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ON POSITIVE FUNCTION SERIES

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ABSTRACT

Function series of the form $f(x) = \sum_{n=0}^N c_n f_n(x)$ are considered under the constraint $f(x) \geq 0$ in a given interval $a \leq x \leq b$. The cone in the space R^{N+1} of the coefficients c_n which is determined by the positivity constraint is approximated numerically by a polyhedral cone. A numerical estimate for the error involved is given and it is shown how it may be reduced. A special series of Jacobi polynomials is discussed and new estimates for the range of parameters for which this series is non-negative are obtained.

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1. STATEMENT OF THE PROBLEM

We consider a series

$$f(x) = \sum_{n=0}^N c_n f_n(x) \quad (1.1)$$

in the interval $a \leq x \leq b$, where the $f_n(x)$ are a set of continuous, continuously differentiable and linearly independent functions, for example orthogonal polynomials. The c_n are real coefficients and $\vec{c} = (c_0, \dots, c_N) \in R^{N+1}$. In several fields of application (for example in approximation theory and in the construction of unitary scattering amplitudes in high-energy physics [9]), there arises the problem: under what conditions on $\vec{c}$ is the function $f(x)$ non-negative, i.e. $f(x) \geq 0$, in the whole interval $[a, b]$?

It is easy to see that the points $\vec{c}$ represent a convex homogeneous cone in R^{N+1} . If $\vec{c}$ makes $f(x) \geq 0$ for all $x \in [a, b]$, then so also does $\lambda \vec{c}$ with $\lambda > 0$. Similarly, if two vectors $\vec{c}'$ and $\vec{c}''$ make $f(x) \geq 0$, then so also does $\lambda \vec{c}' + (1 - \lambda) \vec{c}''$ with $0 \leq \lambda \leq 1$. We call this cone the positivity cone C_{ab} of (1.1), and try to find an approximate description for it by discretizing the problem in the following way. We consider the series (1.1) at a certain number of points $x_m \in [a, b]$, $m = 1, \dots, M$, and thus obtain a system of M linear homogeneous inequalities

$$\sum_{n=0}^N c_n f_n(x_m) \geq 0 \quad (1.2)$$

in the $N + 1$ unknown variables c_n . The theory of these systems is well known [10]; we state only the basic result. The general solution of a system (1.2) is a convex polyhedral cone in R^{N+1} . Denoting its basis vectors by $\vec{v}_i$, $i = 1, \dots, I$, any coefficient vector $\vec{c}$ satisfying (1.2) may be written in the form

$$\vec{c} = \sum_{i=1}^I \lambda_i \vec{v}_i, \quad (1.3)$$

with scalar coefficients $\lambda_i \geq 0$. Solving the system (1.2) therefore means determining the basis vectors $\vec{v}_i$. Their number I is generally not known in advance. The polyhedral cone defined by (1.2) contains the exact non-polyhedral cone C_{ab} .

If M is chosen large as compared to N , one may expect the polyhedral cone to be a good approximation to C_{ab} . A method for solving systems (1.2) has been described in [8].

Geometrically, the exact cone arises in the following way. The inequality

$$\sum_{n=0}^N c_n f_n(x) \geq 0 \quad (1.4)$$

defines for each $x \in [a, b]$ a half-space, and the equation in (1.4) a hyperplane. In general, these hyperplanes determine an envelope, which can be calculated by eliminating x from the equation in (1.4) and its derivative with respect to x . The envelope specified by $a < x < b$, together with the "boundary" hyperplanes specified by $x = a$ and $x = b$, namely

$$\sum_{n=0}^N c_n f_n(a) = 0, \quad \sum_{n=0}^N c_n f_n(b) = 0, \quad (1.5)$$

determines the cone C_{ab} . The elimination of x can be performed only for special sequences $\{f_n(x)\}$, and for small N . For $N \leq 6$, it has been possible to solve the problem approximately, and to obtain numerical error estimates depending on M .

The cone C_{ab} may be empty. However, if the sequence $\{f_n(x)\}$ contains a constant, this is not so. Assume, for example, that $f_0(x) = 1$. Then $(c_0, 0, \dots, 0)$ clearly satisfies (1.4) for all $c_0 > 0$.

2. THE POSITIVITY CONE OF A GIVEN SERIES

It turns out that several questions connected with the positivity cone may be investigated in the particular case $N = 2$, i.e. if the series (1.1) is restricted to only three terms. The advantage of this simple case is that the corresponding positivity cone may be visualized in three-dimensional space. We begin with the example of a non-negative Legendre series with three terms

$$c_0 + c_1 P_1(x) + c_2 P_2(x) \geq 0 \quad (2.1)$$

in the interval $-1 \leq x \leq 1$, where $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$. We assume that $c_0 > 0$, and introduce homogeneous coordinates $\xi = c_1/c_0$ and $\eta = c_2/c_0$. Formula (2.1) then becomes

$$1 + \xi P_1(x) + \eta P_2(x) \geq 0, \quad (2.2)$$

giving as envelope the ellipse

$$\frac{\xi^2}{3} + (\eta - 1)^2 = 1. \quad (2.3)$$

The interval end points $x = \pm 1$ give the boundary lines

$$\omega_{\pm 1}: \eta = \mp \xi - 1. \quad (2.4)$$

These touch the ellipse at the points $T_{\pm 1} = (\pm 3/2, 1/2)$. The region bounded by the longer arc of the ellipse connecting the points T_{-1} and T_1 and the lines $\omega_{\pm 1}$ is the intersection $C(\xi, \eta)$ of the exact positivity cone of (2.1) with the plane $c_0 = 1$. We have plotted this region in Fig. 1. For a fixed value of x the inequality (2.1) determines a half-space in R^3 . The plane separating the two half-spaces touches the positivity cone along a ray starting at the origin. The intersection of such a plane with the plane $c_0 = 1$ is a tangent to the ellipse. Therefore a unique value of x may be assigned to any tangent. For $x = -1$ the tangent starts at ω_{-1} at the right-hand side, moves round the ellipse as x increases, and terminates at ω_{+1} on the left-hand side when $x = 1$. From this consideration it is obvious how the cone changes if the positivity interval is changed. For example, if $a = -1$, $b = 2$, the new cone is obtained by cutting off from the original cone that part which lies below the line $\omega_{+2}: \eta = -2(2\xi + 1)/11$ in Fig. 1. On the other hand, for intervals $[a, b]$ smaller than $[-1, +1]$ we may obtain intersections $C(\xi, \eta)$ which are open. Finally, the cone corresponding to the whole real axis is the elliptic cone whose intersection with the plane $c_0 = 1$ is the complete ellipse (2.3).

