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GHOSTS AND GEOMETRY

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A B S T R A C T

We comment on recent attempts
to give a geometrical inter-
pretation of the Faddeev-Popov
ghost fields.

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In a manifestly Lorentz covariant diagrammatic ¹⁾ formulation of gauge theories it is necessary to introduce "probability eating" ghosts ²⁾ in order to recover a unitary S matrix. From a path integral quantization point of view this can be understood ³⁾ as a consequence of the fact that the correct measure on the space of paths (after fixing the gauge) is not generally the "flat" product measure

$$\prod_{x,\mu} dA^\mu(x)$$

but has to be supplemented with an additional weight factor $\det \Delta[A]$ given by the Faddeev-Popov prescription. The ghost fields are introduced only because it is convenient to write the determinant in an exponential form :

$$\det \Delta[A] = Z^{-1} \int \prod_x dc(x) d\bar{c}(x) \exp[i \int dx dy \bar{c}^a(x) \Delta^{ab}(x,y) c^b(y)] \quad (1)$$

where Z is a field independent constant and $c^a(x)$, $\bar{c}^a(x)$ are anti-commuting Grassmann fields. They obey the algebraic constraints

$$\{c^a(x), c^b(y)\} = \{\bar{c}^a(x), c^b(y)\} = \{\bar{c}^a(x), \bar{c}^b(y)\} = 0 \quad (2)$$

and no further relations. This is necessary for the above integration formula, with integration in the sense of Berezin ⁴⁾, to hold. With a replacement $\Delta \rightarrow \Delta^{-1}$ in the exponent, we could have used ordinary c number fields as integration variables, but this would have been impractical for explicit calculations.

Thus, as with any ghosts, those of Faddeev and Popov only exist in our minds as convenient substitutions for a more bothersome reality. Recently, however, there has been a lot of activity ⁵⁾⁻⁷⁾ in identifying them with concrete (classical) geometrical objects, namely with certain differential forms on the principal fibre bundle associated with the gauge theory. Do the ghosts have such a simple meaning? Clearly, since "anti-commuting c numbers" are notoriously obscure objects, such an explicit model for them would be extremely interesting.

Can the Grassmann fields of path integration really be modelled by differential forms on a suitable classical (i.e., finite dimensional) manifold? We will argue that this is not possible, since they do not generate isomorphic algebras. To illustrate this point let us take as an example a (real) manifold with co-ordinates x^a , and (real) Grassmann fields $c^a(x)$ defined on it. With $\{f_a^n\}$ a complete set of functions, in the sense that

$$f_a^n(x) * f_b^n(y) = \delta_{ab} \delta(x-y) \quad (3)$$

we can define a basis $\{\omega^n\}$ for the cotangent bundle by the relation

$$\omega^n = f_a^n dx^a \quad (4)$$

Similarly, we can introduce the quantities

$$c^n = \int dx f_a^n(x) c^a(x) \quad (5)$$

which define a basis in the linear space spanned by the Grassmann fields. Clearly, we can set up a vector space isomorphism between the spaces spanned by the two bases $\{\omega^n\}$ and $\{c^n\}$. The algebras generated by these objects also share the anticommuting property

$$\omega^m \wedge \omega^n + \omega^n \wedge \omega^m = c^m c^n + c^n c^m = 0 \quad (6)$$

but this does not prove that they are equal. In fact they are not since the differential forms have the additional nilpotency property

$$\prod_{i=1}^N \omega^{n_i} = 0 \quad (7)$$

when N exceeds the dimension of the manifold.

For gauge theories (in ordinary space-time) the dimension of the principal bundle is equal to $k+4$, where k is the dimension of the gauge group. Thus the product of N differential forms on this manifold vanishes when $N > k+4$ [$= n^2 + 3$ for $SU(n)$]. Since the absence of additional relations of this kind is crucial for the integration formula (1) of the ghost fields to hold, the functional integrals written down by the authors in Refs. 5), 6) (although not always very precisely defined) are not identical to the conventional (and correct) ones.

Therefore, the ghost fields cannot be identified with differential forms on the principal fibre bundle ⁵⁾ or on some finitely extended construction thereof ⁶⁾. We must also add that the identifications, even when viewed only as vector space isomorphisms, look rather artificial to us. To press this point let us first review the concepts involved in a simple fashion.

Fields $\psi^a(x)$ with an internal symmetry index actually should be regarded as the components of a geometric (i.e., co-ordinate independent) object $\underline{\psi}(x)$ decomposed on a given local basis $\{\underline{e}_a(x)\}$,

$$\underline{\psi}(x) = \underline{e}_a(x) \psi^a(x) \quad (8)$$

A gauge transformation is nothing but a local change of this basis frame

$$\underline{e}_a(x) \rightarrow \underline{e}'_a(x) = \underline{e}_b(x) (g(x)^{-1})^b_a \quad (9)$$

which must be accompanied by a compensating change

$$\psi^a(x) \rightarrow \psi'^a(x) = g(x)^a_b \psi^b(x) \quad (10)$$

A gauge potential, or connection field, is introduced to describe the change of local frames as one traverses the manifold,

$$\underline{D} \underline{e}_a = \underline{e}_b A^b_a = \underline{e}_b A^b_{\mu a} dx^\mu \quad (11)$$

