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DUALITY ROTATIONS FOR INTERACTING FIELDS

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*Dedicated to Andrei D. Sakharov
on the occasion of his
60th birthday*

A B S T R A C T

We study the properties of interacting field theories which are invariant under duality rotations which transform a vector field strength into its dual. We consider non-abelian duality groups and find that the largest group for n interacting field strengths is the non-compact $Sp(2n, R)$, which has $U(n)$ as its maximal compact subgroup. We show that invariance of the equations of motion requires that the Lagrangian change in a particular way under duality. We use this property to demonstrate the existence of conserved currents, the invariance of the energy momentum tensor, and also in the general construction of the Lagrangian. Finally we comment on the existence of zero mass spin one bound states in $N = 8$ supergravity, which possesses a non-compact E_7 dual invariance.

1. INTRODUCTION

It has long been known that the free Maxwell's equations are invariant under a rotation of the electric field and the magnetic field into each other. In relativistic notation this means that the electromagnetic field strength $F_{\mu\nu}$ and its dual

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

rotate into each other. For this reason the transformation has been called a "duality rotation". It is easy to see^{1),2)} that duality rotation invariance can be extended to the case when the electromagnetic field interacts with the gravitational field, which does not transform under duality. On the other hand, it is obvious that duality invariance is violated by electromagnetic couplings of the minimal type. Related to this is the fact that there is no non-abelian generalization of duality under which the pure Yang-Mills equations could be invariant²⁾.

Non-minimal couplings of the magnetic moment type can however be made duality-invariant, and this invariance can in fact be generalized to a non-abelian group. This is the situation³⁾ in extended supergravity theories without gauging. The assumption that the theory is invariant under duality rotations has been used^{4),5),6)} to simplify the search for the correct supersymmetric Lagrangian. For $N = 4$ supergravity it was discovered⁶⁾ that the $U(4)$ duality could be extended to a larger $SU(4) \otimes SU(1,1)$ non-compact duality invariance. Similar situations occur for $N > 4$ supergravities; in the particular case of $N = 8$ supergravity, Cremmer and Julia⁷⁾ have shown that the theory is invariant under a non-compact E_7 duality. A non-compact duality invariance is possible only when there are scalar fields in the theory, and is related to non-linear transformations of the scalars.

Our purpose in this paper is to clarify the structure of theories admitting both compact and non-compact duality. Our analysis will not make explicit reference to supersymmetry, although of course we have in mind the application to supergravity theories. In fact, we have been strongly influenced by the work of Cremmer and Julia⁷⁾ on $N = 8$ supergravity, and some of our results can be found in their paper although in a less explicit and systematic form.

The first point we wish to emphasize is that the requirement that the equations of motion be duality invariant is not identical to the invariance of the Lagrangian. This is already apparent in the case of the free electromagnetic field, where the Lagrangian is

$$L = \frac{1}{2} (E^2 - H^2)$$

and is obviously not invariant under rotations^{*)} of E into H . Nevertheless, when the equations of motion are duality invariant, the Lagrangian has some special properties; in particular it changes under a duality transformation in a specific way which will be described below.

A second point we make is that the invariance of the equations of motion under duality rotations implies the existence of conserved currents, both for the compact and the non-compact cases. These currents are constructed in terms of the basic fields of the theory and in terms of a set of dual vector potentials, and are not invariant under the abelian gauge transformations up to which these potentials are determined. However the currents change under these gauge transformations by the divergence of an antisymmetric tensor, and therefore the integrated charges are gauge invariant. These charges are in fact the generators of the duality rotations.

In Section 2 we discuss the general properties of duality transformations. We find that the most general group which can be realized with n field strengths is the non-compact real symplectic group $Sp(2n, R)$, which has $U(n)$ as its maximal compact subgroup. In the absence of scalar fields, $U(n)$ is the largest group of duality transformations^{**)} . In specific examples the actual group of duality transformations can be smaller. For example in $N = 8$ supergravity there are 28 field strengths. The non-compact invariance is the $E_{7(+7)}$ subgroup of $Sp(56, R)$, and its maximal compact subgroup is the $SU(8)$ subgroup of $U(28)$. We derive the transformation property of the Lagrangian which is required for the equations of motion to be duality invariant, and show that this property implies the existence of conserved currents and the invariance of the energy momentum tensor. We further exploit this property for the explicit construction of the Lagrangian, which we illustrate first by specializing to the compact case in Section 3.

In order to generalize to the non-compact case it is necessary to consider non-linear realizations of the symmetry group and the corresponding scalar Lagrangians and currents which we discuss in Section 4. In Section 5 we describe in more detail non-compact duality invariant theories. In Section 6 we conclude with some comments relevant to the $N = 8$ supergravity theory.

*) It has been remarked²⁾ that the electromagnetic Lagrangian appears to be instead invariant under hyperbolic rotations. However these turn out to play no rôle in the generalized case of interacting fields.

**) For the free Maxwell equations with n field strengths, the largest duality group is $GL(n, C)$, the non-compact general linear group of complex $n \times n$ matrices, which operate on the complex field strengths $F + i\tilde{F}$. In the presence of interactions without scalars this is reduced to its maximal compact subgroup $U(n)$ which can be enlarged to $Sp(2n, R)$ in the presence of interacting scalars. Observe however that, even for the free Maxwell theory, the energy momentum tensor is only invariant under $U(n)$.

2. GENERAL DUALITY ROTATIONS

We consider a Lagrangian which is a function of n field strengths $F_{\mu\nu}^a$ and of a certain number of additional fields χ^i (scalars and fermions) and their derivatives $\chi_{,\mu}^i = \partial_\mu \chi^i$:

$$L = L(F^a, \chi^i, \chi_{,\mu}^i) \quad (2.1)$$

The field strengths are the curl of vector potentials

$$F_{\mu\nu}^a = \partial_\mu \mathcal{A}_\nu^a - \partial_\nu \mathcal{A}_\mu^a \quad (2.2)$$

and therefore they satisfy

$$\partial^\mu \tilde{F}_{\mu\nu}^a = 0. \quad (2.3)$$

If we define the antisymmetric tensors $G_{\mu\nu}^a$ by^{*)}

$$\tilde{G}_{\mu\nu}^a = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{a\rho\sigma} \equiv 2 \frac{\partial L}{\partial F^{a\mu\nu}} \quad (2.4)$$

the equations of motion obtained by varying $\mathcal{A}_\mu^a$ are

$$\partial^\mu \tilde{G}_{\mu\nu}^a = 0. \quad (2.5)$$

The system of equations (2.3) and (2.5) is invariant under linear transformations among the F 's and G 's. We therefore consider an infinitesimal transformation of the form

$$\delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \quad (2.6a)$$

$$\delta \chi^i = \xi^i(\chi) \quad (2.6b)$$

where A, B, C, D are arbitrary real $n \times n$ matrices and $\xi^i(\chi)$ non derivative functions of the additional fields χ^i .

