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REVIEW OF RECENT DEVELOPMENTS IN MAGNET COMPUTATIONS

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Abstract - An attempt is made to classify the magnetic field problems according to their degree of difficulty, i.e. according to

1. their number of dimensions,
2. their linearity,
3. their time dependency.

For the different classes of problems, approaches are presented which are believed to be best suited for the problem.

Special emphasis is given to recent trends. This paper is mainly based on the proceedings of the first two COMPUMAG conferences (Oxford, 1976 [1], Grenoble, 1978 [2]).

INTRODUCTION

In recent years progress has been made towards a better understanding of the characteristic difficulties arising in magnetic field computations. Numeric instabilities have been noticed in certain methods of formulating the problem [1], and ways have been searched to avoid them. The existence of instabilities has forced programmers to concentrate on different approaches depending on the type of the problem to be solved.

Other difficulties reside in the size of the matrix equations arising from potential problems. Advantage must be taken of recent advances in numerical methods which allow stable and efficient solution of such systems.

Much effort has been spent in the development of easy-to-use programs, in particular of programs offering interactive graphics facilities.

This paper concentrates on static magnetic fields, as well as on eddy current problems, but it does not treat problems involving displacement current. It attempts to summarize the recent trends in magnetic field computations, but it does not claim to present the computing methods with full mathematical rigor.

Symbols used

A	magnetic vector potential
B	magnetic flux density
c_k	k^{th} coefficient in series expansion
D	electric flux density
E	electric field strength
H	magnetic field strength
I	electric current
J	electric current density
j	square root of -1
M	magnetization vector
S	closed surface
s	arc length

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s_k	k^{th} member of a set of shape functions
T	electric vector potential
T_n	component of T normal to a given surface
t	time
V	volume of a finite body
W	weight function
x,y,z	space coordinates
ϵ	dielectric constant
μ	permeability
ρ	space charge density
ρ_m	fictitious magnetic space charge density
σ	electric conductivity
σ_m	fictitious magnetic surface charge density
ϕ	electric scalar potential
ω	angular frequency = $2\pi f$
Ω	magnetic scalar potential
$\Delta\phi$	Laplace operator = $\nabla^2\phi$
$\delta\Omega$	first variation of Ω
grad ϕ	gradient operator = $\nabla\phi$
div A	divergence operator = $\nabla \cdot A$
curl A	curl operator = $\nabla \times A$
d/dt	total derivative with respect to time
$\partial/\partial n$	partial derivative with respect to normal direction of a surface
dS	surface element, defined as a vector normal to the surface
dV	volume element

DIFFERENTIAL EQUATIONS FOR MAGNETIC FIELD PROBLEMS

Maxwell's Equations

For the purpose of this paper, Maxwell's equations will be written in the form

$$\begin{aligned} \text{div } D &= \rho, \\ \text{div } B &= 0, \\ \text{curl } E &= -\frac{dB}{dt}, \\ \text{curl } H &= J + \frac{dD}{dt}. \end{aligned} \quad (1)$$

The field vectors are related by the material property equations

$$\begin{aligned} D &= D(E) = \epsilon(E) E, \\ B &= B(H) = \mu(H) H, \\ J &= J(E) = \sigma(E) E. \end{aligned} \quad (2)$$

It should be noted that the equation

$$\text{div } J = -\frac{d\rho}{dt} \quad (3)$$

is an immediate consequence of (1).

Magnetostatic Vector Potential Method

In the magnetostatic case (1) reduces to

$$\text{div } \mathbf{B} = 0,$$

$$\text{curl } \mathbf{H} = \mathbf{J}. \quad (4)$$

The usual approach is to introduce the magnetic vector potential $\mathbf{A}$, defined by

$$\text{curl } \mathbf{A} = \mathbf{B}, \quad (5)$$

and to solve

$$\text{curl } \left(\frac{1}{\mu} \text{curl } \mathbf{A} \right) = \mathbf{J}. \quad (6)$$

To make the solution unique, a further conditions must be imposed. Usually one uses the Coulomb gauge

$$\text{div } \mathbf{A} = 0. \quad (7)$$

In two-dimensional problems the vector potential can be constrained such as to have one component only which depends only on two coordinates. In this case the vector potential approach is very efficient. It has been used with success in several two-dimensional computer programs.

Magnetostatic Scalar Potential Method

When using the magnetic vector potential in three dimensions the number of unknowns is three, namely the three components of the vector potential $\mathbf{A}$, and it is often difficult to ensure (7). Usually one therefore prefers to use a scalar magnetic potential Ω defined by

$$\mathbf{H} = \mathbf{T} - \text{grad } \Omega. \quad (8)$$

The first term represents an electric vector potential due to the currents. In virtue of (1) and (3) it obeys the equation

$$\text{curl } \mathbf{T} = \mathbf{J} \quad (9)$$

with suitable conditions to make it a simple function. Note that $\mathbf{T}$ can be computed independently of Ω . Having found $\mathbf{T}$, the remaining equation states for the potential Ω

$$\text{div } (\mu \text{ grad } \Omega) = \text{div } (\mu \mathbf{T}). \quad (10)$$

In regions where the permeability μ is large, the two terms on the right-hand side of (8) tend to cancel. This can cause the numeric solution to become badly unstable.