This relation in fact defines the internal geometry. The corresponding (covariant) variation in $\underline{\psi}$ is given by

$$\begin{aligned} \underline{D} \underline{\psi} &= \underline{e}_a d\psi^a + \underline{D} \underline{e}_a \psi^a \\ &= \underline{e}_a (d\psi^a + A^a_b \psi^b) \end{aligned} \quad (12)$$

It is readily verified that the (matrix valued) gauge potential transforms as

$$A \rightarrow A' = g A g^{-1} + g \partial_\mu g^{-1} dx^\mu \quad (13)$$

under the gauge transformations described above.

Since the gauge transformations are merely changes of the internal co-ordinate systems, all gauge equivalent potentials correspond to the same geometric situation. To parametrize the latter in a unique way one has to impose a gauge fixing condition which picks out one and only one potential from each class of gauge equivalent potentials. One possible way to do this for gauge fields in Minkowski (or Euclidean) space is to impose the radial gauge constraint ⁸⁾

$$x^\mu A_\mu(x) = A_\mu(0) = 0 \quad (14)$$

This choice amounts to defining the local group frame at each point x by a parallel transportation of the frame at the origin along the radial directions. Since the relative orientation of frames at different points is now fixed, the geometry will determine a unique connection. (On a topological non-trivial manifold one can impose a similar gauge constraint over each co-ordinate patch, but in addition one must take care of the gauge transforms connecting the different patches).

It may not be practical to stick to the gauge choice given above, but we can parametrize any other configuration of frames by the gauge transformation $g(x)$ away from the radial gauge. In fact, one may consider all possible frame orientations simultaneously by extending the space-time manifold to a larger geometrical structure. Its points are characterized by a pair (x, g) specifying a space-time point x and the orientation g of the basis frame at x . This new manifold is called the principal fibre bundle. Its virtue is that we can define the previous gauge transforming quantities as absolute objects on it. For instance, we have

$$\psi^a(x, g) = g^a_b \psi^b(x) \quad (15)$$

with $\psi^b(x)$ referring to the original gauge choice (the radial gauge). The gauge potential $A(x)$ can, in a related way, be extended to a connection field $\Omega(x, g)$ defined on the principal bundle. With z^α as group co-ordinates, i.e., $g = g(z)$, the connection field Ω is defined by

$$(dx^\mu \frac{\partial}{\partial x^\mu} + dz^\alpha \frac{\partial}{\partial z^\alpha} + \Omega) \psi(x, g) = g (dx^\mu \frac{\partial}{\partial x^\mu} + A) \psi(x) \quad (16)$$

which means that the covariant derivatives of ψ also satisfy Eq. (15). This implies

$$\begin{aligned}\Omega(x,g) &= g A(x) g^{-1} + g d_z g^{-1} \\ &= g A_\mu g^{-1} dx^\mu + g(z) \frac{\partial}{\partial z^\alpha} g(z)^{-1} dz^\alpha\end{aligned}\quad (17)$$

The second part in this decomposition of the connection $\Omega(x,g)$ is the object which the authors in Ref. 5) attempt to identify with the ghost fields. However, the ghost fields are space-time dependent objects, while the term gdg^{-1} apparently does not have any such dependence. To get around this problem, the authors of Ref. 5) enforce an x dependence on the object, by making the group parametrization x dependent. However, this seems to us a purely artificial construction, and it masks the essential point that the x -dependence of the ghost fields $c(x)$ and of the one-form gdg^{-1} have completely different meanings. Thus, whereas the ghost fields $c(x)$, at different points x , take their values in the same Grassmann algebra, the one-form gdg^{-1} , when decomposed into local objects, takes its values in different spaces for each point (x,g) , namely in the cotangent spaces associated with these points. The authors of Ref. 6) have recognized the problem with the x -dependence of the forms, and have tried to repair the situation by a more elaborate construction. Thus, they consider an extension of the principal bundle P to a manifold $P \times G \times G$, where G is the gauge group. We admit that it may be possible to give x -dependence to the actual forms in this way. However, it is not possible to see how this manifold arises in any natural way from the geometry. And the fundamental problem of identification of ghosts and geometrical forms, which comes from the finite dimensionality of the manifold, is also present in this case.