Since by construction Eqs (2.3) and (2.5) are invariant under (2.6), we shall study the conditions under which the definition (2.4) of G and the equations of motion for the χ^i are invariant. Notice that we do not impose the invariance of the Lagrangian itself: we shall see that the system of the equations of motion can be invariant only if δL does not vanish. Instead the variation of L is required

^{*)} We differ by a factor of 2 from the definition of Cremmer and Julia⁷⁾.

to have a specific form which can be used to demonstrate the existence of a conserved current and leads to an essentially unique construction of the vector field couplings.

2.1 Covariance of the equations of motion

The variation of the Lagrangian under (2.6) is

$$\delta L = \left[\xi^i \frac{\partial}{\partial \chi^i} + \chi_\mu^j \frac{\partial \xi^i}{\partial \chi^j} \frac{\partial}{\partial \chi_\mu^i} + (F^c A^{bc} + G^c B^{bc}) \frac{\partial}{\partial F^b} \right] L. \quad (2.7)$$

We differentiate this equation with respect to F^a

$$\begin{aligned} \frac{\partial}{\partial F^a} \delta L &= A^{ba} \frac{\partial L}{\partial F^b} + \frac{\partial G^c}{\partial F^a} B^{bc} \frac{\partial L}{\partial F^b} + \delta \frac{\partial L}{\partial F^a} \\ &= \frac{1}{2} A^{ba} \tilde{G}^b + \frac{1}{2} \frac{\partial G^c}{\partial F^a} B^{bc} \tilde{G}^b + \frac{1}{2} \delta \tilde{G}^a \end{aligned} \quad (2.8)$$

If we now require (2.6a) or

$$\delta \tilde{G}^a = C^{ab} \tilde{F}^b + D^{ab} \tilde{G}^b, \quad (2.9)$$

we obtain

$$\begin{aligned} 2 \frac{\partial}{\partial F^a} \delta L &= C^{ab} \tilde{F}^b + (D^{ab} + A^{ba}) \tilde{G}^b + \frac{\partial G^c}{\partial F^a} B^{bc} \tilde{G}^b \\ &= \frac{1}{2} \frac{\partial}{\partial F^a} (FC\tilde{F} + GB^T\tilde{G}) + 2(D^{ab} + A^{ba}) \frac{\partial L}{\partial F^b} \\ &\quad + \frac{1}{2} \left[(C^{ab} - C^{ba}) \tilde{F}^b + \frac{\partial G^c}{\partial F^a} (B^{bc} - B^{cb}) \tilde{G}^b \right] \end{aligned} \quad (2.10)$$

where we have used the property $F_1 \tilde{F}_2 = \tilde{F}_1 F_2$. Since the left hand side of (2.10) is a derivative with respect to F^a , so must be the right hand side. In general $\tilde{F}$ and $\tilde{G}$ are independent (an exception is the free field case, $\tilde{G} = -F$); therefore we must impose conditions on the infinitesimal transformation matrices

$$C = C^T, \quad B = B^T \quad (2.11)$$

$$D^{ab} + A^{ba} = \epsilon \delta^{ab} \quad (2.12)$$

and obtain

$$\frac{\partial}{\partial F^a} \delta L = \frac{\partial}{\partial F^a} \left(\frac{1}{4} FC\tilde{F} + \frac{1}{4} GB\tilde{G} + \epsilon L \right). \quad (2.13)$$

Next we study the equations of motion for χ^i using the conditions (2.11) and (2.12). We define

$$E_i = \frac{\partial L}{\partial \chi^i} - \partial_\mu \frac{\partial L}{\partial \chi_\mu^i} \quad (2.14)$$

so that the equations of motion are just

$$E_i = 0. \quad (2.15)$$

The covariance of these equations under the transformation (2.6) requires

$$\delta E_i = - \frac{\partial \xi^j}{\partial \chi^i} E_j. \quad (2.16)$$

From (2.7) we obtain after some algebra

$$\left(\frac{\partial}{\partial \chi^i} - \partial_\mu \frac{\partial}{\partial \chi_\mu^i} \right) \delta L = \frac{\partial \xi^j}{\partial \chi^i} E_j + \delta E_i + \frac{\partial G^c}{\partial \chi^i} B^{bc} \frac{\partial L}{\partial F^b} - \partial_\mu \left(\frac{\partial G^c}{\partial \chi_\mu^i} B^{bc} \frac{\partial L}{\partial F^b} \right) \quad (2.17)$$

where we have used the fact that the operations ∂_μ and δ commute, since the effect of δ is to replace a derivative $\partial_\mu \chi^i$ by $\partial_\mu (\delta \chi^i)$. Imposing (2.16) and using (2.11) and (2.4) we have

$$\left(\frac{\partial}{\partial \chi^i} - \partial_\mu \frac{\partial}{\partial \chi_\mu^i} \right) \left(\delta L - \frac{1}{4} GB\bar{G} \right) = 0. \quad (2.18)$$

We see that this equation is consistent with (2.13) only if

$$\epsilon = 0, \quad D = -A^T \quad (2.19)$$

and obtain the transformation property

$$\delta L = \frac{1}{4} (FC\bar{F} + GB\bar{G}). \quad (2.20)$$

The matrix conditions (2.11) and (2.19) tell us that (2.6a) is an infinitesimal transformation of $Sp(2n, R)$, the real non compact form of $Sp(2n)$ in a real basis for the $2n$ dimensional representation. Given two vectors F, G and H, I which transform in the same way as given in (2.6a) one can construct an invariant

$$(H \ I) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = IF - HG \quad (2.21)$$

a property that will prove useful in the construction of the Lagrangian. In specific examples the group of duality transformations leaving invariant the equations of motion may be a subgroup of $Sp(2n, R)$.