Total Magnetostatic Scalar Potential Method

Recently a solution to the above stability problem was published [12]. In this approach (8) is used for air and conductor regions only, while for current-free iron regions the magnetic field $\mathbf{H}$ is computed from

$$\mathbf{H} = - \text{grad } \Omega. \quad (11)$$

The potential Ω is multi-valued in annular iron regions which enclose any electric currents, but it can always be made single-valued by suitable cuts. If the electric vector potential $\mathbf{T}$ is source-free, the equation reduces to

$$\text{div}(\mu \text{ grad } \Omega) = 0. \quad (12)$$

It does not contain $\mathbf{T}$ explicitly. The interface conditions for Ω are defined by $\mathbf{T}$ on the interfaces only :

$$-\mu_2 \frac{\partial \Omega_2}{\partial n} = \mu_1 \left(-\frac{\partial \Omega_1}{\partial n} + \mathbf{T}_n \right)$$

$$\Omega_2 = \Omega_1 - \int \mathbf{T} \cdot d\mathbf{s} \quad (13)$$

The integral in the second condition must be taken along a line lying in the interface. From this condition it follows that the potential Ω has a jump on the interface.

Several papers on eddy currents [13,14,15] suggest a different normalization for the electric vector potential $\mathbf{T}$. Let the current carrying regions be enclosed by simply connected domains (see Figs. 1 and 2). Outside these domains $\mathbf{H}$ is curl-free and (11) is valid with a single-valued potential Ω . Inside the domains (8) must be used. No current flows out of the domain boundaries. $\mathbf{T}$ can therefore be constrained on the boundaries such as to have only a component normal to the surface. We are left with the scalar equation (10), with the right-hand side zero everywhere except in the domains enclosing the conductors. The interface conditions are simplified to

$$-\mu_2 \frac{\partial \Omega_2}{\partial n} = \mu_1 \left(-\frac{\partial \Omega_1}{\partial n} + \mathbf{T}_n \right)$$

$$\Omega_2 = \Omega_1 \quad (14)$$

Fig. 1. Region surrounding a current filament.

Fig. 2. Region surrounding a solenoid coil.

For a filament loop the enclosing domain can be shrunk to a double layer surface spanned by the loop (see Fig. 1). Within the double layer $\mathbf{T}$ is normal to the spanned surface, and its magnitude, integrated across the layer, is equal to the current of the loop. The effect is that the scalar potential Ω has a jump on the surface equal to the current of the loop. These considerations show that the current loop might as well be replaced by an equivalent layer of magnetic dipole charges. The value of the dipole charges is given by the vector $\mathbf{T}$, integrated across the double layer.

A thick coil, like a solenoid, can be built up from many filaments (see Fig. 2). T is found by integration over the contributions of all filaments. Again, one may replace the coil by equivalent dipole charges. This time T gives directly the equivalent volume density of magnetic dipoles.

Eddy Currents, Magnetic Vector Potential Method

In low-frequency applications one puts

$$\epsilon = 0 \Rightarrow D = 0, \rho = 0. \quad (15)$$

Using the magnetic vector potential A and the electric scalar potential ϕ defined by

$$B = \text{curl } A, \\ E = -\text{grad } \phi - \frac{dA}{dt}. \quad (16)$$

One has

$$\text{curl} \left(\frac{1}{\mu} \text{curl } A \right) + \sigma \frac{dA}{dt} = -\sigma \text{grad } \phi. \quad (17)$$

The right-hand side can be interpreted as the impressed current density. To make the solution unique, one usually imposes the Coulomb gauge (7). The vector potential A is non-zero all over space, and it is frequently difficult to find boundary conditions on the conductor limits. This means that one has to compute the three components of A in a volume which may be considerably larger than the conductor volume. In three-dimensional problems it is often difficult to ensure that the vector potential remains source-free.

In two-dimensional problems however it is possible to constrain A such that it has only one component. In this case the solution with the vector potential can be quite economic.

Eddy Currents, Electric Vector Potential Method

Several authors have proposed an alternate approach for eddy current problems [13,14,15]. In virtue of (1) and (3) an electric vector potential T can be defined by

$$\text{curl } T = J \quad (18)$$

with suitable conditions to make T a simple function. Putting

$$H = T - \text{grad } \Omega \quad (19)$$

the two remaining equations for the potentials T and Ω state

$$\text{div} (\mu \text{grad } \Omega) = \text{div} (\mu T), \\ \text{curl} \left(\frac{1}{\sigma} T \right) + \frac{d}{dt} (\mu T) = \frac{d}{dt} (\mu \text{grad } \Omega). \quad (20)$$

Let all conductors be enclosed in singly connected domains (see Fig. 3). Then T can be made zero everywhere except inside these domains. On the boundaries of these domains the components of T parallel to the surface can be made to vanish. In the general case this approach requires four unknowns in conductor regions, namely Ω and the three components of T , and one unknown in free space, namely Ω .

In many problems the eddy currents are confined to flow in known surfaces only. This is true for purely two-dimensional problems, but also in three-dimensional problems including laminated iron cores. In these problems it is possible to constrain T such that it has only one component perpendicular to the current surfaces. This reduces the number of unknowns

in conductor regions to two. In three-dimensional problems the single component of T perpendicular to the laminations however still depends on all three space coordinates.

Fig. 3. Region enclosing a conductor.

