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IMPLICATIONS OF ANOMALIES IN TWO-DIMENSIONAL VECTOR MODELS *)

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A B S T R A C T

We investigate classes of two-dimensional models in which low mass mesons needed to saturate anomalies are accompanied by low mass fermions. This peculiar mode of chiral behaviour is rather general in two dimensions.

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In this note we discuss the anomaly constraints in two-dimensional vector models. We show that the appearance of surprisingly light fermions and mesons is a rather general phenomenon. We start by studying $(QCD)_2$.

In $(QCD)_2$ with N_c colours and N_f flavours it is possible to study the low-energy spectrum of hadrons in various approximation schemes. 't Hooft ¹⁾ has studied the spectrum of the system in the limit of $N_c \rightarrow \infty$ with $g^2 N_c$ fixed and with $N_f = 1$. The results were derived for any value of $x = (m^2/g^2 N_c)$, the dimensionless parameter of $(QCD)_2$. In particular for $x \rightarrow 0$ a massless odd parity decoupled particle emerged, the rest of the colour singlets having masses of order $g\sqrt{N_c}$.

The singlet axial current in this model has an anomaly coming from the single fermion loop diagram (Figure). In four-dimensional confining theories it follows that anomalies generated by the massless constituents should be saturated by massless bound states in the overlying theory ^{2),3)}. In the 't Hooft limit of $(QCD)_2$ and with $x = 0$ the massless meson, as well as the singlet axial current, decouple from the massive states. The massless meson saturates the anomaly on the bound state level, in accordance with the above discussion. Note that for small x only one light particle with mass of order m is formed, the rest of the spectrum consisting of states with masses of order $g\sqrt{N_c}$.

The existence of the zero mass decoupled particle results from the conservation of both the axial and vector singlet currents. The conservation equations imply

$$\square J = 0 \quad (1)$$

for both the vector and axial-vector currents.

For a general number N_f of flavours, we can construct N_f^2 flavour currents leading to N_f^2 massless particles [actually, (N_f^2-1) charges are generators of $SU(N_f)$ and one is a singlet]. These particles saturate all flavour anomalies. This argument, valid strictly at $m = 0$, does not depend on the large N_c limit and is valid for any N_c .

For finite m , Baluni ⁴⁾ has pointed out a way in which $(QCD)_2$ could be analyzed in the limit $(m^2/g^2) \ll 1$ without applying the $1/N_c$ expansion. The spectrum of the system in this limit was studied by Baluni ⁴⁾ and also Steinhardt ⁵⁾ for $N_f = 1$ and by Amati and Rabinovici ⁶⁾ for general N_f .

The system has two mass scales, m and g , and its spectrum is divided into light (vanishing when $m \rightarrow 0$) and heavy (order g) particles. As mentioned above, the anomalies are saturated in the chiral limit $m \rightarrow 0$ by decoupled bosons. It turns out that for finite mass the composite fermions of the system (the baryons) actually belong to the light spectrum.

Let us write down the Lagrangian in the bosonic variables. We follow the notation of Ref. 6). In the limit of $(g/m) \rightarrow \infty$,

$$\begin{aligned}
 H \approx N_m' \left\{ \frac{1}{2} [\pi_x^2 + (\partial_x \chi)^2] + \frac{1}{2} \sum_{\beta} \left[\pi_{(\chi_{\beta})}^2 + (\partial_x \chi_{\beta})^2 \right] \right. \\
 + \frac{1}{2} \sum_{\beta j} \left[\pi_{(\chi_{\beta j})}^2 + (\partial_x \chi_{\beta j})^2 \right] \\
 + m'^2 \sum_{\alpha i} \left[1 - \cos 2\sqrt{\pi} \left(\frac{1}{\sqrt{N_c N_f}} \chi + \right. \right. \\
 \left. \left. + \frac{1}{\sqrt{N_c}} \sum_{\beta} \chi_{\beta} \tau_{\alpha\alpha}^{\beta} + \sum_{\beta j} \chi_{\beta j} \tau_{\alpha\alpha}^{\beta} \lambda_{ii}^j \right) \right] \Big\} \quad (2)
 \end{aligned}$$

where i are colour indices, β flavour indices, only the diagonal λ and τ appear ($\alpha=1\dots N_f$, $\beta=1\dots N_f-1$, $i=1\dots N_c$, $j=1\dots N_c-1$), and