We have obtained several polyhedral cones for the series (2.1) by solving the system of inequalities (1.2) with $f_n(x) = P_n(x)$. The intersections of these polyhedral cones for $M = 5, 13$, and 46 are also plotted in Fig. 1. The increasing accuracy of approximation is clearly seen. The cone for $M = 46$ already coincides with the exact cone within the accuracy of the plot. The small edges bounded by two neighbouring faces and the exact cone correspond to sets of coefficients for which $f(x)$ may be negative for some values of x . The size of these regions decreases if M increases. A qualitative measure for the accuracy of approximation will be given later in this section.

We have also computed approximate positivity cones for the series

$$f(x) = \sum_{n=0}^N c_n \sin(n+1)x \geq 0 \quad (0 \leq x \leq \pi) \quad (2.5)$$

$$f(x) = \sum_{n=0}^N c_n \cos nx \geq 0 \quad (0 \leq x \leq \pi) \quad (2.6)$$

$$f(x) = \sum_{n=0}^N c_n P_n(x) \geq 0 \quad (-1 \leq x \leq 1) \quad (2.7)$$

with $c_0 = 1$. In each case we have taken $M = 46$, thus dividing the positivity interval into 45 subintervals of equal length, and have solved the inequalities by means of the program published in [8]. The number of terms in the expansions varies from $N + 1 = 3$ to $N + 1 = 6$. A common feature of all these problems is the rapid increase of the number I of basis vectors $\vec{v}_i$ of the resulting cones if M and N are increased. The choice $M = 46$ seemed to be a reasonable compromise between accuracy and computational feasibility. It turned out that the number of basis vectors is the same function of M and N in all three cases under consideration. For $M = 46$ we get

Dimension $N + 1$	3	4	5	6
Basis vectors I	46	88	989	1806

For $N = 3$, the basis vectors always occur in pairs, the components of which may differ in sign. For higher N , certain symmetries in the $\vec{v}_i$ also occur, but we were not able to sort these vectors into a plausible order.

A problem related to (2.6) has been considered by Geiger and Opfer [6]. They show that the series (2.6), if the summation starts with $n = 1$, cannot be positive in an interval larger than $0 \leq x \leq N\pi/(N + 1)$. This provides an example of a problem with an empty positivity cone, as mentioned at the end of Section 1.

As with any approximation method, it is important to know how to estimate the accuracy and how to improve it. If we set

$$\vec{f}(x) = [f_1(x), \dots, f_n(x)]^T$$

and use (1.3), we may write (1.1) as

$$f(x) = \vec{c}^T \vec{f}(x) = \sum_{i=1}^I \vec{v}_i^T \vec{f}(x) \lambda_i = \sum_{i=1}^I u_i(x) \lambda_i. \quad (2.8)$$

As a measure of the accuracy of the approximation of the cone we may use the quantity

$$\epsilon = \left| \min_{x \in [a,b]} \left(\min_{\substack{0 \leq \lambda_j \leq 1 \\ j=1,2,\dots,I}} \sum_{i=1}^I u_i(x) \lambda_i \right) \right|. \quad (2.9)$$

For each fixed $x \in [a,b]$ the inner minimum problem is then a linear programming problem which may be solved by the simplex algorithm. In general, ϵ can be determined, approximately, by discretizing the outer minimum problem. We have calculated the error measure ϵ for the three series (2.5) to (2.7). The results are listed in Table 1. For all cases, ϵ gets larger as the number of terms in the series is increased. This is to be expected, because the absolute value of the derivative of the terms of the series increases with n .

To improve the polyhedral cone for a series with a fixed number N of terms the number of inequalities has to be increased. An effective strategy is to proceed as follows. First choose a reasonably large number of inequalities in the interval under consideration and determine the corresponding cone. Then compute the error measure (2.9) by choosing sufficient numbers of points $x \in [a,b]$. For that value x_ϵ which satisfies (2.9) add a new inequality to the system (1.2), and compute the corresponding cone. Then repeat the procedure. The effect of the new inequalities may be visualized as a planing away of the initial cone more and more until it approaches the exact positivity cone with the desired accuracy.

We applied this procedure to the four-term Laguerre series [1, p. 773]

$$1 + c_1 L_1(x) + c_2 L_2(x) + c_3 L_3(x) \geq 0 \quad (0 \leq x \leq \infty). \quad (2.10)$$

using 45 values for x_m unequally spaced between 0 and 100, and the asymptotic condition $-c_3 \geq 0$ as inequality 46. New inequalities 47 to 50 were introduced successively as described above. The results are shown in Table 2. We see that adding only four inequalities to the initial set reduces the minimum by almost an order of magnitude.

3. A SPECIAL SERIES OF JACOBI POLYNOMIALS

The positivity of a series of Jacobi polynomials

$$\sum_{n=0}^N c_n P_n^{(\alpha, \beta)}(x) \geq 0 \quad (-1 \leq x \leq 1, N = 1, 2, \dots) (\alpha > -1, \beta > -1) \quad (3.1)$$

where [1, p. 793]

$$\begin{aligned} P_0^{(\alpha, \beta)}(x) &= 1, \\ P_1^{(\alpha, \beta)}(x) &= \frac{1}{2} [2(\alpha + 1) + (\alpha + \beta + 2)(x - 1)], \\ P_2^{(\alpha, \beta)}(x) &= \frac{1}{8} [4(\alpha + 1)_2 + 4(\alpha + \beta + 3)(\alpha + 2)(x - 1) + (\alpha + \beta + 3)_2(x - 1)^2], \\ P_3^{(\alpha, \beta)}(x) &= \frac{1}{48} [8(\alpha + 1)_3 + 12(\alpha + \beta + 4)(\alpha + 2)_2(x - 1) \\ &\quad + 6(\alpha + \beta + 4)_2(\alpha + 3)(x - 1)^2 + (\alpha + \beta + 4)_3(x - 1)^3], \dots, \\ &\dots, \\ P_n^{(\alpha, \beta)}(1) &= \frac{(\alpha + 1)_n}{n!}, \quad P_n^{(\alpha, \beta)}(-1) = (-1)^n \frac{(\beta + 1)_n}{n!}, \end{aligned} \quad (3.2)$$

and $(z)_k = z(z + 1) \dots (z + k - 1)$ is an important problem from both a theoretical and practical point of view, because it covers several expansions occurring in mathematical physics. The special series

$$s_N(x, \lambda) = \sum_{n=0}^N \frac{(\lambda + 1)_{N-n} (\lambda + 1)_n}{(N - n)! n!} \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\beta, \alpha)}(1)} \geq 0 \quad (-1 \leq x \leq 1; N = 1, 2, \dots; \lambda > -1) \quad (3.3)$$

has received considerable attention in the literature. Numerous results on values $\lambda(\alpha, \beta)$ which make (3.3) true for given α and β have been obtained analytically by several authors. Results published before 1975 have been compiled in lecture notes by Askey [2]. More recently, Gasper [4, 5], and Bustoz and Savage [3] have improved and generalized some of these results. Gasper [4] also specifies regions where (3.3) fails. The constraints on $\lambda(\alpha, \beta)$ obtained by these authors are valid for any N . However, they are in general not necessary, but only sufficient. The boundaries of the corresponding regions in the space of α , β , and λ are quite complicated. For $\beta = 1$, these regions are shown in Fig. 2 (hatched area).