One important point remains. There exists a formal similarity between parts of the BRS symmetry ⁹⁾, given by the transformation formulae

$$\begin{aligned}\delta A_\lambda^a &= \xi D_\lambda^a{}_b c^b \\ \delta c^a &= -\frac{1}{2} \xi f_{bc}^a c^b c^c\end{aligned}\quad (18)$$

(where ξ is an "infinitesimal constant anticommuting parameter" and f_{bc}^a are the structure constants of the gauge group) and some of the structural equations of the principal bundle

$$d_z A_\mu(x,g) = D_\mu \omega \quad (19a)$$

$$d_z \omega = -\omega \wedge \omega \quad (19b)$$

where

$$\begin{aligned} A_\mu(x, g) &= g A_\mu(x) g^{-1} \\ D_\mu &= \partial_\mu + [A_\mu(x, g), \cdot] \\ \omega &= g d_z g^{-1} \end{aligned} \quad (20)$$

This alone clearly does not prove equivalence of the objects involved, not least in view of the fact that symmetry transformations are usually quite different from the identities they imply. Nevertheless, it gives a strong motivation to search for a geometrical interpretation and for more explicit models for the ghost fields and the BRS symmetry. This would be very useful for a better conceptual understanding of their nature, since, strictly speaking, the BRS (and any other supersymmetry) transformations are merely formal symbolic manipulations until the spaces on which they act are clearly defined.

However, it should be clear from the previous discussion that if we want to model Grassmann variables by differential forms on some manifold, its dimension must equal the number of independent generators in the Grassmann algebra. Otherwise we will encounter unwanted relations due to nilpotency. Therefore, to model the ghost fields we have to consider an infinite dimensional manifold. We will end this paper with a brief outline of how it may be possible to carry out such an idea ⁷⁾. An obvious choice for the infinite dimensional manifold is the "large gauge group" $\mathcal{G}$, consisting of all continuous gauge transformations $g(x)$. This group is roughly identical to a direct product of the gauge groups G_x associated with each point x in space-time

$$\mathcal{G} = \prod_x G_x \quad (21)$$

Formally we can introduce the exterior derivative on $\mathcal{G}$ as the sum

$$d_g = \sum_x dz^\alpha(x) \frac{\partial}{\partial z^\alpha(x)} = \sum_x d_{g(x)} \quad (22)$$

where $z^\alpha(x)$ gives a parametrization of G_x . We note that the x -co-ordinate here plays quite a different rôle from that of the principal bundle. It defines different directions in cotangent space, and therefore the exterior product of forms associated with different x values, e.g.,

$$dz^\alpha(x) \wedge dz^\beta(y)$$

is a meaningful operation.

The Maurer-Cartan form $\chi(x)$ of the gauge group G_x is defined as

$$\chi(x) = g(x) dg(x) g(x)^{-1} \quad (23)$$

This form can now be considered as a component of the Maurer-Cartan form χ of the full group $\mathcal{G}$, i.e.,

$$\chi = \sum_x \chi(x) \quad (24)$$

and as such it may be possible to associate it with the ghost field $c(x)$. In particular, we note that it satisfies the anticommutativity property for different values of x , and due to the infinite dimensionality of the cotangent space no constraints similar to that of Eq. (7) are imposed on $\chi(x)$. In addition it satisfies the structural equation similar to that of Eq. (19b).

$$d_g \chi(x) + \chi(x) \wedge \chi(x) = 0 \quad (25)$$

We may go one step further and introduce the gauge potential A_μ as a function on $M \times \mathcal{G}$, with M as the space-time manifold. It is defined in the following way

$$A_\mu(x, g(\cdot)) = g(x) A_\mu(x) g(x)^{-1} + g(x) \partial_\mu g(x)^{-1} \quad (26)$$

with $A_\mu(x)$ referring to a particular gauge choice, e.g., the radial gauge as above. Note that, even if the form of this equation looks similar to that of Eq. (17), the meaning is quite different. The definition of the gauge potential in this form is in line with the general philosophy which also motivated the construction of the principal bundle : gauge transforming quantities can be made absolute by sufficiently enlarging the space on which they are defined.

Since A_μ is defined as a field on $M \times \mathcal{G}$, it is possible to make the exterior differentiation with respect to the group co-ordinates. Furthermore, the differential forms defined on $\mathcal{G}$ have an obvious extension to $M \times \mathcal{G}$. Taking some care to interpret the operation of d_g on the different quantities in Eq. (26) correctly, we get

$$\begin{aligned} d_g A_\mu(x, g(\cdot)) &= d_g g(x) A_\mu(x) g(x)^{-1} \\ &+ g(x) A_\mu(x) d_g g(x)^{-1} + d_g g(x) \partial_\mu g(x)^{-1} + g(x) d_g \partial_\mu g(x)^{-1} \\ &= A_\mu(x, g(\cdot)) \omega(x) - \omega(x) A_\mu(x, g(\cdot)) + \partial_\mu \omega(x) \\ &\equiv D_\mu \omega(x) \end{aligned}$$

(27)

Thus, in this way we obtain an equation which has exactly the same form as that of Eq. (19a), although it should be interpreted quite differently.

We conclude from this that it may be possible to find a connection between ghost fields $c^a(x)$ and geometrical objects, which also satisfy the same algebraic relations. In such a construction the large gauge group $\mathcal{G}$ seems to play an essential rôle, rather than the principal bundle. However, we would like to stress the point that to make a real identification of the ghost fields with geometrical objects requires more than showing a formal connection between the BRS transforms and geometrical equations, as we have sketched above. In addition, one should be able to explain the rôle played by the ghost fields in the functional integral formalism from the geometrical point of view.

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