$$\begin{aligned}
 \chi &= \frac{1}{\sqrt{N_c N_f}} \sum_{\alpha i} \phi_{\alpha i} \\
 \chi_{\beta} &= \frac{1}{\sqrt{N_c}} \sum_{\alpha i} \phi_{\alpha i} \tau_{\alpha\alpha}^{\beta} \\
 \chi_{\beta j} &= \sum_{\alpha i} \phi_{\alpha i} \tau_{\alpha\alpha}^{\beta} \lambda_{ii}^j \quad (3)
 \end{aligned}$$

where $\phi_{\alpha i}$ are the bosonic variables directly related to the original fermions through

$$\partial_x \phi_{\alpha i} = \sqrt{\pi} \psi_{\alpha i}^{\dagger} \psi^{\alpha i} \quad (4)$$

and finally

$$m' = \frac{g \sqrt{N_f}}{2 \sqrt{\pi}} \left(\frac{m \sqrt{\pi}}{g \sqrt{N_f}} \right)^{\left(\frac{N_c N_f}{N_c (N_f + 1) - 1} \right)} \quad (5)$$

Take now the case $N_f = 1$. Then

$$\begin{aligned} H &\approx N_{m'} \left\{ \frac{1}{2} [\pi_x^2 + (\partial_x \chi)^2] + m'^2_{(N_f=1)} N_c \left[1 - \cos \left(2 \sqrt{\pi} \frac{\chi}{\sqrt{N_c}} \right) \right] \right\} \\ &= N_{m''} \left\{ \frac{1}{2} [\pi_x^2 + (\partial_x \chi)^2] + m''^2 \left[1 - \cos \left(2 \sqrt{\pi} \frac{\chi}{\sqrt{N_c}} \right) \right] \right\} \quad (6) \end{aligned}$$

where

$$m'' = m'_{(N_f=1)} N_c^{\left(\frac{N_c}{2N_c-1} \right)} = \frac{g}{2 \sqrt{\pi}} \left(\frac{m N_c \sqrt{\pi}}{g} \right)^{\left(\frac{N_c}{2N_c-1} \right)} \quad (7)$$

The system Eq. (6) has a well-known spectrum ⁷⁾, consisting of a fermion and an antifermion which correspond to soliton configurations of χ , and of their bosonic bound states whose number and masses depend on

$$\beta = 2 \sqrt{\frac{\pi}{N_c}} \quad (8)$$

The masses of the fermions and bosons are obviously proportional to m'' , the only mass scale in the theory. In the context of $(QCD)_2$ the fermions are the baryons of the system, their bound states are the baryonium states and all these states are light in mass (namely proportional to m'' ^{*)}). The existence of light particles in the spectrum is required for the saturation of anomalies ^{2),3)}. In four dimensions one usually expects either light bosons, corresponding to the mode of broken chiral symmetry, or light fermions with a realization of the chiral symmetry. Surprisingly, in the two-dimensional case we get both light bosons and light fermions. The strength of the anomaly may be distributed somehow among all the light particles.

In four dimensions, once the quantum numbers of a light fermionic particle are known, its contribution to the imaginary part of the triangle diagram, at zero q^2 for the external currents, is fixed (zero q^2 is taken in a certain limiting procedure ³⁾). This is sufficient to compute the anomaly ³⁾. When non-zero quark masses are involved, the massless bound states will, in general, also

^{*)} We took $(g/m) \rightarrow \infty$ to arrive at Eq. (6). In this limit one obtains $(m''/g) \rightarrow 0$.

acquire a mass ^{*)}. We thus need couplings of the currents at non-zero q^2 , and now the axial current is no longer conserved. Hence arguments of charge coupling [as used in Ref. 3)] for computing the contribution of bound states to the anomaly, do not apply. In four dimensions, we expect the deviations to be of order m/Λ , where Λ is the QCD scale. The analogy in two dimensions would be m/g . However, in two dimensions the zero mass limit is believed to be more singular, and the above smoothness assumption may not apply. In fact, in our two-dimensional case, a determination such as the one above of the contributions of the bound state fermions to the imaginary part of the loop diagram leads to a contradiction. On the underlying level each quark contributes equally to the imaginary part of the loop diagram (Figure). The anomaly will therefore be proportional to N_c (there will also be a factor N_f , but we take now $N_f = 1$). On the overlying level we have to saturate this imaginary part by all light particles, each of them contributing positively. The spectrum contains a light baryon, the charge of which is N_c . If we are allowed to use smoothness in mass we would get a contribution proportional to N_c^2 from this baryon-antibaryon pair alone. Let us now take $N_c \rightarrow \infty$. We want to keep M_B/g small (M_B is the baryon mass). In the limit $\beta \rightarrow 0$ we have ⁷⁾ $M_B \sim (8m''/\beta)$ and thus $(M_B/g) \propto \sqrt{(m/g)} N_c$. We thus take $(m/g) \sim N_c^{-\alpha}$, $\alpha > 2$ and $N_c \rightarrow \infty$. In this limit the baryon mass remains small and we get a contradiction just from the contribution of the $B\bar{B}$ pair to the imaginary part of the loop diagram.

