}{2})$	$\frac{5}{2}$
	$56 + 504^* + 1008^* + 56^* + 504 + 1008$	$(\frac{3}{2}, 0) + (0, \frac{3}{2})$	$\frac{3}{2}$
1	$63 + 720 + 2352'$	$(\frac{1}{2}, \frac{1}{2})$	1
	$28^* + 28$	$(1, 0) + (0, 1)$	3
	$70 + 378^* + 70 + 378$	$(1, 0) + (0, 1)$	2
	$28 + 1512^* + 420^* + 28^* + 1512 + 420$	$(1, 0) + (0, 1)$	1

Table 4 (contd.)

Spin	SU(8)	SL(2,C)	Dimension
$\frac{1}{2}$	$56^* + 56$		$\frac{5}{2}$
	$56 + 504^* + 56^* + 504$	$(\frac{1}{2}, 0) + (0, \frac{1}{2})$	$\frac{3}{2}$
	$216^* + 1344^* + 216 + 1344$		$\frac{1}{2}$
0	$70 + 70$		2
	$420^* + 420$	$(0, 0)$	1
	720		0

Table 5

N = 8, spin 8, dimension = spin

Spin	SU(8)	SL(2,C)
8	1	(4,4)
$1\frac{5}{2}$	$8 + 8^*$	$(4, \frac{7}{2}) + (\frac{7}{2}, 4)$
7	$1 + 63$ $28 + 28^*$	$(\frac{7}{2}, \frac{7}{2})$ $(4, 3) + (3, 4)$
$1\frac{3}{2}$	$8 + 216^* + 8^* + 216$ $56 + 56^*$	$(\frac{7}{2}, 3) + (3, \frac{7}{2})$ $(4, \frac{5}{2}) + (\frac{5}{2}, 4)$
6	$1 + 63 + 720$ $28 + 420^* + 28^* + 420$ $70 + 70$	$(3, 3)$ $(\frac{7}{2}, \frac{5}{2}) + (\frac{5}{2}, \frac{7}{2})$ $(4, 2) + (2, 4)$
$1\frac{1}{2}$	$8 + 216^* + 1344^* + 8^* + 216 + 1344$ $56 + 504^* + 56^* + 504$ $56^* + 56$	$(3, \frac{5}{2}) + (\frac{5}{2}, 3)$ $(\frac{7}{2}, 2) + (2, \frac{7}{2})$ $(4, \frac{3}{2}) + (\frac{3}{2}, 4)$
5	$1 + 63 + 720 + 2352'$ $28 + 420^* + 1512^* + 28^* + 420 + 1512$ $70 + 378^* + 70 + 378$ $28^* + 28$	$(\frac{5}{2}, \frac{5}{2})$ $(3, 2) + (2, 3)$ $(\frac{7}{2}, \frac{3}{2}) + (\frac{3}{2}, \frac{7}{2})$ $(4, 1) + (1, 4)$
$\frac{9}{2}$	$8 + 216^* + 1344^* + 2352^* + 8^* + 216 + 1344 + 2352$ $56 + 504^* + 1008^* + 56^* + 504 + 1008$ $56^* + 168^* + 56 + 168$ $8^* + 8$	$(\frac{5}{2}, 2) + (2, \frac{5}{2})$ $(3, \frac{3}{2}) + (\frac{3}{2}, 3)$ $(\frac{7}{2}, 1) + (1, \frac{7}{2})$ $(4, \frac{1}{2}) + (\frac{1}{2}, 4)$
4	$1 + 63 + 720 + 2352' + 1764$ $28 + 420^* + 1512^* + 1176^* + 28^* + 420 + 1512 + 1176$ $70 + 378^* + 336^* + 70 + 378 + 336$ $28^* + 36^* + 28 + 36$ $1 + 1$	$(2, 2)$ $(\frac{5}{2}, \frac{3}{2}) + (\frac{3}{2}, \frac{5}{2})$ $(3, 1) + (1, 3)$ $(\frac{7}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{7}{2})$ $(4, 0) + (0, 4)$
$\frac{7}{2}$	$8 + 216^* + 1344^* + 2352^* + 8^* + 216 + 1344 + 2352$ $56 + 504^* + 1008^* + 56^* + 504 + 1008$ $56^* + 168^* + 56 + 168$ $8^* + 8$	$(2, \frac{3}{2}) + (\frac{3}{2}, 2)$ $(\frac{5}{2}, 1) + (1, \frac{5}{2})$ $(3, \frac{1}{2}) + (\frac{1}{2}, 3)$ $(\frac{7}{2}, 0) + (0, \frac{7}{2})$
3	$1 + 63 + 720 + 2352'$ $28 + 420^* + 1512^* + 28^* + 420 + 1512$ $70 + 378^* + 70 + 378$ $28^* + 28$	$(\frac{3}{2}, \frac{3}{2})$ $(2, 1) + (1, 2)$ $(\frac{5}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{5}{2})$ $(3, 0) + (0, 3)$

Table 5 (contd.)

Spin	SU(8)	SL(2,C)
$\frac{5}{2}$	$8 + 216^* + 1344^* + 8^* + 216 + 1344$ $56 + 504^* + 56^* + 504$ $56^* + 56$	$(\frac{3}{2}, 1) + (1, \frac{3}{2})$ $(2, \frac{1}{2}) + (\frac{1}{2}, 2)$ $(\frac{5}{2}, 0) + (0, \frac{5}{2})$
2	$1 + 63 + 720$ $28 + 420^* + 28^* + 420$ $70 + 70$	$(1, 1)$ $(\frac{3}{2}, \frac{1}{2}) + (\frac{1}{2}, \frac{3}{2})$ $(2, 0) + (0, 2)$
$\frac{3}{2}$	$8 + 216^* + 8^* + 216$ $56 + 56^*$	$(1, \frac{1}{2}) + (\frac{1}{2}, 1)$ $(\frac{3}{2}, 0) + (0, \frac{3}{2})$
1	$1 + 63$ $28 + 28^*$	$(\frac{1}{2}, \frac{1}{2})$ $(1, 0) + (0, 1)$
$\frac{1}{2}$	$8 + 8^*$	$(\frac{1}{2}, 0) + (0, \frac{1}{2})$
0	1	(0, 0)