Diffusion Equation for the Magnetic Field

To solve eddy current problems one may also use the vector diffusion equation for the magnetic field H [16]:

$$\Delta H - \sigma \frac{dB}{dH} \frac{dH}{dt} + \text{grad} \left(H \cdot \frac{\text{grad } \mu}{\mu} \right) + \frac{\text{grad } \sigma}{\sigma} \times \text{curl } H = 0. \quad (21)$$

This approach is specially suited to problems whose boundary conditions are given in terms of the magnetic field H . It requires however three unknowns all over space, and is economic only if the problem region is limited or if the problem is two-dimensional.

Steady-State Eddy Currents

For all the above eddy current equations steady-state solutions exist for sinusoidal excitation if the material properties are linear or if the oscillation amplitudes are small. Under these conditions the relevant field quantities can all be written in phasor form

$$H = \bar{H} \exp(j\omega t) \quad (22)$$

and the time derivatives are replaced according to

$$\frac{d}{dt} = j\omega. \quad (23)$$

The resulting equations are time-independent, but they contain complex coefficients. They can be solved by methods similar to the ones used in magnetostatics, provided the procedures are adapted to allow complex quantities.

SOLUTION METHODS

Finite Difference Methods

The finite difference method is the classical solution method for magnetic field problems. Usually the differential operators are approximated by finite difference operators based on a grid built up from coordinate planes. The method is very simple if all domains occurring in the problem are limited by grid planes, and if the material properties are isotropic.

In the finite difference method approximations are needed for non-linear second-order differential operators. In particular for anisotropic materials these are not always easy to find. On irregular material boundaries the interface conditions demand

rather complicated approximations. Modern efficient methods for solving large matrix equations usually require the matrix to be symmetric. This symmetry is by no means guaranteed in the finite difference method. For these reasons many authors try to avoid using this method. The finite difference method has been well known for a long time ; this paper will not deal with it further.

Finite Element Methods

In recent years the theory of finite element methods has made considerable progress [6,7,8]. This method is one of the most powerful means to solve partial differential equations.

The first authors to use finite elements in magnetic field problems have applied the familiar variational integral [17,18]

$$\int_V (H \cdot dB - J \cdot A) dV = \text{minimum} \quad (24)$$

subject to the conditions

$$B = \text{curl } A, \quad H = H(B). \quad (25)$$

The analytic solution of the above problem leads to the Euler equation

$$\text{curl } H = J. \quad (26)$$

For the magnetostatic scalar potential the variational integral is

$$\int_V (B \cdot dH) dV = \text{minimum}, \quad (27)$$

subject to the conditions

$$H = T - \text{grad } \Omega, \quad B = B(H). \quad (28)$$

It has the Euler equation

$$\text{div } B = 0. \quad (29)$$

In the variational form of the problem the order of the differential operators occurring is at most one. It suggests the following solution procedure :

1. The problem region is divided into elementary volumes, the "finite elements".
2. The variational integral is approximated numerically on each of these elements in terms of the node values of the potential.
3. The contributions to the total integral are summed over all elements.
4. The sum is minimized in terms of the node values of the potential. In the general case the resulting system of equations is non-linear. It must be solved by iteration. In magnetostatics the coefficients of the linearized system usually form a real symmetric matrix. For steady-state eddy current problems the matrix is complex.
5. The magnetic field is derived from the potential by numeric differentiation.

An alternate approach leads to the following procedure :

1. The potential, say Ω , is developed in a series of arbitrary "shape functions" s_k

$$\Omega = \sum_k c_k s_k \quad (30)$$

2. This series is substituted in the variational integral.
3. The variational integral is minimized in terms of the coefficients c_k .

Approaches similar to the above are obviously also possible when using the vector potential.

How to find Variational Integrals

We have seen above that many different partial differential equations arise in magnetic field problems. In order to apply the finite element method, we need a way to find a variational integral whose Euler equation is the partial differential equation to be solved. To illustrate how this is done, consider the magnetostatic scalar potential problem

$$\begin{aligned} H &= T - \text{grad } \Omega, \\ B &= B(H), \\ \text{div } B &= 0. \end{aligned} \quad (31)$$

Multiplication of the last equation by an arbitrary function $\delta\Omega$ and integration over the problem region gives

$$\int_V (\delta\Omega \text{ div } B) dV = 0, \quad \text{for all } \delta\Omega. \quad (32)$$

Using identities for the divergence and Gauss's theorem, this can be transformed to

$$\int_S (\delta\Omega B) \cdot dS - \int_V (B \cdot \text{grad } \delta\Omega) dV = 0 \quad (33)$$

If $\delta\Omega$ is interpreted as an arbitrary variation of the potential Ω this equation is equivalent to the variational problem

$$\int_S (\Omega B) \cdot dS - \int_V (B \cdot dH) dV = \text{minimum}, \quad (34)$$

Depending on the boundary conditions for the potential and/or for the flux density, the first term looks different. For the natural boundary condition $B_n = 0$ it vanishes. The second term is the integral whose Euler equation is the original partial differential equation.