2.2 The conserved current

In order to construct a conserved current we extract the variation of L due to the variation of the χ 's alone:

$$\delta_\chi L = \delta\chi^i \frac{\partial L}{\partial\chi^i} + \delta\chi_\mu^i \frac{\partial L}{\partial\chi_\mu^i} = \delta L - \delta_F L, \quad (2.22)$$

where

$$\delta_F L = \delta F^a \frac{\partial L}{\partial F^a} = \frac{1}{2} (FA^T \tilde{G} + GB\tilde{G}) . \quad (2.23)$$

Together with (2.20) this gives

$$\delta_\chi L = \frac{1}{4} (\tilde{F}CF - \tilde{G}BG) + \frac{1}{2} \tilde{F}DG . \quad (2.24)$$

The equations of motion (2.5) imply that we can introduce a vector potential $\mathcal{B}_\mu$ of which $G_{\mu\nu}$ is the curl:

$$G_{\mu\nu}^a = \partial_\mu \mathcal{B}_\nu^a - \partial_\nu \mathcal{B}_\mu^a . \quad (2.25)$$

Then the current

$$\hat{J}^\mu = \frac{1}{2} (\tilde{G}^{\mu\nu} A \mathcal{A}_\nu - \tilde{F}^{\mu\nu} C \mathcal{A}_\nu + \tilde{G}^{\mu\nu} B \mathcal{B}_\nu - \tilde{F}^{\mu\nu} D \mathcal{B}_\nu) \quad (2.26)$$

has divergence

$$\partial_\mu \hat{J}^\mu = \frac{1}{4} (\tilde{G} A F - \tilde{F} C F + \tilde{G} B G - \tilde{F} D G) = -\delta_\chi L . \quad (2.27)$$

Now, by the usual Noether argument, the equations of motion (2.14), (2.15) for χ^i imply that

$$\partial_\mu \left(\delta\chi^i \frac{\partial L}{\partial\chi_\mu^i} \right) = \delta_\chi L . \quad (2.28)$$

Therefore the current

$$J^\mu = \xi^i \frac{\partial L}{\partial\chi_\mu^i} + \hat{J}^\mu \quad (2.29)$$

is conserved:

$$\partial_\mu J^\mu = 0 . \quad (2.30)$$

Notice that the current (2.29) is not invariant under the gauge transformations

$$\begin{aligned}\mathcal{A}_\mu^a &\rightarrow \mathcal{A}_\mu^a + \partial_\mu \alpha^a \\ \mathcal{B}_\mu^a &\rightarrow \mathcal{B}_\mu^a + \partial_\mu \beta^a\end{aligned}\quad (2.31)$$

which leave invariant (2.2) and (2.25). It changes by the divergence of an anti-symmetric tensor

$$J^\mu \rightarrow J^\mu + \frac{1}{2} \partial_\nu (\tilde{G}^{\mu\nu} A\alpha - \tilde{F}^{\mu\nu} C\alpha + \tilde{G}^{\mu\nu} B\beta - \tilde{F}^{\mu\nu} D\beta). \quad (2.32)$$

Therefore the corresponding charge $\int J^0 d^3x$ is gauge invariant and is actually the generator of the duality rotations, as one can see by constructing the potentials $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$ in a special gauge. A conserved current of this kind does not preclude^{8),9)} the existence of massless spin one states which couple to the charge operator, as we shall discuss in more detail in Section 6.

2.3 Invariance of the energy momentum tensor

It is easy to see that the (in general non symmetric) energy momentum tensor

$$\Theta^\mu_\lambda = -\chi_\lambda^i \frac{\partial L}{\partial \chi_\mu^i} + \delta_\lambda^\mu L + \tilde{G}^{\mu\nu} F_{\nu\lambda}^a \quad (2.33)$$

is conserved as a consequence of the equations of motion (2.3), (2.5) and (2.15):

$$\partial_\mu \Theta^\mu_\lambda = 0. \quad (2.34)$$

We will show that (2.20) implies also that Θ^μ_λ is invariant under duality rotations. We must evaluate

$$\delta \Theta^\mu_\lambda = -\delta \left(\frac{\partial L}{\partial \chi_\mu^i} \right) \chi_\lambda^i - \frac{\partial L}{\partial \chi_\mu^i} \delta \chi_\lambda^i + \delta_\lambda^\mu \delta L + \delta \tilde{G}^{\mu\nu} F_{\nu\lambda} + \tilde{G}^{\mu\nu} \delta F_{\nu\lambda}. \quad (2.35)$$

Now, from (2.7),

$$\delta \frac{\partial L}{\partial \chi_\mu^i} = \frac{\partial}{\partial \chi_\mu^i} \delta L - \frac{\partial \xi^j}{\partial \chi_\mu^i} \frac{\partial L}{\partial \chi_\mu^j} - \frac{1}{2} \frac{\partial G^{\rho\nu}}{\partial \chi_\mu^i} B \tilde{G}_{\rho\nu}. \quad (2.36)$$

Therefore

$$\delta \frac{\partial L}{\partial \chi_\mu^i} \chi_\lambda^i + \frac{\partial L}{\partial \chi_\mu^i} \delta \chi_\lambda^i = \chi_\lambda^i \frac{\partial}{\partial \chi_\mu^i} (\delta L - \frac{1}{4} GB\tilde{G}). \quad (2.37)$$

In order to evaluate the other terms of (2.35) we take δF from (2.6a) and $\delta \tilde{G}$ from (2.6a) or (2.9). Using also (2.19) we obtain

$$\delta \tilde{G}^{\mu\nu} F_{\nu\lambda} + \tilde{G}^{\mu\nu} \delta F_{\nu\lambda} = \tilde{F}^{\mu\nu} C F_{\nu\lambda} + \tilde{G}^{\mu\nu} B G_{\nu\lambda} . \quad (2.38)$$

This expression can be simplified if one remembers that, for any two antisymmetric tensors $F_{1\mu\nu}$ and $F_{2\mu\nu}$,

$$\tilde{F}_1^{\mu\nu} F_{2\nu\lambda} + \tilde{F}_2^{\mu\nu} F_{1\nu\lambda} = -\frac{1}{2} \delta_\lambda^\mu \tilde{F}_1^{\rho\sigma} F_{2\rho\sigma} . \quad (2.39)$$

Since the matrices C and B are symmetric we can use (2.39) in (2.38). Combining the result with (2.37) we obtain finally

$$\delta \Theta_\lambda^\mu = \left(-\chi_\lambda^i \frac{\partial}{\partial \chi_\mu^i} + \delta_\lambda^\mu \right) \left(\delta L - \frac{1}{4} \tilde{F} C F - \frac{1}{4} \tilde{G} B G \right) , \quad (2.40)$$

since the field F does not depend on χ_μ . This equation shows that our basic relation (2.20) implies

$$\delta \Theta_\lambda^\mu = 0 . \quad (2.41)$$

The invariance of the symmetric energy momentum tensor will be shown in Appendix B.

