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BOUNDARY CONDITIONS AND SPURIOUS SINGULARITIES
IN CLASSICAL GAUGE THEORIES

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A B S T R A C T

We show, at the classical level, that the so-called spurious singularities, characteristic of the axial gauges, are related to the boundary conditions one can impose on the potentials. In the spacelike case $A_3 = 0$ they are needed in order to avoid non-physical constraints on the external currents.

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A class of gauge conditions, the so-called "physical" or "axial" gauges

$$\eta_\mu A^\mu = 0 \quad , \quad (1)$$

is widely used in perturbative QCD calculations¹⁾. In fact, one can show that leading log contributions in those gauges involve only planar diagrams¹⁾.

Therefore Yang-Mills theories, quantized with condition (1), have been studied by several authors²⁾. In particular, the condition

$$A_3(t, \vec{x}) = 0 \quad (2)$$

was firstly considered in order to solve explicitly the Gauss law in the Hamiltonian formulation of the theory³⁾.

It has been noted, however, that the choice of subsidiary condition (2), together with the very natural request of the finiteness of the total energy momentum of the Yang-Mills field, leads to additional constraints whose meaning is rather obscure⁴⁾; to be eventually imposed on the physical states of the quantized theory.

In this letter we discuss the classical equations of motion of the Maxwell and Yang-Mills fields coupled with given external sources in the gauge (1). It is easy to see that, in general, the condition (1) does not fix univocally the potentials; on the other hand, the choice (1) entails the appearance of "spurious" poles in the Green functions. We show that, when η_μ is spacelike, suitable choices of boundary conditions can remove that ambiguity, providing at the same time a prescription for handling the singularities.

We first discuss the Abelian case with the condition (2). The Maxwell equations are

$$\partial^2 A_0 - \partial_0 \partial_\alpha A_\alpha = J_0 \quad , \quad (3.a)$$

$$\square A_\alpha - \partial_\alpha (\partial_0 A_0 - \partial_\beta A_\beta) = -J_\alpha \quad , \quad (3.b)$$

$$\partial_3 (\partial_0 A_0 - \partial_\beta A_\beta) = J_3 \quad , \quad (3.c)$$

$$A_3 \equiv 0, \quad \alpha, \beta = 1, 2. \quad (3.d)$$

We notice that Eq. (3.a) is the well-known Gauss law constraint, which is conserved in time thanks to the continuity equation for the current and both Eqs. (3.b) and (3.c).

The request that the total electromagnetic energy momentum is finite implies that

$$\lim_{|\bar{x}| \rightarrow \infty} F_{\mu\nu}(t, \bar{x}) = 0, \quad (4)$$

$F_{\mu\nu}$ being the electromagnetic field. Surprisingly, a similar condition is not generally allowed for the potentials. Integrating Eq. (3.c) we get

$$\begin{aligned} (\partial_0 A_0 - \partial_\beta A_\beta)(t, x_\alpha, z) &= (\partial_0 A_0 - \partial_\beta A_\beta)(t, x_\alpha, z_0) + \\ &+ \int_{z_0}^z J_3(t, x_\alpha, z) dz. \end{aligned} \quad (5)$$

If the potentials vanish in the limit $|\bar{x}| \rightarrow \infty$ we obtain the constraint

$$\int_{-\infty}^{+\infty} J_3(t, x_\alpha, z) dz = 0 \quad (6)$$

which, in general, need not be true.

As a matter of fact, Eq. (4) entails the weaker condition

$$\lim_{|\bar{x}| \rightarrow \infty} [A_\mu(t, \bar{x}) - \partial_\mu \Gamma(t, \bar{x})] = 0, \quad (7)$$

namely, in particular,

$$\lim_{Z \rightarrow \pm \infty} A_{0,\alpha}(t, \bar{x}) = \partial_{0,\alpha} \Gamma_{\pm}(t, x_{\beta}) . \quad (8)$$

In order to avoid Eq. (6) we must have

$$\Gamma_{+}(t, x_{\beta}) \neq \Gamma_{-}(t, x_{\beta}) . \quad (9)$$

At this point we can arbitrarily choose one of these quantities, the other one being subjected to satisfy the equation

$$\square(\Gamma_{+} - \Gamma_{-}) = \int_{-\infty}^{+\infty} J_3(t, x_{\alpha}, z) dz . \quad (10)$$

A choice of Γ_{+} (Γ_{-}) removes the residual ambiguity expressed by the transformation

$$A_{\mu}(t, \bar{x}) \rightarrow A'_{\mu}(t, \bar{x}) = A_{\mu}(t, \bar{x}) + \partial_{\mu} \gamma(x_{\alpha}) . \quad (11)$$

In fact, if we impose on any such potential the boundary condition

$$\lim_{Z \rightarrow +\infty} A_{\mu}(t, \bar{x}) = \partial_{\mu} \Gamma_{+}(t, x_{\alpha}) , \quad (12)$$

we get

$$\partial_{\mu} \gamma(x_{\alpha}) = 0 . \quad (13)$$

On the other hand, we show that a choice of Γ_{+} (Γ_{-}) gives a prescription to handle the so-called "spurious" singularities in the Green functions. The particular solution of the inhomogeneous Eqs. (3) is, taking the continuity equation into account,

$$A_0(t, \bar{x}) = \int \frac{e^{ikx} d^4k}{k^2} \left(\tilde{J}_0 - \frac{k_0}{k_3} \tilde{J}_3 \right),$$

$$A_\alpha(t, \bar{x}) = \int \frac{e^{ikx} d^4k}{k^2} \left(\tilde{J}_\alpha - \frac{k_\alpha}{k_3} \tilde{J}_3 \right), \quad (14)$$

where $\tilde{J}_\mu$ are the Fourier transforms of J_μ and a prescription is understood concerning the singularities $k^2 = 0$ and $k_3 = 0$ *).