Thus we must conclude that the small mass limit of the form factors of baryons in the time-like region has some non-trivial dependence on N_c , which is different from the simple result that one gets for the space-like zero momentum limit.

Let us now take another limit, still with $N_c \rightarrow \infty$, but now with $(M_B/g) \rightarrow \infty$. This can be arranged by taking $(m/g) \sim N_c^{-\alpha}$ with $\alpha < 2$ ^{**)}. Now the anomaly has to be saturated by the bosons alone. In the extreme limit $\beta \rightarrow 0$, only one boson should contribute with mass $M_{mes.} \sim m''\beta \sim \pi^{1/4}(mg)^{1/2}$ [obviously $(M_{mes.}/g) \rightarrow 0$]. In this limit, Eq. (6) actually describes a single free scalar

^{*)} For other possibilities like that some bound states although containing massive quarks remain massless in a certain range, see Ref. 8).

^{**)} We arrived at Eq. (6) in the limit of $(g/m) \rightarrow \infty$ and N_f, N_c fixed. Now we take $N_c \rightarrow \infty$. Thus the neglect of the F term [in the notation of Ref. 6)] has to be justified since it involves a factor m'^2 , to be compared to $(g^2/8 N_c)$ (the coefficient of the term considered leading). The approximation holds when

$$\frac{4}{3} \pi^{3/2} N_f N_c \left(\frac{\pi m^2}{g^2 N_f} \right)^{\frac{N_c N_f}{N_c N_f + 1}} < < 1$$

In our case of $N_f = 1$ and $(m/g) \sim N_c^{-\alpha}$ with $N_c \rightarrow \infty$ this condition means $\alpha > 1$.

field which then has to saturate the $U(1)$ anomaly all by itself. This coincides with the result in the 't Hooft limit ^{*}).

We should emphasize that although at $m = 0$ all states degenerate into a decoupled boson (for arbitrary N_c), it does not mean that one can smoothly relate $f_\pi^{(0)}$ of the decoupled pion ($f_\pi^{(0)} = 1/\sqrt{\pi}$) to f of the lowest bosonic excitation that occurs for $m \neq 0$. In the case $m \neq 0$ bosonic and fermionic excitations contribute, and all may tend to the decoupled pions as $m \rightarrow 0$.

Let us now take the region of finite β . For $4\pi < \beta^2 < 8\pi$ (which corresponds to the unphysical values $\frac{1}{2} < N_c < 1$), there are no bosonic states at all. Thus the baryon-antibaryon pair contribution should equal that of the underlying quark diagram, up to order m/g corrections.

Actually, in all the above analyses, we have really studied the sine-Gordon model, which is equivalent to the massive Thirring model ⁹⁾. We found that there should be non-trivial coupling constant dependence in the contribution of the various states to the anomaly. We then used the fact that the baryons of $(QCD)_2$ in the large g/m limit are subject to the dynamics of the massive Thirring model.

The appearance of these (to some extent unexplained) light fermions seems to be quite a general phenomenon in two-dimensional theories. In the theories dealt with above [namely $(QCD)_2$ and the massive Thirring model] the light fermions were the only fermionic states of these systems. There are other two-dimensional systems whose spectrum contains heavy fermions even in the limit of strict chiral symmetry. In this limit we also have the decoupled massless pseudoscalar needed for the anomaly saturation. Such a system is the chiral invariant $SU(N)$ Thirring model ^{10),11)}. Suppose now that we slightly break chiral symmetry by adding a small mass term. We claim that in addition to the heavy fermions with dynamically generated masses of order of the renormalization scale, new light fermions of mass of order m will accompany the light bosons of the model.