Galerkin's Method

Another solution method, which is related to the finite element method, is due to Galerkin. Substituting the series (30) in the differential equation and multiplying the latter by an arbitrary weight function W , integration over the problem region gives for the magnetostatic problem

$$\int_V W \text{ div } B dV = 0, \quad \text{for all } W. \quad (35)$$

This can be transformed to

$$\int_S (W B) \cdot dS - \int_V (B \cdot \text{grad } W) dV = 0, \quad \text{for all } W, \quad (36)$$

subject to the conditions

$$B = B(H), \quad H = T - \sum_k c_k \text{ grad } s_k. \quad (37)$$

If the above integrals are evaluated for as many weight functions as there are shape functions, the number of equations is equal the number of unknowns. Being non-linear, the equations have to be solved by iteration. If the weight functions are taken from the same set as the shape functions, the linearized system of equations can often be made symmetric.

The volume integrals occurring in Galerkin's method are to be evaluated over the regions where both the shape function and the weight functions are non-zero. By a suitable choice of the functions these regions can be made small and disjoint, and the resulting linearized system of equations has a sparse matrix. Such a set of functions is known as locally based.

Fast Poisson Solvers

Several methods are known which allow to solve Poisson's equation very efficiently [19,20]. Most of them are based directly or indirectly on the properties of the discrete Fourier transform. These methods are best suited for rectangular problem regions with constant material properties. The basic idea is to transform Poisson's equation

$$\Delta\phi = f(x,y,z) \quad (38)$$

in a rectangle by applying the Fourier transform to one or more of the space coordinates in order to reduce the dimensionality of the problem.

Each Fourier transform along one of the coordinate axes replaces the partial derivatives with respect to this coordinate direction by simple algebraic factors which depend on the corresponding frequency variable. Such a transform is equivalent to the separation of variables in analytic solutions. After one transform in case of a two-dimensional problem, or after two transforms in case of a three-dimensional problem, one is left with a one-dimensional problem depending parametrically on one or two frequency variables, and which is solved easily by a direct method. Application of the inverse transform to the solution of the one-dimensional problem yields the solution of the original problem. It is clear that the Fourier transform may also be applied to time.

Since the Fourier transform replaces the multiplication by a space function by a convolution integral, problems with more general elliptic partial differential operators cannot be treated this way. Consider the magnetostatic equation

$$\operatorname{div}(\mu \operatorname{grad} \Omega) = \operatorname{div}(\mu \mathbf{T}) \quad (39)$$

This can be rewritten as

$$\mu \Delta\Omega + \operatorname{grad} \mu \cdot \operatorname{grad} \Omega = \mu \operatorname{div} \mathbf{T} + \operatorname{grad} \mu \cdot \mathbf{T}. \quad (40)$$

If the variations of μ are not too large, a convergent iteration procedure is defined by splitting the operator:

$$\Delta\Omega^{(k+1)} = \operatorname{div} \mathbf{T} + \frac{\operatorname{grad} \mu}{\mu} \cdot (\mathbf{T} - \operatorname{grad} \Omega^{(k)}). \quad (41)$$

In this method interface conditions, as well as boundary conditions on irregular outer boundaries, must be replaced by equivalent single or double layers, which have to be determined at each iteration step from the potential found so far. When the material properties depend strongly on the magnetic field, the operator on the right-hand side of (41) changes strongly from one iteration to the next, and it is unlikely that the above method converges. Other methods may be much more efficient for such problems, even if a very fast Poisson solver is available.

Volume Integral Methods

In many magnetic field problems a difficult problem is caused by unlimited problem regions. For example, if one wants to find the leakage field of a magnet, it is not known a priori how far away from the magnet the computation must be extended. An interesting solution to this problem is the use of integral methods.

It is well known that in absence of magnetic material the magnetic field is found using the Biot-Savart integral

$$4\pi\mathbf{T} = \int_{\text{Cu}} \frac{\mathbf{J} \times \mathbf{r}}{r^3} dV \quad (42)$$

In presence of magnetic material the gradient of the magnetic scalar potential due to the magnetization of the iron must be added. The magnetic field becomes

$$4\pi\mathbf{H} = 4\pi\mathbf{T} - \operatorname{grad} \int_{\text{Fe}} \frac{\mathbf{M} \cdot \mathbf{r}}{r^3} dV. \quad (43)$$

Using the relation

$$\mathbf{M} = \mathbf{B}(\mathbf{H}) - \mathbf{H} \quad (44)$$

a vector integral equation can be set up for $\mathbf{H}$ or for $\mathbf{M}$ [21]. The advantage of this method is that (43) gives the leakage field over all of space; and that no boundary conditions are required. It suffers however strongly from the instabilities mentioned above for the magnetostatic scalar potential method.

Boundary Integral Methods

The integral (43) can be transformed to [22]

$$4\pi\mathbf{H} = 4\pi\mathbf{T} - \operatorname{grad} \int_{\text{S}} \frac{\mathbf{M}}{r} dS + \operatorname{grad} \int_{\text{V}} \frac{1}{r} \operatorname{div} \mathbf{M} dV. \quad (45)$$

For constant permeability regions

$$\operatorname{div} \mathbf{M} = 0 \quad (46)$$

and the volume integral vanishes. The unknown normal component of $\mathbf{M}$ on the interfaces may be interpreted as a fictitious surface charge σ_m . (45) becomes a scalar integral equation for this unknown surface charge. This drastically reduces the number of unknowns compared to the volume integral method. In the same time the instability problem completely disappears.