2.4 Construction of the Lagrangian

We start by noting the identity

$$\delta L = \frac{1}{4} \delta(F \tilde{G}) \quad (2.42)$$

which follows from the comparison of the transformation (2.6a) subject to the constraints (2.11) and (2.19), with the property (2.20). As a consequence of (2.42) the Lagrangian can be written in the form

$$L = \frac{1}{4} F \tilde{G} + L_{\text{inv.}} . \quad (2.43)$$

Under the most general group of duality transformations, namely $Sp(2n, R)$, the only^{*)} invariant which can be constructed from vectors in the fundamental $2n$ -dimensional representation is an antisymmetric bilinear. If we introduce two antisymmetric Lorentz-tensors function of the fields χ^i , $(H_{\mu\nu}(\chi), I_{\mu\nu}(\chi))$, which transform under (2.6) like $(F_{\mu\nu}, G_{\mu\nu})$, then the Lagrangian

$$L = \frac{1}{4} F \tilde{G} + \frac{1}{4} (FI - GH) + L_{\text{inv.}}(\chi) , \quad (2.44)$$

where $L_{\text{inv.}}(\chi)$ is an invariant function of the χ , has the property that the equations of motion are invariant. The definition (2.4) of $\tilde{G}$:

^{*)} If the true invariance group is a subgroup of $Sp(2n, R)$ other invariants may exist.

$$\frac{\partial L}{\partial F} = \frac{1}{2} \tilde{G} = \frac{1}{4} \tilde{G} + \frac{1}{4} F \frac{\partial \tilde{G}}{\partial F} + \frac{1}{4} I + \frac{1}{4} \frac{\partial \tilde{G}}{\partial F} \tilde{H} , \quad (2.45)$$

where we have used $F_1 F_2 = -\tilde{F}_1 \tilde{F}_2$, gives a differential equation for $\tilde{G}$:

$$\tilde{G} - I = (F + \tilde{H}) \frac{\partial \tilde{G}}{\partial F} . \quad (2.46)$$

At this point it is convenient to introduce the operator j which changes an anti-symmetric tensor into its dual

$$j T_{\mu\nu} = \tilde{T}_{\mu\nu} \quad (2.47)$$

$$(j)^2 = -1$$

and to rewrite (2.46) in the form

$$jG - I = (F + jH) \frac{\partial jG}{\partial F} . \quad (2.48)$$

The similarity of the operator j with the usual imaginary unit i has been noted and used before^{1),7)}. The general solution of (2.48) is

$$jG - I = -K(\chi)(F + jH) \quad (2.49)$$

where $K(\chi)$ is a priori an arbitrary $2n \times 2n$ symmetric matrix function of the χ and can contain j . To verify that (2.49) is a solution of (2.48) observe that the definition (2.4) implies that

$$\frac{\partial \tilde{G}_{\mu\nu}^a}{\partial F_{\rho\sigma}^b} = \frac{\partial \tilde{G}_{\rho\sigma}^b}{\partial F_{\mu\nu}^a} . \quad (2.50)$$

The transformation law of K under (2.6) is determined by those of (F, G) and (H, I) . Varying both sides of (2.49) one finds

$$\delta K(\chi) = -jC - jKBK + DK - KA \quad (2.51)$$

where the matrices A, B, C, D satisfy (2.11) and (2.19). In addition, it is clear that the form of the kinetic energy term for the vector fields requires

$$K(\chi) = 1 + f(\chi) \quad (2.52)$$

with $f(0) = 0$. As we shall see in Section 5, these properties permit the determination of the matrix K as a function of the fields χ . Substituting (2.49) in (2.44) we find the Lagrangian

$$L = -\frac{1}{4} FKF + \frac{1}{2} F(I - jKH) + \frac{1}{4} jH(I - jKH) + L_{\text{inv.}}(\chi) . \quad (2.53)$$

3. COMPACT DUALITY ROTATIONS

As an example, let us consider the case where $f(X) = 0$ in (2.52), i.e. $K(X) = 1$ in (2.49). Then, since $\delta K = 0$, the transformation law (2.51) restricts further the matrices:

$$\begin{aligned} B &= -C = B^T \\ A &= D = -A^T. \end{aligned} \quad (3.1)$$

The subgroup of $Sp(2n, R)$ determined by (3.1) is just its maximal compact subgroup $U(n)$. This becomes obvious if one writes (2.6a) in the complex basis given by the combinations $F \pm iG$. One finds

$$\delta \begin{pmatrix} F + iG \\ F - iG \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & T^* \end{pmatrix} \begin{pmatrix} F + iG \\ F - iG \end{pmatrix} \quad (3.2)$$

where

$$T = A - iB = -T^\dagger \quad (3.3)$$

is an anti-hermitean $n \times n$ matrix. Whether or not the invariance group is actually $U(n)$ or some (compact) subgroup thereof, one sees that the tensors $F + iG$ and $F - iG$ transform according to an n -dimensional representation (which may be reducible) and to its conjugate.

The complex basis defined in (3.2) allows a simple physical interpretation: spin one states of opposite helicity transform according to conjugate representations of the duality group, just as massless fermions do under chiral transformations. However the equations become more concise and are more easily generalized to the non-compact case if we choose a similar basis with $i \rightarrow j$, where j is the operator defined in Eqs (2.47), i.e.

$$\left. \begin{aligned} F \pm iG &\rightarrow F \pm jG \\ \text{Re}T \pm i\text{Im}T &\rightarrow \text{Re}T \pm j\text{Im}T \end{aligned} \right\} \quad (3.4)$$

in eq.(3.2). Then if we introduce antisymmetric tensors function of X

$$jH_\pm \equiv (H \pm jI) = \pm j(I \mp jH) \quad (3.5)$$

transforming like $F \pm jG$, the Lagrangian (2.53) with $K = 1$ becomes

$$L = -\frac{1}{4} F^2 + \frac{1}{2} F H_+ - \frac{1}{8} H_+^2 - \frac{1}{8} H_+ H_- + L_{\text{inv.}}(X) \quad (3.6)$$

Clearly, without loss of generality, we may set $H_- = 0$: since this field does not couple to F , its couplings must be themselves invariant (and indeed $H_+ H_-$ is an invariant) and can be absorbed in $L_{\text{inv.}}(X)$. Then (H, I) reduces to $(H, -jH)$, and we see that the correct transformation properties for L are obtained in the compact case by introducing a tensor $H = \frac{1}{2} jH_+$ which transforms according to the same j -complex representation of the unitary group as does the field $F + jG$. In practice H is constructed from fermion fields ψ , e.g.:

$$H_{\mu\nu}^a = C_{ij}^a \bar{\psi}^i \sigma_{\mu\nu} \psi^j \quad (3.7)$$

and the transformation (2.6b) is chiral. Finally we remark that by setting $H_- = 0$ we can rewrite the invariant

$$FI - GH = \frac{1}{2} (F^2 + G^2) = \frac{1}{2} (F - iG)(F + iG) \quad (3.8)$$

which is manifestly invariant under linear unitary transformations among F and G . The models of Ref. 3 are examples of the developments of this subsection.

In order to generalize the above construction to the case of non-compact duality transformations without introducing ghosts, we must add scalar fields in the way recalled in the following section.

4. NON-LINEAR REALIZATIONS

In this section we recall the well known description of non-linear models for scalars valued in the quotient space of a group by a subgroup^{10),11)}. Although the formalism is very general, and can be applied to a compact group as well, we have in mind the special case when the semi-simple group $\mathcal{G}$ is non-compact but the subgroup $\mathcal{K}$ is its maximal compact subgroup. As pointed out in Refs 12-14), this case leads to a non-linear scalar Lagrangian without ghosts.