We consider here a slightly modified series, namely

$$S_N(x, \lambda) = \frac{N!}{(\lambda + 1)_N} s_N(x, \lambda) \quad \lambda > -[(N + 3)/2]. \quad (3.4)$$

For $\lambda > -1$, $S_N(x, \lambda) \geq 0$ is equivalent to $s_N(x, \lambda) \geq 0$. For any fixed value of N the series (3.4) is a special case of (3.1). If one knows the positivity cone $C_{-1,1}^{(\alpha, \beta)}$ of (3.1) or an approximating polyhedral cone for it, the allowed range of λ is obtained from the intersection of the surface of $C_{-1,1}^{(\alpha, \beta)}$ with the surface in R^{N+1} which is defined by the components

$$c_n = c_n(\alpha, \beta, \lambda; \mu) = \mu \frac{(\lambda + 1)_{N-n} (\lambda + 1)_n}{(N - n)! (\lambda + 1)_N (\beta + 1)_n} \quad \begin{matrix} (n = 0, 1, \dots, N; \\ \lambda > -[(N + 3)/2], \mu \geq 0) \end{matrix} \quad (3.5)$$

The values of λ corresponding to these intersections obviously cannot be determined analytically for arbitrary N . In order to get some insight into this problem, we have solved the cases $N = 2$ and $N = 3$, which turn out to be rather instructive. We consider first the series (3.1) for $N = 2$, $c_0 > 0$, $-\infty < x < \infty$, introduce homogeneous coordinates ξ, η , and prove

Theorem 1:

Let

$$\eta_1(\xi) = [\alpha + \beta + 4 + (\alpha - \beta)\xi] / [(\alpha + 2)(\beta + 2)]$$

$$\eta_2(\xi) = (\alpha + \beta + 4) \{ \alpha + \beta + 4 + 2(\alpha - \beta)\xi - [(\alpha + 1)(\beta + 1)(\alpha + \beta + 4) / (\alpha + \beta + 3)] \xi^2 \} / [(\alpha + 2)(\beta + 2)]^2$$

$$\xi_1(\eta) = [(\alpha + 1)(\alpha + 2) - (\beta + 1)(\beta + 2)] \eta / (\alpha + \beta + 2)^2$$

$$\xi_2(\eta) = \frac{(\alpha + \beta + 3)(\alpha + \beta + 4) [2(\alpha + \beta + 2)^2 \eta - (\alpha + 1)(\beta + 1)(\alpha + \beta + 4) \eta^2]}{(\alpha + \beta + 2)^4}$$

$$\xi_{m,M} = \frac{(\alpha + 1)(\alpha + 2) - (\beta + 1)(\beta + 2) \mp (\alpha + \beta + 2) \sqrt{(\alpha + 2)(\beta + 2)(\alpha + \beta + 3)}}{(\alpha + 1)(\beta + 1)(\alpha + \beta + 4)}$$

$$\eta_M = 2(\alpha + \beta + 2)^2 / [(\alpha + 1)(\beta + 1)(\alpha + \beta + 4)] .$$

For given values of α, β ($\alpha > -1, \beta > -1$), and $-\infty < x < \infty$, the inequality

$$1 + \xi P_1^{(\alpha, \beta)}(x) + \eta P_2^{(\alpha, \beta)}(x) \geq 0 \quad (3.6)$$

holds

a) for given ξ , if and only if $\xi_m \leq \xi \leq \xi_M$ and

$$\eta_1(\xi) - \sqrt{\eta_2(\xi)} \leq \eta \leq \eta_1(\xi) + \sqrt{\eta_2(\xi)}$$

or, equivalently,

b) for given η , if and only if $0 \leq \eta \leq \eta_M$ and

$$\xi_1(\eta) - \sqrt{\xi_2(\eta)} \leq \xi \leq \xi_1(\eta) + \sqrt{\xi_2(\eta)}. \quad (3.7)$$

Proof:

As in the previous section, we eliminate x from the equation in (3.6) and its derivative with respect to x , and get

$$\begin{aligned} E(\xi, \eta) = & (\alpha + \beta + 2)^2 \xi^2 - 2[(\alpha + 1)(\alpha + 2) - (\beta + 1)(\beta + 2)]\xi\eta \\ & + (\alpha + 2)(\beta + 2)(\alpha + \beta + 3)\eta^2 - 2(\alpha + \beta + 3)(\alpha + \beta + 4)\eta = 0. \end{aligned} \quad (3.8)$$

The quadric $E(\xi, \eta) = 0$ defines an ellipse in the $\xi\eta$ -plane, as can be seen by considering the relevant determinants

$$\begin{aligned} \Delta &= -(\alpha + \beta + 2)^2(\alpha + \beta + 3)^2(\alpha + \beta + 4)^2 < 0 \\ \delta &= (\alpha + 1)(\beta + 1)(\alpha + \beta + 3)(\alpha + \beta + 4)^2 > 0. \end{aligned} \quad (3.9)$$

This ellipse is the intersection of the positivity cone of (3.1) for $N = 2$, $-\infty < x < \infty$, with the plane $c_0 = 1$. By solving $E(\xi, \eta) = 0$ for η and ξ respectively, we obtain the limits given in the theorem. From $E(0, 0) = 0$ and $\eta_M > 0$, it follows that the ellipse lies in the upper half of the $\xi\eta$ -plane.