If the permeability is variable, one introduces a fictitious space charge

$$\rho_m = -\operatorname{div} \mathbf{M} \quad (47)$$

and sets up a scalar integral equation for both unknown charges.

Exterior Finite Elements

Two approaches have been used so far to model exterior regions with finite elements:

The first version uses special elements for the exterior region [23]. These elements extend from the problem boundary up to infinity. The shape functions for the potentials are selected such as to obey the boundary conditions at infinity. With a suitable choice of node values, a solution for the potential may be modelled which is correct out to infinity.

The second version [24] has the objective to fill the exterior region with finite elements, and to eliminate all additional nodes generated by the filling before the solution for the interior region is started.

The external space is divided into a series of concentric layers, each geometrically similar to all others. The first two layers are shown in Fig. 4. Elimination of the nodes on the common interface between the two layers generates a matrix for the coupling between the nodes on the inner and outer boundaries of the two layers. The generated annular region can again be scaled and combined with its scaled image. Repeating this process several times allows to move the outer boundary away from the problem region in an exponential progression. After a few steps it is sufficiently far away that the boundary potential can be set to zero safely. The final matrix simply represents the relations between the nodes on the original problem boundary.

Fig. 4. Boundary matching method

It turns out that the matrix elements obey some simple scaling laws and that at each step the matrix for the outer layer can be obtained easily from the matrix for the inner layer. This makes the recursive process to generate the boundary coupling very fast. Since the outer region is usually free space, the problem is linear in this region, and the coupling matrix does not depend on field strengths. It is computed only once before solving in the interior region.

Hybrid Methods

For variable permeability material, it is usually most efficient to use the finite element method. If the surrounding air region is infinite, one may still save the advantage of the integral method by treating this air region by a boundary integral method [25]. Treating for example the interior by the T-Ω method, the boundary potentials can be connected by Green's formula

$$4\pi\Omega = \int_S \frac{1}{r} \text{grad } \Omega \cdot dS - \int_S \Omega \text{ grad } \frac{1}{r} \cdot dS \quad (48)$$

This method has the advantage of simplicity for the non-linear material region, combined with the absence of boundary conditions at infinity. The drawback is that the evaluation of Green's formula is not always easy.

Network Models

Eddy current problems are not always easy to solve with finite elements, since the differential equation is not self-adjoint due to energy dissipation. Using finite element [26,27] or integral equations [28] it is possible to replace the differential equations for eddy currents by difference equations which can be interpreted as the Kirchhoff equations for an electric network. The advantage of this method is that the network equations can be derived from the finite element equations without reference to energy considerations. The network model also contributes to a better understanding of the physics of the problem. Network models can be used for both steady-state or transient eddy current problems. Very accurate results are reported for calculations based on network models.

LARGE MATRIX EQUATIONS

When high precision is asked for, the magnetic fields must be discretized on very fine grids. The result is a matrix equation of very large order. Unless integral equation methods are used the matrix is sparse. A great deal of efficiency can be gained, if the sparsity is exploited properly.

Successive Point-Overrelaxation

Originally most computer programs to solve partial differential equations used successive point overrelaxation methods [31]. It was however recognized that point overrelaxation becomes less efficient, the finer the grid is made. The convergence of successive point-overrelaxation also strongly depends on the regularity of the grid, as well as on the linearity of the problem. Many computer programs have therefore abandoned its use in favour of more efficient methods.

Successive Line-Overrelaxation

In most problems it is possible to relax the grid points in groups. Such a group could for example consist of a grid line. In this case several residuals are cancelled at the same time. To achieve this goal, suitable diagonal blocks of the problem matrix are inverted. In a similar way as for point relaxation the convergence is greatly improved if the corrections obtained are multiplied by a relaxation factor greater than unity. The number of iteration steps may be reduced compared to point-relaxation by about an order of magnitude.

Incomplete Cholesky Method

For symmetric matrices a better method has been proposed in [32]. The idea is to execute a Gauss-Cholesky triangle decomposition, but to allow fill-in only in those positions where the original matrix had non-zero elements. Sometimes a few more elements are allowed. The resulting triangular matrices can be inverted easily, they represent an approximate inverse of the original problem matrix. The number of operations required is however much smaller than for an exact inversion.

The linearized system of equations is solved by iteration. Starting with the system of equations

$$A x + b = 0, \quad (49)$$

one has the approximate triangle decomposition

$$A \approx LL'. \quad (50)$$

At each iteration step the residuals are computed using the original matrix, while the corrections are found by premultiplying the residuals by the approximate inverse :

$$x^{(k+1)} = x^{(k)} - L^{-1}L^{-1} (A x^{(k)} + b) \quad (51)$$

Since the original matrix was symmetric, one may rewrite the equation as

$$(L^{-1}AL^{-1}) (L'x) + (L^{-1}b) = A' x' + b' = 0. \quad (52)$$

The modified matrix A' is also symmetric, and there is no difficulty to apply the conjugate gradients method [33] to the modified system. The condition of A' is much better than the condition of A , and in most cases the eigenvalues of A' will come in clusters, such as to make the conjugate gradients method converge very rapidly.