The scalars can be described by a group element $g(x) \in \mathcal{G}$ (in some representation) and are therefore at first as many as the parameters of $\mathcal{G}$. However, we consider as equivalent all group elements which differ by multiplication (on the right) by an element of $\mathcal{K}$. This reduces the scalars to as many as the parameters of the quotient space (coset space). To implement the equivalence we require the Lagrangian to be invariant under a gauge transformation which we choose to write as

$$g(x) \rightarrow g(x) [k(x)]^{-1}. \quad (4.1)$$

We require it also to be invariant under the rigid transformation

$$g(x) \rightarrow g_0 g(x) \quad (4.2)$$

with $g_0 \in \mathcal{G}$.

In order to construct the Lagrangian, we introduce a gauge field Q_μ which belongs to the Lie algebra of $\mathcal{K}$ and which transforms as

$$Q_\mu \rightarrow k Q_\mu k^{-1} - \partial_\mu k k^{-1} \quad (4.3)$$

under the gauge transformation (4.1). With it one can construct a covariant derivative

$$D_\mu g = \partial_\mu g - g Q_\mu \quad (4.4)$$

which transforms as

$$D_\mu g \rightarrow (D_\mu g) [k(x)]^{-1} \quad (4.5)$$

so that

$$g^{-1} D_\mu g \rightarrow k g^{-1} D_\mu g k^{-1} . \quad (4.6)$$

Under (4.2), $g^{-1} D_\mu g$ is invariant. Therefore the Lagrangian

$$L = -\frac{1}{2} \text{Tr}(g^{-1} D_\mu g)^2 \quad (4.7)$$

is invariant under both (4.1) and (4.2). Here Tr can be defined as the trace in some representation, suitably normalized. The field Q_μ enters the Lagrangian without derivatives. If one varies Q_μ keeping g fixed, one obtains the equation of motion

$$\delta L = \text{Tr} \delta Q_\mu (g^{-1} \partial_\mu g - Q_\mu) = 0 \quad (4.8)$$

Since δQ_μ , like Q_μ , is in the Lie algebra of $\mathcal{K}$, this means that $P_\mu \equiv g^{-1} \partial_\mu g - Q_\mu$ is in the orthogonal complement. The element $g^{-1} \partial_\mu g$ of the Lie algebra of $\mathcal{G}$ is decomposed as

$$g^{-1} \partial_\mu g = Q_\mu + P_\mu \quad (4.9)$$

and Q_μ is its part in the Lie algebra of $\mathcal{K}$. This equation can be thought of as giving Q_μ and P_μ in terms of g and its derivatives. One can rewrite (4.9) as

$$P_\mu = g^{-1} D_\mu g \quad (4.10)$$

and give (4.7) the form

$$L = -\frac{1}{2} \text{Tr} P_\mu^2 . \quad (4.11)$$

Substitution into (4.7) of Q_μ expressed in terms of g from (4.9) gives a Lagrangian which depends only on g and its derivatives. Clearly it is given by (4.11) in which P_μ is expressed in terms of g from (4.9).

To obtain the equations of motion for the scalar fields one must vary g in the Lagrangian keeping Q_μ fixed. With a little algebra and up to a total derivative one finds

$$\delta L = \text{Tr } \delta g g^{-1} \partial_\mu (D_\mu g g^{-1}) = 0 . \quad (4.12)$$

Now, $\delta g g^{-1}$ is an arbitrary element of the Lie algebra of G , therefore one has the equations of motion

$$\partial_\mu (D_\mu g g^{-1}) = 0 \quad (4.13)$$

or

$$\partial_\mu (g P_\mu g^{-1}) = 0 . \quad (4.14)$$

These equations can also be written as

$$D_\mu P_\mu \equiv \partial_\mu P_\mu - [P_\mu, Q_\mu] = 0 . \quad (4.15)$$

One could have obtained these equations of motion also by varying g in the Lagrangian (4.11) where Q_μ no longer appears as an independent field.

Our Lagrangian is invariant under (4.1) and (4.2). In particular it is invariant under the rigid transformation

$$g \rightarrow g k^{-1} \quad (4.16)$$

with k independent of x . According to a general argument due to Emmy Noether, the current corresponding to (4.16) vanishes identically as a consequence of the gauge invariance (4.1). The current corresponding to (4.2) does not vanish, however, and the current corresponding to the subtransformation

$$g \rightarrow k g \quad (4.17)$$

is the same as that for

$$g \rightarrow k g k^{-1} . \quad (4.18)$$

According to the Noether prescription, we construct the currents by replacing in the Lagrangian the derivatives of the scalar fields by the variation of the same fields under the corresponding transformations. This amounts to varying the

Lagrangian (4.11)

$$\delta L = - \text{Tr } P_\mu \delta P_\mu \quad (4.19)$$

and replacing δP_μ by δP (and δQ_μ by δQ when it appears in the Lagrangian, as below) defined from

$$g^{-1} \delta g = \delta Q + \delta P \quad (4.20)$$

with the same decomposition of the Lie algebra. So the current is

$$J_\mu = - \text{Tr } P_\mu \delta P \quad (4.21)$$

where δ is the infinitesimal transformation in question. This equation can also be written

$$\begin{aligned} J_\mu &= - \text{Tr } P_\mu (\delta Q + \delta P) \\ &= - \text{Tr } P_\mu g^{-1} \delta g \end{aligned} \quad (4.22)$$

since the term we have added vanishes by orthogonality. For (4.2) let

$$\delta g = (q + p)g \quad (4.23)$$

where q and p are infinitesimals in the Lie algebra of $\mathcal{K}$ and perpendicular to it respectively. We find

$$J_\mu = - \text{Tr } g P_\mu g^{-1} (q + p) . \quad (4.24)$$

So the current of (4.2) is given by the operator

$$J_\mu = - g P_\mu g^{-1} \quad (4.25)$$

which is in the Lie algebra of $\mathfrak{g}$. Its part in the Lie algebra of $\mathcal{K}$ gives the current of (4.17) and also of (4.18). The equations of motion, in the form (4.14), state just the conservation of the current in agreement with Noether's theorem. We can also check directly that the current of (4.16) vanishes identically. Indeed we must now use (4.20) and (4.21) with

$$\delta g = - g q . \quad (4.26)$$

This gives

$$\delta Q = - q \quad \delta P = 0 . \quad (4.27)$$

Coupling to other fields, which we denote by ψ , can be introduced as follows. Let ψ be invariant under (4.2) and let it belong to some unitary representation of $\mathcal{K}$ so that it transforms under (4.1) as

$$\psi(x) \rightarrow k(x)\psi(x) . \quad (4.28)$$

It is easy to construct Lagrangians invariant under (4.1), (4.28) by means of the covariant derivative

$$D_\mu \psi = \partial_\mu \psi + Q_\mu \psi , \quad (4.29)$$

where $k(x)$ and Q_μ are matrices in the appropriate representation. For instance, the kinetic term of the Dirac Lagrangian would be

$$- \frac{i}{2} \bar{\psi} \gamma^\mu (\vec{D}_\mu - \overleftarrow{D}_\mu) \psi . \quad (4.30)$$

One can also have derivative coupling interaction terms such as

$$\bar{\psi}_1 \gamma^\mu P_\mu \psi_2 \quad (4.31)$$

where P_μ is now a matrix in the appropriate representation of the Lie algebra of $\mathcal{K}$.