For the case $-1 \leq x \leq 1$, we find

Theorem 2:

Let

$$\begin{aligned} T_{-1} &= (\tilde{\xi}_{-1}, \tilde{\eta}_{-1}) = \left[\frac{2(\alpha + \beta + 3)}{(\beta + 1)(\alpha + \beta + 4)}, \frac{2(\alpha + \beta + 2)}{(\beta + 1)(\beta + 2)(\alpha + \beta + 4)} \right] \\ T_{+1} &= (\tilde{\xi}_{+1}, \tilde{\eta}_{+1}) = \left[\frac{-2(\alpha + \beta + 3)}{(\alpha + 1)(\alpha + \beta + 4)}, \frac{2(\alpha + \beta + 2)}{(\alpha + 1)(\alpha + 2)(\alpha + \beta + 4)} \right], \\ P^* &= (\xi^*, \eta^*) = \left[\frac{(\alpha + 1)(\alpha + 2) - (\beta + 1)(\beta + 2)}{(\alpha + 1)(\beta + 1)(\alpha + \beta + 4)}, \frac{-2(\alpha + \beta + 2)}{(\alpha + 1)(\beta + 1)(\alpha + \beta + 4)} \right]; \\ \bar{\eta}_{-1}(\xi) &= 2\xi/(\beta + 2) - 2/[(\beta + 1)(\beta + 2)] \\ \bar{\eta}_{+1}(\xi) &= -2\xi/(\alpha + 2) - 2/[(\alpha + 1)(\alpha + 2)] \\ \bar{\xi}_{-1}(\eta) &= (\beta + 2)\eta/2 + 1/(\beta + 1) \\ \bar{\xi}_{+1}(\eta) &= -(\alpha + 2)\eta/2 - 1/(\alpha + 1), \end{aligned}$$

and let $\eta_1(\xi)$, $\eta_2(\xi)$, $\xi_1(\eta)$, $\xi_2(\eta)$, ξ_m , ξ_M , and η_M be as defined in Theorem 1.

For given values of α , β ($\alpha > -1$, $\beta > -1$), and $-1 \leq x \leq 1$, the inequality (3.6) holds

a) for given ξ if and only if $\xi_m \leq \xi \leq \xi_M$ and

$$\eta'_m(\xi) \leq \eta \leq \eta_1(\xi) + \sqrt{\eta_2(\xi)}$$

where

$$\eta'_m(\xi) = \begin{cases} \eta_1(\xi) - \sqrt{\eta_2(\xi)} & \text{if } \xi_m \leq \xi \leq \tilde{\xi}_{+1} \\ \bar{\eta}_{+1}(\xi) & \text{if } \tilde{\xi}_{+1} \leq \xi \leq \xi^* \\ \bar{\eta}_{-1}(\xi) & \text{if } \xi^* \leq \xi \leq \tilde{\xi}_{-1} \\ \eta_1(\xi) - \sqrt{\eta_2(\xi)} & \text{if } \tilde{\xi}_{-1} \leq \xi \leq \xi_M, \end{cases}$$

or, equivalently,

b) for given η if and only if $\eta^* \leq \eta \leq \eta_M$ and

$$\xi'_m(\eta) \leq \eta \leq \xi'_M(\eta),$$

where

$$\xi'_m(\eta) = \begin{cases} \tilde{\xi}_{+1}(\eta) & \text{if } \eta^* \leq \eta \leq \tilde{\eta}_{+1} \\ \xi_1(\eta) - \sqrt{\xi_2(\eta)} & \text{if } \tilde{\eta}_{+1} \leq \eta \leq \eta_M \end{cases} \quad (3.10)$$

$$\xi'_M(\eta) = \begin{cases} \tilde{\xi}_{-1}(\eta) & \text{if } \eta^* \leq \eta \leq \tilde{\eta}_{-1} \\ \xi_1(\eta) + \sqrt{\xi_2(\eta)} & \text{if } \tilde{\eta}_{-1} \leq \eta \leq \eta_M. \end{cases} \quad (3.11)$$

Proof:

In this case, the two straight lines $\omega_{\pm 1}$: $\eta = \bar{\eta}_{\pm 1}(\xi)$ which define the "boundary" for $x = \pm 1$ have to be taken as parts of the contour $C(\xi, \eta)$ which defines the intersection of the positivity cone of (3.1) with the plane $c_0 = 1$. These lines touch the ellipse at $T_{\pm 1}$ and intersect in P^* (Fig. 3 for $\alpha = 3/2$, $\beta = -1/2$). $C(\xi, \eta)$ is therefore composed of the longer arc of the ellipse (3.8) connecting the two points T_{-1} and T_{+1} , together with those parts of $\bar{\eta}_{\pm 1}(\xi)$ which lie between $T_{\pm 1}$ and P^* , respectively. Considering the facts that $E(\xi, \eta) = 0$ lies in the upper half of the $\xi\eta$ -plane, that $\eta^* < 0$, $\tilde{\xi}_{+1} < \tilde{\xi}_{-1}$, and that the slopes of $\eta_{-1}(\xi)$ and $\eta_{+1}(\xi)$ are positive and negative, respectively, one finds the results as stated.

We now turn to the special series (3.4) for $N = 2$, $-\infty < x < \infty$ and prove

Theorem 3:

Let

$$\begin{aligned} u(\alpha, \beta) &= (\alpha + \beta + 4)[(\beta + 1)(\alpha + \beta + 2) - 1]/(\beta + 2) , \\ v(\alpha, \beta) &= (\beta + 1)(\alpha + \beta + 3)(\alpha + \beta + 4)v^*(\alpha, \beta)/(\beta + 2)^2 , \\ v^*(\alpha, \beta) &= (\beta + 2)(\alpha + \beta + 2)^2 - (\alpha + 1)(\alpha + \beta + 4) . \end{aligned}$$

For given values of α, β ($\alpha > -1, \beta > -1$), $-\infty < x < \infty$, and $\lambda > -2$, the inequality

$$S_2(x, \lambda) = 1 + \frac{2(\lambda + 1)}{(\lambda + 2)(\beta + 1)} P_1^{(\alpha, \beta)}(x) + \frac{2}{(\beta + 1)(\beta + 2)} P_2^{(\alpha, \beta)}(x) \geq 0 \quad (3.12)$$

holds if and only if $v^*(\alpha, \beta) \geq 0$ and

$$\lambda_- = u(\alpha, \beta) - \sqrt{v(\alpha, \beta)} \leq \lambda \leq u(\alpha, \beta) + \sqrt{v(\alpha, \beta)} = \lambda_+ .$$

Proof:

Let

$$S_2(x, \lambda) = 1 + \hat{\xi}(\lambda) P_1^{(\alpha, \beta)}(x) + \hat{\eta} P_2^{(\alpha, \beta)}(x) ,$$

where

$$\hat{\xi}(\lambda) = 2(\lambda + 1)/[(\lambda + 2)(\beta + 1)] , \quad \hat{\eta} = 2/[(\beta + 1)(\beta + 2)] > 0 .$$

Replacing $\hat{\eta}$ by η in Theorem 1 (b), it follows that $v^*(\alpha, \beta) < 0$ leads to a contradiction. Further, from (3.7), we find $\xi_- \leq \xi \leq \xi_+$, where

$$\xi_{\pm} = 2 \frac{(\alpha + 1)(\alpha + 2) - (\beta + 1)(\beta + 2) \pm \sqrt{v(\alpha, \beta)}}{(\beta + 1)(\beta + 2)(\alpha + \beta + 2)^2} . \quad (3.13)$$

Equating these values to $\hat{\xi}(\lambda)$ and solving for λ yields two values $\lambda_{\pm}$ with $\lambda_- \leq \lambda_+$, and thus the result stated.