Preconditioned Conjugate Gradients Method

In (52) the original matrix was replaced by a different matrix with a significantly better condition number. A different choice of the transformation matrix L leads to a similar improvement. Let the matrix L represent line-relaxation (not overrelaxation!) [34]. This transform will also improve the condition of A . In addition the matrix A' has a 2-cyclic structure, also known as Young's "property A" [31]. In this case the conjugate gradients method can be programmed in a particularly simple way [35]. The operation count for one iteration step is comparable with one line-overrelaxation step, but the convergence speed in terms of iterations is improved by up to two orders of magnitude compared to point-overrelaxation.

Band Matrix Methods

Several authors have tried direct inversion methods. The most obvious one uses the band properties of the problem matrix [36]. For this method the number of arithmetic operations depends only on the problem geometry. Unless the matrix is very badly conditioned, the computing time is as long as for iteration methods. For badly conditioned matrices however, the direct inversion can be unstable. This requires solution by iterative refinement, thus losing any advantage compared to iteration methods.

Another direct inversion method is based on the block properties of the problem matrix. The matrix is inverted by blocks and the single blocks are kept on disk. Similar remarks apply as for the band matrix methods.

Tearing or Marching Methods

A third class of direct methods finds a direct solution for most of the node values in terms of a small subset of node values. This is for example possible in a regular rectangular grid, if one row or column of unknown node values along a boundary is taken as the subset. These methods are known as "tearing" or "marching" methods [37,38]. A straightforward Gaussian elimination reduces the problem to a matrix equation for the small subset, having a nearly full matrix. Inversion of this smaller matrix yields the solution for the "teared" subset. The other node values can then be obtained recursively from this solution.

The computing work is strongly reduced compared to the classical methods. The elimination order deviates however very much from the strategy of optimal pivoting.

In most magnetic field problems this will cause bad instabilities. A precise answer can only be found by iterative refinement, and the saving in time is lost.

Dissection Methods

A potentially very efficient and stable method has been proposed recently [39,40]. It is known as the dissection method. It is a direct inversion method based on a special ordering. For large problems this ordering gives the asymptotically smallest operation count possible. Unfortunately the break-even point compared to other direct methods is reached for very large problems only. Also the book-keeping operations for the special ordering required take away some time.

COMPUTER AIDED DESIGN

Five stages can be distinguished in designing a magnet by computer [41,42,43].

Mesh Generation

When the problem geometry is complicated the work to set up a finite difference grid or a set of finite elements can become prohibitive for the user. This is particularly true for three-dimensional problems. Much effort has therefore been spent in simplifying the data input. Commercial as well as problem-tailored programs are available which generate all the information needed for the solution program from a simple problem description language. Sometimes these pre-processor programs even generate part of the solution program, like FORTRAN dimension statements, material property subroutines, or output routines. In order to avoid input errors, they also display the geometric structures, such as to make immediately visible any input errors, like for example displaced points.

Some programs combine optimal mesh generation with the solution process [44]. The problem is first modeled on a very coarse grid. Subsequently the grid is successively refined such as to reduce the discretization error. Subdivision occurs only in regions where the potential has rapid changes. This keeps the computing time and memory space low without causing any sacrifice in accuracy.

Material Properties

Mathematical models are needed for magnetic materials, i.e. for the dependency of field quantities according to (2). For hard magnetic materials, as well as for superconductors, completely satisfactory models are not yet available. Much work is still needed to find better models. Some progress has however been made, mainly for hard magnetic materials [45]. A useful feature exists in certain computer programs, which contain a library of material properties. The program users need only specify the name of the material to be used to fill a given region.

Solution

The next step in the process is the solution of the field equations. As has been said above, often parts of the solution program are generated by a preprocessor program, allowing an optimization of the solution process. Storage allocation can be minimized,

TABLE 1
Recommended Methods

and it is for example possible to automatically generate the optimal order in which the grid points must be treated. These developments will certainly go on in future.

Edition of Results

Once the solution is found in terms of node potentials or of other parameters, the results must be presented to the user in useful form. This part of the solution is well suited for an interactive postprocessor program. Graphs of field distributions, or analysis of the field in terms of a given set of orthogonal functions, or computation of derived quantities like forces torques etc. are frequently desired [29, 30].

Optimization

Often the user wants to optimize the magnet design such as to obtain a given field distribution and/or dynamic behaviour. Some programs allow to feed the results back to the mesh generator. This will adapt the problem geometry as a function of the desired field pattern. Such a feedback has been proposed in [46]. It has not been used very widely so far, but its use will certainly be developed.

SUMMARY

For the solution of Maxwell's equations in two dimensions, the most suitable potential to be used seems to be the magnetic vector potential A. In three dimensions however, the electric vector potential T, combined with the magnetic scalar potential Ω , should be preferred.

The most efficient methods to solve general static magnetic field problems seem to be the incomplete Cholesky methods or preconditioned conjugate gradients methods combined with a finite element formalism. As for time-stepping methods in transient magnetic field problems, many versions are proposed in the literature. A judgement on their relative merits cannot be given at this time.

The table 1 tries to summarize the recommended methods. The problems get progressively harder when stepping down in the table. Correspondingly the uncertainty grows about which is the best method.