We immediately notice that, without the singularity at $k_3 = 0$, we would obtain

$$\lim_{|\bar{x}| \rightarrow \infty} A_\mu(t, \bar{x}) = 0 \quad (15)$$

(we assume, for simplicity, that the currents are regular at $k = 0$). It is easy to realize that the prescription $k_3 + i\epsilon$ ($-i\epsilon$) leads to a vanishing Γ_- (Γ_+). Finally, the principal value prescription proposed in Ref. 2) leads to the symmetric condition $\Gamma_+ = -\Gamma_-$. It is also simple to recover from Eqs. (14) the consistency relation (10).

We recall that in Eqs. (14) one must add a solution of the homogeneous equation in order to satisfy an arbitrarily assigned boundary condition when z tends to $+\infty$ or $-\infty$; obviously Eq. (10) is not affected.

One can relate the case $A_3 \equiv 0$ to the "timelike" case $\hat{A}_0 \equiv 0$ by means of the gauge transformation

$$\hat{A}_\mu = A_\mu - \partial_\mu \left[\int_{t_0}^t A_0(\tau, \bar{x}) d\tau + \chi(\bar{x}) \right], \quad (16)$$

*) Different prescriptions in handling the singularities at $k^2 = 0$ give rise, as is well known, to different Green functions.

where the arbitrariness in t_0 and χ restores a residual $\bar{x}$ dependent ambiguity in $\hat{A}_\mu$, which cannot be eliminated by imposing boundary conditions. This is also consistent with the particular solution of the Maxwell equations in this gauge

$$\hat{A}_j(t, \bar{x}) = \int \frac{e^{ikx}}{k^2} d^4k \left(\tilde{J}_j - \frac{k_j}{k_0} \tilde{J}_0 \right), \quad (17)$$

if one realizes that the singularity at $k_0 = 0$ does not affect the behaviour when $|\bar{x}| \rightarrow \infty$.

When η_μ is lightlike, for instance $\check{A}_0 = \check{A}_3$, we get

$$\check{A}_0(t, \bar{x}) = \int \frac{e^{ikx}}{k^2} d^4k \left[\tilde{J}_0 - \frac{k_0}{k_0 - k_3} (\tilde{J}_0 - \tilde{J}_3) \right], \quad (18)$$

$$\check{A}_\alpha(t, \bar{x}) = \int \frac{e^{ikx}}{k^2} d^4k \left[\tilde{J}_\alpha - \frac{k_\alpha}{k_0 - k_3} (\tilde{J}_0 - \tilde{J}_3) \right].$$

In this case we can show that, for a given Green function, there is always a choice of contour of integration over k_0 , which correctly defines the potentials and such that no constraint ensues on the behaviour when $|\bar{x}| \rightarrow \infty$ *).

The results we obtained can be extended to the non-Abelian case. We write the inhomogeneous Yang-Mills equations in the form

$$\square A_\mu - \partial_\mu (\partial A) = -j_\mu \quad (19)$$

$$j_\mu = J_\mu - g [A_\nu, F_\mu^\nu] - g \partial_\nu [A_\mu, A^\nu],$$

*) An arbitrary prescription of handling the singularity at $k_0 = k_3$ would lead, in general, to a divergent integral, depending only on x_0, x_- which can acquire a meaning, for instance through dimensional regularization.

where j_μ obeys the continuity equation we used in our previous reasoning. In particular, in the case $A_3 = 0$, we get the constraint

$$\begin{aligned} & \partial_0 (\omega_+^{-1} \partial_0 \omega_+) - \partial_\alpha (\omega_+^{-1} \partial_\alpha \omega_+) - \partial_0 (\omega_-^{-1} \partial_0 \omega_-) + \\ & + \partial_\alpha (\omega_-^{-1} \partial_\alpha \omega_-) = \int_{-\infty}^{+\infty} j_3(t, x_\alpha, z) dz, \end{aligned} \quad (20)$$

where

$$\omega_\pm^{-1} \partial_\mu \omega_\pm(t, x_\alpha) = \lim_{z \rightarrow \pm\infty} A_\mu(t, x_\alpha, z). \quad (21)$$

The vanishing of the quantity on the left-hand side of Eq. (20) (in particular $\omega_+ = \omega_-$) would impose the implicit constraint on the potentials

$$\int_{-\infty}^{+\infty} j_3(t, x_\alpha, z) dz = 0 \quad (22)$$

which, in general, is not conserved in time and cannot be satisfied. The analogues of the solutions (14), (17) and (18) are now implicit equations for the Yang-Mills potentials. We can repeat for those equations the discussion and the conclusions we reached in the Abelian case.

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