We consider now the case $-1 \leq x \leq 1$, which leads to

Theorem 4:

Let

$$\begin{aligned} h(\alpha, \beta) &= (\alpha + 1)(\alpha + 2)(\alpha + \beta + 4) - (\beta + 1)(\beta + 2)(\alpha + \beta + 2) , \\ w(\alpha, \beta) &= -2 \frac{(\alpha + 1)(\alpha + 2) + (\alpha + 1)(\beta + 2) + (\beta + 1)(\beta + 2)}{(\alpha + 1)(\alpha + 2) + 2(\alpha + 1)(\beta + 2) + (\beta + 1)(\beta + 2)} , \end{aligned}$$

and let $u(\alpha, \beta), v(\alpha, \beta), v^*(\alpha, \beta)$ be as defined in Theorem 1. For given values of α, β ($\alpha > -1, \beta > -1$), $-1 \leq x \leq 1$, and $\lambda > -2$, the inequality (3.12) holds if and only if $v^*(\alpha, \beta) \geq 0$ and $\lambda'_- \leq \lambda \leq \lambda_+$, where

$$\lambda'_- = \begin{cases} u(\alpha, \beta) - \sqrt{v(\alpha, \beta)} & \text{if } h(\alpha, \beta) \geq 0 \\ w(\alpha, \beta) & \text{if } h(\alpha, \beta) \leq 0, \end{cases}$$

and $\lambda_+ = u(\alpha, \beta) + \sqrt{v(\alpha, \beta)}$.

Proof:

As in the proof of Theorem 3, we substitute $\hat{\eta}$ for η in Theorem 2 and find from (3.11) that $\xi'_M(\hat{\eta}) = \xi_+$ as defined in (3.13), the inequality $\hat{\eta} \leq \tilde{\eta}_{-1}$ being false. The condition $\hat{\eta} \leq \tilde{\eta}_{+1}$ is equivalent to $h(\alpha, \beta) \leq 0$, and in this case we have from (3.10) $\xi'_m(\hat{\eta}) = \xi'_- \leq \xi \leq \xi_+$, where

$$\xi'_- = - \frac{(\alpha + 1)(\alpha + 2) + (\beta + 1)(\beta + 2)}{(\alpha + 1)(\alpha + 2)(\beta + 2)}. \quad (3.14)$$

Equating this value to $\hat{\xi}(\lambda)$ and solving for λ yields $\lambda'_- = w(\alpha, \beta)$.

Figure 3 shows the contour $C(\xi, \eta)$, the line $\eta = \hat{\eta}$, and the intersection points $\xi_{\pm}$ for $\alpha = 3/2$, $\beta = -1/2$, together with the approximations to the contour for $M = 5$ and $M = 13$. Note that the intersection points in the $\xi\eta$ -plane correspond to straight lines on the surface of the positivity cone in the three-dimensional $c_0 c_1 c_2$ -space.

We have plotted the intervals for $\lambda(\alpha, \beta)$ obtained from Theorem 4 in Fig. 4. The dots on some of the curves indicate that, from this point onwards with decreasing α , λ'_- is defined by $\lambda' = w(\alpha, \beta)$. As an example, note that for $\alpha = 0$, $\beta = 1$, the relation

$$1 + \frac{\lambda + 1}{\lambda + 2} P_1^{(0,1)}(x) + \frac{1}{3} P_2^{(0,1)}(x) \geq 0 \quad (3.15)$$

holds for $-\infty < x < \infty$ if and only if $\lambda_- = -1.5549 = (25 - 4\sqrt{55})/3 \leq \lambda \leq (25 + 4\sqrt{55})/3 = 18.2216 = \lambda_+$, and for $-1 \leq x \leq 1$ if and only if $\lambda'_- = -1.5714 = -1^{1/7} \leq \lambda \leq \lambda_+$.

There is a minor discrepancy with the results given by Askey: for $\alpha = \beta = -1/2$ we obtain $\lambda_- = \lambda_+ = -1$, whereas his condition IV [2, p. 78] results in $-1 < \lambda \leq -1/2$.

For the case $N = 3$ we only consider the special series (3.4) and find

Theorem 5:

Let

$$u(\alpha, \beta) = \frac{1}{3} (\alpha + \beta + 6) [(\beta + 2)(\alpha + \beta + 2) - 3] / (\beta + 3)$$

$$v(\alpha, \beta) = \frac{1}{9} (\beta + 2)(\alpha + \beta + 5)(\alpha + \beta + 6) v^*(\alpha, \beta) / (\beta + 3)^2$$

$$v^*(\alpha, \beta) = \alpha^2(\beta + 2) + \alpha(2\beta + 1)(\beta + 4) + \beta^3 + 7\beta^2 + 16\beta + 6.$$

For given values of α, β ($\alpha > -1, \beta > -1$), $-1 \leq x < \infty$, and $\lambda > -3$, the inequality

$$S_3(x, \lambda) = 1 + \frac{3(\lambda + 1)}{(\lambda + 3)(\beta + 1)} P_1^{(\alpha, \beta)}(x) + \frac{6(\lambda + 1)}{(\lambda + 3)(\beta + 1)(\beta + 2)} P_2^{(\alpha, \beta)}(x) + \frac{6}{(\beta + 1)(\beta + 2)(\beta + 3)} P_3^{(\alpha, \beta)}(x) \geq 0 \quad (3.16)$$

holds if and only if $v^*(\alpha, \beta) \geq 0$ and

$$\lambda_- = u(\alpha, \beta) - \sqrt{v(\alpha, \beta)} \leq \lambda \leq u(\alpha, \beta) + \sqrt{v(\alpha, \beta)} = \lambda_+.$$

For $-\infty < x < -1$, the inequality (3.16) is false; and for $-\infty < x < -1$, $S_3(x, \lambda) < 0$.