In presence of other fields, the derivatives of the scalars enter also in terms like (4.30) and (4.31) through Q_μ and P_μ . Therefore the current receives additional contributions. For instance (4.30), which contains

$$- i \bar{\psi} \gamma^\mu Q_\mu \psi \quad (4.32)$$

gives, from (4.20) and (4.23)

$$J_\mu' = - i \bar{\psi} \gamma_\mu \delta Q \psi = - i \bar{\psi} \gamma_\mu q \psi + \dots , \quad (4.33)$$

where the dots represent terms of higher order in the scalar fields. Similarly (4.31) gives

$$J_\mu'' = \bar{\psi}_1 \gamma_\mu p \psi_2 + \dots . \quad (4.34)$$

We have assumed that the fields ψ are invariant under (4.2) because the group $\mathcal{G}$ is non-compact: if we had attributed them to a linear representation of $\mathcal{G}$ we would have obtained a Lagrangian with ghosts.

The gauge field Q_μ in (4.29) can be taken as given by (4.9), a function of the scalar fields and their derivatives. On the other hand, Q_μ could be

introduced as an independent field both in the scalar Lagrangian and in the Lagrangian of the other fields ψ . The field Q_μ can still be determined from its own equation of motion but it has now additional terms which are functions of the fields ψ .

It is often convenient to use the gauge equivalence (4.1) to choose a special gauge. For instance, one can always transform any group element g by (4.1) to the form

$$g^* = e^P = g k^{-1} \quad (4.35)$$

where P is an element of the Lie algebra of $\mathcal{G}$ perpendicular to those in the Lie algebra of $\mathcal{K}$. The scalars are now described by $P(x)$, the coset space $\mathcal{G}/\mathcal{K}$ is parametrized by P and g^* represents the equivalence class. A transformation (4.2) is now represented by

$$(g_0 g)^* = (g_0 e^P)^* = e^{P'} = g_0 e^P k^{-1} \quad (4.36)$$

where $P'(g_0, P)$ and $k(g_0, P)$ are functions of the variables indicated and are determined by the group structure. The transformation

$$P \rightarrow P'(g_0, P) \quad (4.37)$$

is a non-linear realization of (4.2). Correspondingly one must transform the other fields as

$$\psi \rightarrow k(g_0, P) \quad (4.38)$$

in order to maintain the special gauge. Similarly Q_μ transforms as in (4.3), but with the above $k(g_0, P)$. Clearly the Lagrangian is invariant under the non-linear transformations (4.37) and (4.38), while the gauge invariance (4.1) is non longer apparent; it has been used to establish and maintain the special gauge.

5. NON-COMPACT DUALITY TRANSFORMATIONS

In section 2 we derived general formulae in terms of the vector field strength F and the remaining, unspecified fields X . Here we wish to study in more detail the case of non-compact groups, requiring the introducing of scalar fields which are valued in the coset space $\mathcal{G}/\mathcal{K}$ as described in Section 4. We saw that the most general non-compact group of dual rotations for n vectors is $\mathcal{G} = \text{Sp}(2n, \mathbb{R})$ with $\mathcal{K} = \text{U}(n)$ as its maximal compact subgroup. In this case the scalar fields can be represented by an $\text{Sp}(2n, \mathbb{R})$ matrix which is most easily expressed in the

complex basis used in equation (3.2) (see Appendix A) as:

$$g = \begin{pmatrix} \phi_0 & \phi_1^* \\ \phi_1 & \phi_0^* \end{pmatrix} \quad (5.1)$$

where ϕ_0 and ϕ_1 are complex $n \times n$ matrices which satisfy the constraint

$$\phi_0^\dagger \phi_0 - \phi_1^\dagger \phi_1 = 1. \quad (5.2)$$

The transformation law of the scalar fields under $Sp(2n, R)$ is

$$\delta g = \begin{pmatrix} T & V^* \\ V & T^* \end{pmatrix} g \quad (5.3)$$

where

$$\begin{aligned} T &= -T^\dagger = M - iN \\ V &= V^T = R - iS \end{aligned}$$

and the real matrices M, N, R, S are related to those of equation (2.6a) by

$$A = M + R, \quad B = S + N, \quad C = S - N, \quad D = M - R. \quad (5.4)$$

Then equation (2.51) for the transformation law of the matrix K can be written in the form:

$$\delta K = [M, K] - \{R, K\} - jK(S + N)K - j(S - N). \quad (5.5)$$

It is easy to verify that this equation is solved by:

$$K = (\phi_0^\dagger + \phi_1^\dagger)^{-1} (\phi_0^\dagger - \phi_1^\dagger) \quad (5.6)$$

where complex numbers are to be interpreted as having i replaced by the operator j defined in equation (2.47), so that, for instance

$$\phi_i = \text{Re } \phi_i + j \text{Im } \phi_i. \quad (5.7)$$

This trick^{1),7)} allows one to write the equations for K in a very compact form.

Next we introduce couplings to fermions fields ψ in a way which is a straightforward generalization of the compact case. As discussed in Section 4, ψ belongs to some representation of the compact gauge group $\mathcal{K}$. We form an antisymmetric Lorentz tensor $H_{\mu\nu}^0(\psi)$ which transforms under the gauge group $\mathcal{K}$ in the same way as $F + iG$ (or, in terms of the transformation matrices defined with $i \rightarrow j, F + jG$)

transforms under the subgroup of the rigid non-compact symmetry which is isomorphic to $\mathcal{K}$. Then the tensors H and I entering equation (2.44) may, without loss of generality, be constructed as:

$$\begin{pmatrix} H + jI \\ H - jI \end{pmatrix} = g \begin{pmatrix} j H^0(\psi) \\ 0 \end{pmatrix} \quad (5.8)$$

H and I are invariant under the gauge transformation $\mathcal{K}$.