Time dependency	material involved	dim.	unknown	method	solver
Static	air only	2	H	BS	-
			A	FEM	PCCG
			T- Ω	FDM	direct
				BIM	direct
				FFT	-
		3	T- Ω	FEM	PCCG
				BIM	direct
				FFT	-
		linear	H	BS	-
				FFT	-
	linear	2	T- Ω	FEM	PCCG
			A	FDM	direct
		3		FFT	-
			T- Ω	FEM	PCCG
	soft iron	2	A	FEM	PCCG
			T- Ω	FDM	direct
		3	T- Ω	FEM	PCCG
				FDM	direct
	hard iron	2	A	FEM	PCCG
			T- Ω		direct
Steady state eddy current	linear	2	T- Ω	FEM	PCCG
			A- ϕ	FDM	direct
		3	H	network	
			T- Ω	FEM	PCCG
transient eddy current	linear or soft iron	2	T- Ω	FEM	SM
			A- ϕ	FDM	
		3	H	network	
			T- Ω	FEM	SM
	hard iron	2	T- Ω	FEM	SM
			A- ϕ	FDM	
		3	H	network	
			T- Ω	FEM	SM

Legend

BS	Biot-Savart integral
FDM	Finite differences method
FEM	Finite elements method
FFT	Fast Fourier transform
network	Network method
PCCG	Preconditioned conjugate gradients
direct	Direct matrix inversion
SM	Stepping methods

REFERENCES

Conference Proceedings

- [1] First Conference on Computation of Magnetic Fields, COMPUMAG 76, Oxford, 1976. Rutherford Laboratory, Science Research Council, Chilton, Didcot, Oxon OX11 0QX, UK.
- [2] Second Conference on Computation of Magnetic Fields, COMPUMAG 78, Grenoble, 1978. Laboratoire d'Electrotechnique, Ecole Nationale Supérieure d'Electricité et de Génie Civil, F-38000 Grenoble, France.
- [3] Third Conference on Computation of Magnetic Fields, COMPUMAG 81, Chicago, september 1981. Argonne National Laboratory, Argonne, Illinois 60439, USA.

Papers on magnetic field computations have also been published in the following conference proceedings :

[4] International Conferences on Magnet Technology :

1. Stanford, 1965, ed. H. Brechna and H.S. Gordon, United States Atomic Energy Commission.
2. Oxford, 1967, ed. H. Hadley, Rutherford Laboratory.
3. Hamburg, 1970, edited by editorial committee MT3, DESY.
4. Brookhaven, 1972, ed. Y. Winterbottom, United States Atomic Energy Commission.
5. Rome, 1975, ed. H. Sacchetti, M. Spadoni, and S. Stipcich, CNEN, Frascati.
6. Bratislava, 1978, edited by editorial committee MT6, ALFA, Bratislava.
7. Karlsruhe, 1981, IEEE MAG-17.

[5] Intermag Conferences :

3. Washington, 1965, IEEE MAG-1.
4. Stuttgart, 1966, IEEE MAG-2.
5. Washington, 1967, IEEE MAG-3.
6. Washington, 1968, IEEE MAG-4.
7. Amsterdam, 1969, IEEE MAG-5.
8. Washington, 1970, IEEE MAG-6.
9. Denver, 1971, IEEE MAG-7.
10. Kyoto, 1972, IEEE MAG-8.
11. Washington, 1973, IEEE MAG-9.
12. Toronto, 1974, IEEE MAG-10.
13. London, 1975, IEEE MAG-11.
14. Pittsburg, 1976, IEEE MAG-12.
15. Los Angeles, 1977, IEEE MAG-13.
16. Firenze, 1978, IEEE MAG-14.
17. New York, 1979, IEEE MAG-15.
18. Boston, 1980, IEEE MAG-16.

Text Books

- [6] G. Strang and G.J. Fix, An analysis of the finite element method. Prentice-Hall, 1973.
- [7] A.R. Mitchell and R. Wait, The finite element method in partial differential equations. J. Wiley & Sons, 1977.
- [8] O.C. Zienkiewicz, The finite element method. 3rd ed., McGraw Hill, 1977.

- [9] E.D. Brigham, The fast Fourier transform. Prentice Hall, 1974.
- [10] R.L. Stoll, The analysis of eddy currents. Oxford University Press, London, 1974.