Proof:

It is easy to verify that $S_3(-1, \lambda) = 0$, and we define

$$\tilde{S}_3(x, \lambda) = \frac{8(\beta + 1)_3(\lambda + 3)}{\alpha + \beta + 4} \frac{S_3(x, \lambda)}{x + 1} = c_0(\lambda) + \xi(\lambda)x + \eta(\lambda)x^2, \quad (3.17)$$

where

$$\begin{aligned} c_0(\lambda) &= \lambda[(\alpha + \beta)^2 + 11\alpha - \beta - 6] + 3(\alpha^2 - 2\alpha\beta - \alpha + 5\beta^2 + 15\beta + 6) \\ \xi(\lambda) &= 2\lambda[(\alpha + \beta)^2 + 11\alpha + 5\beta + 12] + 6(\alpha + \beta + 6)(\alpha - \beta - 1) \\ \eta(\lambda) &= (\alpha + \beta + 5)(\alpha + \beta + 6)(\lambda + 3) > 0. \end{aligned} \quad (3.18)$$

From (3.17) and (3.18) it follows that for $x \rightarrow -\infty$, $\tilde{S}_3(x, \lambda) \rightarrow \infty$ and $S_3(x, \lambda) \rightarrow -\infty$.

Further, we can see that ($' \equiv d/dx$)

$$\tilde{S}_3(-1, \lambda) = 24(\beta + 2)(\beta + 3) > 0$$

and

$$\tilde{S}_3'(-1, \lambda) = -12(\beta + 3)(\lambda + \alpha + \beta + 6) < 0,$$

which leads to $S_3(x, \lambda) < 0$ and $S_3'(x, \lambda) > 0$ for $-\infty < x < -1$.

For the interval $-1 \leq x < \infty$, we consider the discriminant of $\tilde{S}_3(x, \lambda) = 0$, i.e.

$$\Delta(\lambda) = \xi^2(\lambda) - 4\eta(\lambda)c_0(\lambda) = 144(\beta + 3)^2\lambda^2 + \delta_1\lambda + \delta_0. \quad (3.19)$$

For $S_3(x, \lambda) \geq 0$ in $-1 \leq x < \infty$, it is necessary and sufficient that $\Delta(\lambda) \leq 0$. Because $\Delta(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \pm\infty$, we see that $\Delta(\lambda) \leq 0$ for $\lambda_- \leq \lambda \leq \lambda_+$, where $\lambda_{\pm}$ are the two solutions of $\Delta(\lambda) = 0$, assuming that real solutions exist.

We consider now the case $N = 3$, $-1 \leq x \leq 1$, which leads to

Theorem 6:

Let

$$h(\alpha, \beta) = (\alpha + 1)(\alpha + 2)(\alpha + \beta + 7) - (\beta + 1)(\beta + 2)(\alpha + \beta + 3) - 2(\alpha + 1)(\beta + 1) ,$$

$$w(\alpha, \beta) = -3 \frac{(\alpha + 1)(\alpha + 4) + (\beta + 1)(\beta + 2)}{(\alpha + \beta)^2 + 11\alpha + 5\beta + 12} ,$$

$$z(\alpha, \beta) = (\alpha + \beta + 6)[(\alpha + \beta)^2 + 11\alpha + 5\beta + 12][(\beta + 2)(\alpha + \beta + 2) - 3] \\ + 9(\beta + 3)[(\alpha + 1)(\alpha + 4) + (\beta + 1)(\beta + 2)] ,$$

and let $u(\alpha, \beta)$, $v(\alpha, \beta)$, and $v^*(\alpha, \beta)$ be as defined in Theorem 5. For given values of α , β ($\alpha > -1$, $\beta > -1$, $z(\alpha, \beta) \geq 0$), $-1 \leq x \leq 1$, $\lambda > -3$, the inequality (3.16) holds if and only if $v^*(\alpha, \beta) \geq 0$ and $\lambda'_- \leq \lambda \leq \lambda_+$ where

$$\lambda'_- = \begin{cases} u(\alpha, \beta) - \sqrt{v(\alpha, \beta)} & \text{if } h(\alpha, \beta) \geq 0 \\ w(\alpha, \beta) & \text{if } h(\alpha, \beta) \leq 0 , \end{cases}$$

$$\text{and } \lambda_+ = u(\alpha, \beta) + \sqrt{v(\alpha, \beta)} .$$

Remark:

Numerical evidence shows that $z(\alpha, \beta) < 0$ is indeed possible for $\alpha > -1$, $\beta > -1$, but also that $v^*(\alpha, \beta) \geq 0$ implies $z(\alpha, \beta) \geq 0$, and that the assumption $z(\alpha, \beta) \geq 0$ is therefore likely to be superfluous.

Proof:

It is clear that $S_3(x, \lambda) \geq 0$ for all λ defined by Theorem 5. In order to find the λ -intervals which in addition make $S_3(x, \lambda) \geq 0$ in $-1 \leq x \leq 1$, one has to ensure that the smallest root x_1 of $\tilde{S}_3(x, \lambda) = 0$ satisfies $x_1 \geq 1$, which is equivalent to

$$-\xi(\lambda) - \sqrt{\xi^2(\lambda) - 4\eta(\lambda)c_0(\lambda)} \geq 2\eta(\lambda) . \quad (3.20)$$

This inequality is equivalent to the following three inequalities:

$$f_1(\lambda) = c_0(\lambda) + \xi(\lambda) + \eta(\lambda) = \mu_1(\alpha, \beta)\lambda + v_1(\alpha, \beta) \geq 0 , \quad (3.21)$$

$$f_2(\lambda) = \xi(\lambda) + 2\eta(\lambda) = \mu_2(\alpha, \beta)\lambda + v_2(\alpha, \beta) \leq 0 , \quad (3.22)$$

$$\Delta(\lambda) = \xi^2(\lambda) - 4\eta(\lambda)c_0(\lambda) \geq 0 , \quad (3.23)$$

where

$$\mu_1(\alpha, \beta) = 4[(\alpha + \beta)^2 + 11\alpha + 5\beta + 12] \\ v_1(\alpha, \beta) = 12[(\alpha + 1)(\alpha + 4) + (\beta + 1)(\beta + 2)] \quad (3.24)$$

$$\begin{aligned}\mu_2(\alpha, \beta) &= 4[(\alpha + \beta)^2 + 11\alpha + 8\beta + 21] \\ v_2(\alpha, \beta) &= 12(\alpha + 2)(\alpha + \beta + 6) .\end{aligned}\tag{3.25}$$