We could of course replace the zero in (5.8) by a tensor H' which transforms according to the representation of $\mathcal{K}$ conjugate to that of H^0 . However, as in the compact case described in Section 3, H' does not couple to F so that its couplings to the remaining fields must be themselves invariant and can be absorbed in $L_{\text{inv.}}(x)$. We have verified explicitly this invariance. Equation (5.8) gives two equations for H and I in terms of the scalar and fermion fields:

$$\left. \begin{aligned} I - jKH &= (\phi_0^\dagger + \phi_1^\dagger)^{-1} H^0(\psi) \\ I - jK^*H &= 0 \end{aligned} \right\} \quad (5.9)$$

Once one has specified $H^0(\psi)$ and $L_{\text{inv.}}(\psi, \phi)$, the Lagrangian is uniquely constructed as given by equations (2.53), (5.6) and (5.9). The compact case of Section 3 is recovered by setting $\phi_0 = 1$ and $\phi_1 = 0$.

We would like to observe that the solution (2.49) of the differential equation (2.48) must be invariant under $\text{Sp}(2n, \mathbb{R})$ transformations. Since H and I transform, respectively, in the same way as F and G , an invariant linear relation among them can be obtained by writing

$$g^{-1} \left[\begin{pmatrix} F + jG \\ F - jG \end{pmatrix} + \lambda j \begin{pmatrix} H + jI \\ H - jI \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \dots \end{pmatrix} \quad (5.10)$$

where

$$g^{-1} = \begin{pmatrix} \phi_0^\dagger & -\phi_1^\dagger \\ -\phi_1^T & \phi_0^T \end{pmatrix} \quad (5.11)$$

is the inverse of (5.1). The coefficient λ can be determined by considering the limit $K \rightarrow 1$, $\phi_0 \rightarrow 1$, $\phi_1 \rightarrow 0$, which gives $\lambda = 1$. Then one finds that in the general case, the relation given by the upper component of (5.10) is identical with (2.49) when K is given by equation (5.6). In fact this is how the expression for K was found. K is invariant under gauge transformations of the local $U(n)$ which operates on the matrix g from the right. It can also be expressed in the form

$$K = \frac{1 - Z^*}{1 + Z^*} \quad (5.12)$$

where

$$Z = \phi_1 (\phi_0)^{-1} = Z^T. \quad (5.13)$$

In the special gauge in which

$$g = \begin{pmatrix} \phi_0 & \phi_1^* \\ \phi_1 & \phi_0^* \end{pmatrix} = \exp \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}, \quad (5.14)$$

the symmetry of the non-compact generator matrix P implies that

$$\left. \begin{aligned} \phi_0 &= \phi_0^\dagger \\ \phi_1 &= \phi_1^T \end{aligned} \right\} \quad (5.15)$$

This construction can be applied to the $N = 8$ supergravity Lagrangian of Cremmer and Julia, except that the specific expressions they obtained for $L_{\text{inv.}}(\chi)$ and for $H_{\mu\nu}^0(\psi)$ were determined by the additional constraints of supersymmetry. (The inclusion of the gravitational field presents no difficulty, since it is invariant under duality transformations.) The prescription given in Sections 2.2 and 4 for obtaining the conserved currents can be directly applied for the subgroups $SU(8)$ and even E_7 using their Lagrangian.

The conserved currents associated with the non-compact generators can be constructed only when the interaction with the scalar fields is present. To understand what happens in the limit in which the scalar coupling constant tends to zero, let us recall that this coupling constant κ has the dimension of a reciprocal mass (in supergravity κ is the gravitational constant itself). In the special gauge of Section 4, the coupling constant κ can be introduced explicitly in the Lagrangians by rescaling $P \rightarrow \kappa P$ after which the Lagrangian (4.11) must be divided by κ^2 . The currents associated with the non-compact generators become

$$J_\mu = -\frac{1}{\kappa} \partial_\mu P + \dots \quad (5.16)$$

where the dots denote terms containing the scalar fields which vanish with κ and terms containing the vector field strengths and other fields which have a finite limit as $\kappa \rightarrow 0$. To go to the limit we multiply (5.16) by κ and obtain

$$\lim_{\kappa \rightarrow 0} \kappa J_\mu = -\partial_\mu P. \quad (5.17)$$

All terms involving fields other than the scalars drop out, the scalars become free massless fields and the non-compact part of the group becomes abelian, corresponding to a contraction of the original non-compact group.

6. CONCLUSION

We have shown how a general theory invariant under dual rotations may be constructed. These theories still have a considerable degree of arbitrariness, namely in the choice of the tensor $H_{\mu\nu}^0(\psi)$ and of $L_{\text{inv.}}(X)$. [see sections 4 and 5]. In supergravity theories these quantities, and in fact the field content itself, are fixed by supersymmetry. It appears that the duality invariance of supergravity theories is implied by supersymmetry, a fact which still remains very mysterious.

There has recently been an attempt^{15),16)} to connect $N = 8$ supergravity with gauge theory phenomenology by assuming that the fields of grand unified theories are composites of the fundamental fields of supergravity. This means in particular that the gauge bosons should be zero mass bound states. There are arguments which show that, in a theory where there is a vector current, no zero mass spin-one state can exist which carries the associated charge⁸⁾. A related argument⁹⁾ shows that a vector current operator applied to the vacuum state cannot create a massless spin-one state. In $N = 8$ supergravity, these arguments would forbid the existence of massless states associated with the composite $SU(8)$ gauge fields Q_μ , since, as we have seen, a current associated with the $SU(8)$ generators can indeed be constructed, and is even conserved. However, as we have pointed out, this current is not gauge invariant, and therefore is not a true Lorentz vector. To construct a current operator one has to choose a particular gauge, in which case the Lorentz transformation properties become rather complicated, and the above-mentioned arguments can no longer be applied. Supergravity escapes this difficulty in much the same way as do Yang-Mills theories where one can define a conserved vector current carrying the same quantum numbers as the Yang-Mills field, but which is not gauge invariant.

Finally we wish to clear up some confusion concerning the composite vector fields in the $N = 8$ supergravity theory. In the manifestly gauge invariant formulation, the conserved current J_μ was constructed to be invariant under $SU(8)$ gauge transformations, and to transform linearly under non-compact E_7 . We emphasize that J_μ contains the vector potentials $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$, and is therefore not invariant under their abelian gauge transformation; without these terms it would not integrate to the correct generator of E_7 . On the other hand the composite gauge potential Q_μ was constructed to be invariant under E_7 , and to transform like a gauge field under local $SU(8)$; it does not contain $\mathcal{A}_\mu$ and $\mathcal{B}_\mu$. A source of confusion could be the fact that in the special gauge of sections 4 and 5 these two operators appear similar at the bilinear level in the scalar fields and even in the spinor fields if these are included in Q_μ . However even

in the special gauge the higher non-linear terms are different. In this gauge the non-compact part of E_7 is realized non-linearly, while its $SU(8)$ subgroup is realized linearly. The $SU(8)$ part of J_μ gives a conserved $SU(8)$ current in terms of the physical fields of the special gauge.