Selected Papers

- [11] J. Simkin and C.W. Trowbridge, Which potential, a comparison of the various scalar and vector potentials for the numerical solution of the non-linear poisson problem. Rutherford Laboratory report RL-78-009/B, 1978.
- [12] A.G. Armstrong, C.J. Collie, J. Simkin, and C.W. Trowbridge, The solution of three-dimensional magnetostatic problems using scalar potentials. Proceedings COMPUMAG 78, paper 1.2, Grenoble, 1978.
- [13] C.J. Carpenter and D.H. Locke, Numerical models of three dimensional end winding arrays. Proceedings COMPUMAG 76, pp. 47-55, Oxford, 1976.
- [14] C.J. Carpenter and E.A. Wyatt, Efficiency of numerical techniques for computing eddy currents in two and three dimensions. Proceedings COMPUMAG 76, pp. 242-250, Oxford, 1976.
- [15] T.W. Preston and A.B.J. Reece, Finite element solution of three dimensional eddy current problems in electrical machines. Proceedings COMPUMAG 78, paper 7.4, Grenoble, 1978.
- [16] R.L. Stoll, Solution of transient and steady-state magnetic fields with particular reference to numerical methods. Proceedings COMPUMAG 76, pp. 206-212, Oxford, 1976.
- [17] A.M. Winslow, Numerical solution of the Poisson equation in a non-uniform triangle mesh, University of California report UCRL-7784-T, 1964.
- [18] C. Iselin, Some recent developments in computation of two-dimensional fields in iron magnets. Proceedings Fourth International Conference on Magnet Technology, pp. 83-99, Hamburg, 1970.
- [19] R.W. Hockney, The potential calculation and some applications. Methods of Computational Physics, Vol. 9, pp. 135-211 (1970).
- [20] R.C. Le Bail, Use of fast Fourier transforms for solving partial differential equations in physics. Journal of Computational Physics, Vol. 9, pp. 440-465 (1972).
- [21] J. Simkin and C.W. Trowbridge, GFUN3D user's guide. Rutherford Laboratory report RL-76-029 (1976).
- [22] J. Simkin and C.W. Trowbridge, Magnetostatic fields computed using an integral equation derived from Green's theorems. Proceedings COMPUMAG 76, pp. 5-14, Oxford, 1976.
- [23] P. Bettess, infinite elements. International Journal on Numeric Methods in Engineering, Vol. 11, pp. 53-64 (1977).
- [24] P.P. Sylvester and C.J. Carpenter, Exterior finite elements for two-dimensional field problems with open boundaries. Proceedings IEE, Vol. 124, pp. 1267-1270 (1977).
- [25] O.C. Zienkiewicz, D.W. Kelly, and P. Bettess, The coupling of the finite element and boundary solution procedures. International Journal for Numerical Methods in Engineering, Vol. 11, pp. 355-376 (1977).

- [26] C.J. Carpenter, Finite element network models and their application to eddy current problems. Proceedings IEE, Vol. 122, pp. 455-462 (1975).
- [27] C.J. Carpenter and M. Djurovic, Three-dimensional numerical solution of eddy currents in thin plates. Proceedings IEE, Vol. 122, pp. 681-688 (1975).
- [28] L.R. Turner and R.J. Lari, Developments of the eddy current program EDDYNET. Proceedings COMPUMAG 78, paper 4.4, Grenoble, 1978.
- [29] P. Hammond, Physical basis of the variational method for the computation of magnetic field problems. Proceedings COMPUMAG 76, pp. 28-34, Oxford, 1976.
- [30] K. Reichert, H. Freundl, and W. Vogt, The calculation of forces and torques within numerical magnetic field calculation models. Proceedings COMPUMAG 76, pp. 64-74, Oxford, 1976.
- [31] D. Young, Iterative methods for solving partial differential equations of elliptic type. Transactions American Mathematical Society. Vol. 76, pp. 92-111 (1954).
- [32] H.L. Stone, Iterative solution of implicit approximations of multidimensional partial differential equations. SIAM Journal on Numerical Analysis, Vol. 5, pp. 530-558 (1968).
- [33] M.R. Hestenes and E. Stiefel, Methods of conjugate gradients for solving linear systems, Journal of Research of the National Bureau of Standards, Vol. 49, pp. 409 ff. (1952).
- [34] O. Axelsson, On preconditioning and convergence acceleration in sparse matrix problems. CERN report CERN 74-10 (1974).
- [35] J.K. Reid, The use of conjugate gradients for systems of linear equations possessing "property A", SIAM Journal on Numerical Analysis, Vol. 9, pp. 325-332 (1972).
- [36] A. Jennings, A compact storage scheme for the solution of symmetric linear simultaneous equations. The Computer Journal, Vol. 9, pp. 281-285 (1966).
- [37] R.E. Bank and D.J. Rose, Marching algorithms for elliptic boundary value problems, I : The constant coefficients case. SIAM Journal on Numerical Analysis, Vol. 14, pp. 792-829 (1977).
- [38] R.E. Bank, Marching algorithms for elliptic boundary value problems, II : The variable coefficients case. SIAM Journal on Numerical Analysis, Vol. 14, pp. 950-970 (1977).
- [39] A. George, Nested dissection of a regular finite element mesh. SIAM Journal on Numerical Analysis, Vol. 10, pp. 345-363 (1973).
- [40] A. George, Numerical experiments using dissection methods to solve $n \times n$ grid problems. SIAM Journal on Numerical Analysis, Vol. 14, pp. 161-179 (1977).
- [41] M.J. Newman, Application of interactive graphics techniques to magnet design. Proceedings COMPUMAG 76, pp. 432-445, Oxford, 1976.
- [42] S. Polak, A. de Beer, A. Wachters, and J.V. van Welij, MAGGY2 and PADDY program packages for 2 and 3 dimensional magneto-static problems. Proceedings COMPUMAG 78, paper 2.5, Grenoble, 1978.
- [43] R. Rafinejad and J.C. Coulomb, Interactive computing techniques in 3-dimensional modelisation of field problems by finite elements method. Proceedings COMPUMAG 78, paper 4.5, Grenoble, 1978.
- [44] R. Leyvraz, Iterative generation of optimal triangular grids for the solution of 2-dimensional field problems. Proceedings COMPUMAG 76, pp. 472-485, Oxford, 1976.
- [45] H. Zijlstra, Permanent magnets in magnetic field calculations. Proceedings COMPUMAG 76, pp. 164-167, Oxford, 1976.
- [46] K. Halbach, A programme for inversion of systems analysis and its application to the design of magnets. Proceedings Second International Conference on Magnet Technology, pp. 47-60, Oxford, 1967.