It is easy to verify that $\mu_{1,2}(\alpha, \beta) = 0$ represent parabolas in the $\alpha\beta$ -plane and that $\mu_{1,2}(\alpha, \beta) > 0$ for $\alpha > -1$, $\beta > -1$. Let λ' and λ'' be the solutions of $f_1(\lambda) = 0$ and $f_2(\lambda) = 0$, respectively, where $\lambda' = w(\alpha, \beta)$. Then $\lambda > \lambda''$ and $\lambda < \lambda'$ are excluded by (3.21) and (3.22), and these inequalities are satisfied simultaneously if and only if

$$\lambda'' - \lambda' = -\frac{48(\beta + 3)}{\mu_1(\alpha, \beta)\mu_2(\alpha, \beta)} h(\alpha, \beta) \geq 0 ,\tag{3.26}$$

which is equivalent to $h(\alpha, \beta) \leq 0$. From

$$v^*(\alpha, \beta) + h(\alpha, \beta) = \frac{1}{4}(\alpha + 1)\mu_1(\alpha, \beta) > 0\tag{3.27}$$

we conclude that $h(\alpha, \beta) \leq 0$ implies $v^*(\alpha, \beta) \geq 0$; i.e. λ_- and λ_+ exist. In order to show that $\lambda'' \leq \lambda_+$ we write

$$[\lambda'' - u(\alpha, \beta)]^2 - v(\alpha, \beta) = \frac{16(\alpha + \beta + 3)(\alpha + \beta + 5)(\alpha + \beta + 6)}{\mu_2^2(\alpha, \beta)} h(\alpha, \beta) .\tag{3.28}$$

If $h(\alpha, \beta) \leq 0$, this expression is ≤ 0 , and therefore $\lambda'' - u(\alpha, \beta) \leq \sqrt{v(\alpha, \beta)}$ or $\lambda'' \leq \lambda_+$. An extension of the λ -interval beyond λ_+ is therefore not possible. Similarly, it follows that $u(\alpha, \beta) - \lambda'' \leq \sqrt{v(\alpha, \beta)}$, therefore $\lambda_- \leq \lambda''$ if $h(\alpha, \beta) \leq 0$. It remains to show that, for $h(\alpha, \beta) \leq 0$, $\lambda' \leq \lambda_-$ or

$$\sqrt{v(\alpha, \beta)} \leq u(\alpha, \beta) - w(\alpha, \beta) = \frac{4z(\alpha, \beta)}{3(\beta + 3)\mu_1(\alpha, \beta)} .$$

Under the assumption $z(\alpha, \beta) \geq 0$ (see Remark), we can write

$$v(\alpha, \beta) - [u(\alpha, \beta) - w(\alpha, \beta)]^2 = -16 \frac{h^2(\alpha, \beta)}{\mu_1^2(\alpha, \beta)} \leq 0 .\tag{3.29}$$

This completes the proof.

Most of the calculations in this section have been performed using the algebraic manipulation system REDUCE developed by Hearn [7].

A plot of $\lambda(\alpha, \beta)$ for $N = 3$ as obtained from Theorem 5 ($-1 \leq x < \infty$) is given in Fig. 5.

As a corollary of Theorems 3 and 5 we find that, for $\beta \rightarrow \infty$, $\lambda_- \rightarrow -3/2$ for $N = 2$ and $N = 3$. The proof is easy.

In order to find the λ -interval which makes (3.4) true for any N , one would have to determine analogous results for each N and finally to take the intersection of all these λ -intervals. However, to solve the case $N = 4$, i.e.

$$\begin{aligned} S_4(x, \lambda) = 1 + 4 \frac{\lambda + 1}{(\lambda + 4)(\beta + 1)} P_1^{(\alpha, \beta)}(x) + 12 \frac{(\lambda + 1)_2}{(\lambda + 3)_2(\beta + 1)_2} P_2^{(\alpha, \beta)}(x) \\ + 24 \frac{\lambda + 1}{(\lambda + 4)(\beta + 1)_3} P_3^{(\alpha, \beta)}(x) + \frac{24}{(\beta + 1)_4} P_4^{(\alpha, \beta)}(x) \geq 0 \quad (\lambda > -3) \end{aligned} \quad (3.30)$$

by the approach used for $N = 2$ or 3 does not seem feasible. One can nevertheless try to find common zeros of $S_4(x, \lambda)$ and its derivative by considering the equation $D(S_4^*(x, \lambda)) = 0$, where $D(p)$ is the discriminant of a polynomial (see, e.g. [11, p. 98]), and $S_4^*(x, \lambda) = (\beta + 1)_4(\lambda + 3)_2 S_4(x, \lambda)$.

If one tries to compute $D(S_4^*(x, \lambda))$ which is a determinant of seventh order, one finds that the number of terms to be considered grows so fast that it is difficult to handle them for arbitrary α and β even with an algebraic manipulation system. One sees nevertheless that $D(S_4^*(x, \lambda))$ can be expressed in this case as a polynomial of twelfth degree in λ , which has the factor $(\lambda + 3)^2(\lambda + 4)$, so that $D(S_4^*(x, \lambda)) = 0$ is an equation of ninth degree. For $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, this equation is

$$\begin{aligned} 2\lambda^9 - 341\lambda^8 + 15493\lambda^7 - 182759\lambda^6 - 1205303\lambda^5 + 6315940\lambda^4 \\ + 59802800\lambda^3 + 164448000\lambda^2 + 195840000\lambda + 86400000 = 0. \end{aligned} \quad (3.31)$$

It has only real roots lying in the interval $-\frac{5}{2} \leq \lambda < 106.1$. The largest negative root and the smallest positive root are $\lambda_- = -1.266059$ and $\lambda_+ = 8.605584$, compared to $\lambda_- = 6(2 - \sqrt{5}) = -1.416408$, $\lambda_+ = 6(2 + \sqrt{5}) = 25.416408$ for $N = 2$, and $\lambda_- = -\frac{4}{3}$, $\lambda_+ = 12$ for $N = 3$. A numerical investigation shows that (3.30) is true for $\lambda_- \leq \lambda \leq \lambda_+$, and no other positivity interval was found for $-3 < \lambda \leq 110$. It is possible that the two solutions λ which are nearest to $\lambda = 0$ define the positivity interval also for other values of α and β , even for higher N . We leave this for later investigation.