An unanswered question is the precise definition of the vector operator for which the spectral function may develop a zero mass pole in the spin-one channel. Cremmer and Julia⁷⁾ took this to be Q_μ as defined from the scalar fields alone. As discussed in Section 4, other definitions, including fermion fields, are possible. Presumably the correct combination is one which belongs to an irreducible supermultiplet. The issue we believe to have clarified in this paper is simply that there is no inconsistency in the existence of zero mass bound states which transform like gauge bosons under local $SU(8)$ and therefore couple to the conserved current associated with the rigid $SU(8)$ transformations defined in equation (4.18).

APPENDIX A

We recall briefly some basic properties of the non-compact real symplectic group $Sp(2n, \mathbb{R})$. It can be defined as the group of real $2n \times 2n$ matrices S which satisfy

$$S^T \Omega S = \Omega \quad (\text{A.1})$$

where Ω is a $2n \times 2n$ non-singular antisymmetric matrix which can be taken to be

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.2})$$

Writing

$$S = 1 + X, \quad (\text{A.3})$$

where X is an infinitesimal matrix, (A.1) gives

$$X^T \Omega = -\Omega X. \quad (\text{A.4})$$

If we write

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (\text{A.5})$$

where A, B, C and D are $n \times n$ real matrices, (A.4) is equivalent to

$$B^T = B, \quad C^T = C, \quad A^T = -D. \quad (\text{A.6})$$

It is often convenient to rewrite the infinitesimal transformation of $Sp(2n, \mathbb{R})$

$$\delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \quad (\text{A.7})$$

in the complex basis $F \pm iG$. One obtains

$$\delta \begin{pmatrix} F + iG \\ F - iG \end{pmatrix} = \begin{pmatrix} T & V^* \\ V & T^* \end{pmatrix} \begin{pmatrix} F + iG \\ F - iG \end{pmatrix} \quad (\text{A.8})$$

where

$$\left. \begin{aligned} 2T &= A + D + i(C - B) \\ 2V &= A - D - i(C + B) \end{aligned} \right\} \quad (\text{A.9})$$

so that T is antihermitean and V is symmetric

$$T^\dagger = -T, \quad V^T = V. \quad (\text{A.10})$$

In the complex basis a finite transformation of $\text{Sp}(2n, \mathbb{R})$ takes the form

$$S = \begin{pmatrix} \phi_0 & \phi_1^* \\ \phi_1 & \phi_0^* \end{pmatrix} \quad (\text{A.11})$$

where the $n \times n$ complex matrices ϕ_0 and ϕ_1 satisfy the constraint

$$\phi_0^\dagger \phi_0 - \phi_1^\dagger \phi_1 = 1. \quad (\text{A.12})$$

The inverse matrix is

$$S^{-1} = \begin{pmatrix} \phi_0^\dagger & -\phi_1^\dagger \\ -\phi_1^T & \phi_0^T \end{pmatrix}. \quad (\text{A.13})$$

The maximal compact subgroup $U(n)$ of $\text{Sp}(2n, \mathbb{R})$ is easily recognized in (A.8) for $V = 0$ and in (A.11), (A.12) for $\phi_1 = 0$.

APPENDIX B

We have emphasized in the text that the Lagrangian is not invariant under duality rotations; rather it transforms as given in (2.20). Here we will show that a suitably defined derivative of the Lagrangian with respect to an invariant parameter is invariant. The invariant parameter could be a coupling constant or it could represent an invariant external field like the gravitational field in the matter Lagrangian. An important consequence of this result is that the symmetric energy momentum tensor is invariant under duality rotations, since it is obtained by taking the functional derivative of the matter action with respect to the gravitational field.

Let us assume that the Lagrangian depends upon an invariant parameter λ . If ξ^i is independent of λ , differentiating (2.7) with respect to λ we obtain

$$\frac{\partial}{\partial \lambda} \delta L = \delta \frac{\partial L}{\partial \lambda} + \frac{\partial G}{\partial \lambda} B^T \frac{\partial L}{\partial F}. \quad (\text{B.1})$$

Using (2.4) and (2.11), this equation can be written

$$\delta \frac{\partial L}{\partial \lambda} = \frac{\partial}{\partial \lambda} \left(\delta L - \frac{1}{4} G \tilde{G} - \frac{1}{4} F \tilde{F} \right). \quad (\text{B.2})$$

We are free to add the last term in the right hand side since the field F is independent of λ . We see that, if L satisfies our basic equation (2.20), then

$$\delta \frac{\partial L}{\partial \lambda} = 0 \quad (\text{B.3})$$

This observation provides a method for constructing invariant Lagrangians, by switching on couplings in an invariant way, or for checking if a Lagrangian is invariant.

In deriving (B.1) we assumed that the ξ^i are independent of the parameter λ . When the transformation is non-linear, if it is expressed in terms of scalar fields ϕ with canonical dimensions, the ξ^i will in general have an explicit dependence on some dimensional coupling constant κ . If we wish to apply the above argument to the parameter κ , we must first reexpress the Lagrangian and the transformation laws in terms of the dimensionless fields $\kappa\phi$. The partial derivative $\frac{\partial L}{\partial \kappa}$, taken keeping constant the fields $\kappa\phi$ instead of ϕ , is invariant.

We further wish to point out that the formalism developed in this paper applies directly to the case in which the fields F and χ of Section 2 interact with an external gravitational field, described by a tensor $g_{\mu\nu}$ or by a vierbein

e_μ^a . Now we must distinguish between tensors (denoted by ordinary letters) and densities^{*)} (denoted by script letters). The field $F_{\mu\nu}$ is an antisymmetric tensor with lower indices as in (2.2). The Lagrangian is a density $\mathcal{L}$ and (2.4) must be written more precisely as

$$\tilde{\mathcal{G}}^{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}. \quad (\text{B.4})$$

If we use the tensor

$$G_{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \tilde{\mathcal{G}}^{\rho\sigma} \quad (\text{B.5})$$

and the density

$$\tilde{\mathcal{F}}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad (\text{B.6})$$

we can write (2.20) as

$$\delta \mathcal{L} = \frac{1}{4} (F_{\mu\nu} C \tilde{\mathcal{F}}^{\mu\nu} + G_{\mu\nu} B \tilde{\mathcal{G}}^{\mu\nu}). \quad (\text{B.7})$$

Equation (2.6a) remains correct as written, using tensors with lower indices. Remember that the numerical antisymmetric symbol $\epsilon^{\mu\nu\rho\sigma}$ transforms a tensor into a density, while $\epsilon_{\mu\nu\rho\sigma}$ transforms a density into a tensor.

As already emphasized, the gravitational field does not transform under the duality rotations discussed in this paper. We have not considered here those duality rotations which act on the gravitational field, because they are an invariance only of the linearized gravitational equations.

^{*)} We only need densities of the same weight as the Lagrangian.

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