We shall demonstrate this behaviour in the simple solvable model proposed by Kogut and Sinclair ¹²⁾, which also has the property of heavy chiral invariant fermions. The Lagrangian of this model is

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi + \frac{1}{2} (\partial_\mu \sigma)^2 - \frac{1}{2} \lambda \left[\bar{\psi} (1 + \gamma_5) \psi e^{i \frac{\sigma}{f}} + \bar{\psi} (1 - \gamma_5) \psi e^{-i \frac{\sigma}{f}} \right] \quad (9)$$

^{*}) To make contact with the 't Hooft limit we actually need $g \propto N_c^{-\frac{1}{2}}$.

This Lagrangian has the chiral symmetry

$$\begin{aligned}\psi &\rightarrow e^{i\beta\gamma_5}\psi \\ \sigma &\rightarrow \sigma - 2\beta a\end{aligned}\quad (10)$$

Bosonizing the field ψ one gets the bosonic version

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu c)^2 - \lambda^2 \cos\left(\frac{\sigma}{a} + 2\sqrt{\pi} c\right) \quad (11)$$

where the normal ordering is with respect to λ . Following Witten ¹¹⁾ one introduces the two combinations

$$\begin{aligned}\tilde{c} &= \frac{2\sqrt{\pi} c + \frac{\sigma}{a}}{\sqrt{4\pi + \frac{1}{a^2}}} \\ \tilde{\sigma} &= \frac{-\frac{\sigma}{a} + 2\sqrt{\pi} c}{\sqrt{4\pi + \frac{1}{a^2}}}\end{aligned}\quad (12)$$

and gets

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \tilde{\sigma})^2 + \frac{1}{2}(\partial_\mu \tilde{c})^2 - \lambda^2 \cos\left(\sqrt{4\pi + \frac{1}{a^2}} \tilde{c}\right) \quad (13)$$

The field $\tilde{\sigma}$ describes then the decoupled massless particle which takes care of the anomaly of the original model. The field $\tilde{c}$ has sine-Gordon dynamics with $\beta^2 > 4\pi$ and therefore its spectrum consists entirely of a heavy fermion with mass of order λ .

Let us now add a mass term $m\bar{\psi}\psi$ to the Lagrangian Eq. (9). The bosonized theory will change into

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu \tilde{\sigma})^2 + \frac{1}{2}(\partial_\mu \tilde{c})^2 - \lambda^2 N_\lambda \left\{ \cos\left(\sqrt{4\pi + \frac{1}{a^2}} \tilde{c}\right) \right\} \\ &\quad - 2m\Lambda N_\lambda \left\{ \cos\left(2\sqrt{\pi} \frac{a\tilde{c} - \frac{\tilde{\sigma}}{2\sqrt{\pi}}}{\sqrt{\frac{1}{4\pi} + a^2}}\right) \right\}\end{aligned}\quad (14)$$

Let us now take the case of very large λ . This implies

$$\tilde{C} \sim \frac{\pi}{\sqrt{4\pi + \frac{1}{a^2}}} n$$

Let us take $n = 0$ for simplicity. As in the $(QCD)_2$ case, we can go over to a Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} (2_f \tilde{\sigma})^2 - m'^2 N_{m'} \left\{ \cos \left(\frac{\tilde{\sigma}}{\sqrt{\frac{1}{4\pi} + a^2}} \right) \right\} \\ m' &= (2m)^{\left(\frac{1+4\pi a^2}{1+8\pi a^2}\right)} \left(\lambda \sqrt{4\pi + \frac{1}{a^2}} \right)^{\left(\frac{4\pi a^2}{1+8\pi a^2}\right)} \end{aligned} \quad (15)$$

The spectrum of the massive model will thus always contain light fermions which appear as solitons of the sine-Gordon equation (15). This is analogous to the appearance of light baryons in $(QCD)_2$. The system also contains light breathers, the number of which is $1 + [8\pi a^2]$. These have the same rôle as the baryonium in $(QCD)_2$.

What is to be learned from these results ? The two-dimensionality of the system has indeed a strong impact on the spectrum. The special features are : the decoupling of the massless pseudoscalar, the loop diagram as the anomaly and the resulting possibility of non-smoothness in mass. On the other hand, the models have the advantage of being solvable in certain approximations. We have not found any custodial symmetry to explain the proliferation of light fermions and bosons.

The conclusion that the anomaly analysis is simpler in four dimensions is rather surprising.

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FIGURE

One-loop two-point function of currents.

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