In order to get some results for higher N , we have tried to get insight by numerical computation. Although it is clear that (3.4) cannot be solved numerically in a rigorous way, it turns out that some results can be obtained which

could be useful for further analytical work. We have therefore determined necessary and sufficient constraints upon $\lambda(\alpha, \beta)$ in the case $N = 60$. To do this we tried, by analogy with the analytic computation for $N \leq 3$, to find numerical values for λ which satisfy the system of equations

$$S_N(x, \lambda) = \sum_{n=0}^N \gamma_n(\lambda) P_n^{(\alpha, \beta)}(x) = 0, \quad (3.32)$$

$$\frac{d}{dx} S_N(x, \lambda) = \sum_{n=0}^N \gamma_n(\lambda) \frac{d}{dx} P_n^{(\alpha, \beta)}(x) = 0,$$

where $-1 \leq x \leq 1$ and

$$\gamma_n(\lambda) = \frac{N!(\lambda + 1)_{N-n}(\lambda + 1)_n}{(N - n)!(\lambda + 1)_N(\beta + 1)_n}, \quad (3.33)$$

with the unknowns λ and x . The two equations (3.32) were solved simultaneously by using a combination of Newton's method and numerical tabulation of the function (3.4), disregarding any solution with $|x| > 1$. The values for $P_n^{(\alpha, \beta)}(x)$ and its derivative were calculated with the help of the well-known recurrence formulae [1, p. 782]. For $N = 60$ and $\alpha \geq 1$, $\beta \geq -1/4$, the values of λ_- for $N = 2$ to $N = 60$ were easily found by iteration, taking the values for $N - 1$ as starting values for N . λ_- was found to be monotonically increasing with N in this region, changing more and more slowly for higher N ; and it seems unlikely that it will change significantly for $N > 60$. For other values of α and β , some care is needed to find the extremal λ out of a set of solutions, in particular for λ_+ . In order to check our calculations, we tabulated the function $S_N(x, \lambda)$ for some given values of α and β and varying λ . The results for $N = 60$ are plotted in Fig. 6. We note the remarkable fact that the structure of the plots for $N = 2, 3, 60$ is qualitatively the same, i.e. a smooth behaviour of all curves for $\lambda > 0$ and $\lambda < 0$, $\alpha \geq 1$. The region $\lambda < 0$, $\alpha \leq 1$ is rather complicated. For $\lambda > 0$, $\beta \geq -1/2$, the values for λ appear to be monotonically decreasing with N . This indicates that in this region the limit for $N \rightarrow \infty$ would probably give the solution for the problem (3.4) for all N . A similar conclusion may be drawn if $\lambda < 0$, $\beta \geq -1/4$, and α larger than some critical value which depends on β . Below this value of α

there is a small region where intermediate values of N determine the boundary. If α is still further decreased, the lowest non-trivial case (i.e. the curve for $N = 2$) is the most constraining one and therefore determines the boundary of the general problem. We may add that the behaviour of λ is not monotone with respect to N if $\beta \leq -1/2$; e.g. for $N = 60$, the curves for λ lie outside the plot if $\lambda \leq -0.6$. For $\beta = 1$, the most constraining curves for λ_- and λ_+ are plotted in Fig. 2. They show that the analytical results already give a crude picture of the optimal curves. Note, however, that Askey [2] does not consider values of $\lambda \leq -1$.

4. CONCLUDING REMARKS

It is clear from our discussion that the theory of linear systems of inequalities and the problem of positive function series are closely related. There are numerous questions left unanswered, of which we want to mention only a few. If the problem stated in Section 1 is given, i.e. if the function system $f_n(x)$ and the x -interval are specified and N is fixed, what is the optimal choice of the number M of arguments x_m ? This amounts to looking for the optimal approximation of a given non-polyhedral cone by a polyhedral cone with a prescribed number of faces. The case where M equals the number N of functions is especially important for practical purposes, because it is needed as an initial step [8]. In the limiting case of $M \rightarrow \infty$ there also exists a known theory [10] for systems of linear inequalities, which has a natural application to positive function series. It should be possible, for example, to obtain analytical estimates for the value of ϵ in (2.9) in a given approximate polyhedral positivity cone. Also the question arises as to whether there exists some symmetry of the basis vectors which could be used to reduce the expense in computer time and storage space. Exactly how this could be achieved does not seem to be obvious. The possible number of basis vectors in a given problem seems to be strongly restricted, as our examples in Section 2 show. In addition, those calculated obey a simple law:

$$\frac{\text{number of basis vectors} - \text{dimension}}{\text{number of inequalities} - \text{dimension}} = \text{integer} .$$

Is this property generally true? Certainly only non-redundant inequalities have to be taken into account in this relation. So the question arises as to why, in all the problems considered here, there occurred only non-redundant inequalities.

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Table 1

Values of ϵ (M = 46)

	Dimension N + 1			
	3	4	5	6
(2.5) Sine	0.35×10^{-2}	0.55×10^{-2}	0.98×10^{-2}	0.14×10^{-1}
(2.6) Cosine	0.17×10^{-2}	0.39×10^{-2}	0.72×10^{-2}	0.12×10^{-1}
(2.7) Legendre	0.66×10^{-3}	0.15×10^{-2}	0.36×10^{-2}	0.76×10^{-2}

Table 2

Values of ϵ
[Laguerre (2.10)]

M	I	ϵ
46	88	8.42
47	90	5.01
48	92	1.80
49	94	1.29
50	96	0.94

Figure captions

- Fig. 1 : The cut $c_0 = 1$ through the positivity cone of a three-term Legendre series, with approximations for $M = 5$ and $M = 13$.
- Fig. 2 : Regions $\lambda(\alpha, \beta)$ ($\beta = 1$) where (3.3) is true according to [2]-[5] (hatched area). The most constraining curves for λ_- and λ_+ as found numerically for $N = 2$ and $N = 60$ are also given.
- Fig. 3 : The cut $c_0 = 1$ through the positivity cone of a three-term Jacobi series for $\alpha = 3/2$, $\beta = -1/2$, with approximations for $M = 5$ and $M = 13$.
- Fig. 4 : A plot of λ'_- and λ_+ according to Theorem 4 ($N = 2$, $-1 \leq x \leq 1$). On the left of the dots on some curves, λ'_- is defined by $w(\alpha, \beta)$.
- Fig. 5 : A plot of λ_- and λ_+ according to Theorem 5 ($N = 3$, $-1 \leq x < \infty$).
- Fig. 6 : A plot of numerical solutions of Eq. (3.32) ($N = 60$, $-1 \leq x \leq 1$).

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6