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ON INVARIANT GROUP INTEGRALS IN LATTICE QCD

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A B S T R A C T

A closed expression of the $SU(3)$ [and $U(3)$] one-link invariant group integral in lattice gauge theories is derived. The $U(3)$ result is compared to the expression recently derived by Brower, Rossi and Tan. Applications of our results are briefly touched upon.

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1. INTRODUCTION

In the study of lattice gauge theories^{1,2)} with a gauge group G , one encounters³⁾ invariant group integrals of the form

$$Z(m, m^\dagger) = \int_G dg e^{\text{Tr}(g m^\dagger + g^\dagger m)} \quad (1)$$

where dg is the normalized, invariant Haar measure⁴⁾ on the group G . For convenience, we have absorbed coupling constant parameters in the matrix m . (1) is the generating functional for computing expectation values of any given link for lattice gauge theories or for $(G \times G)$ chiral sigma models⁵⁾. g in (1) then belongs to the fundamental representation of the group G . (1) also enters into the construction of quasi-coherent states appropriate for quantum fields carrying a non-Abelian charge^{6,7)}. In this case, one is interested in other representations of G than the fundamental one. For pion- or gauge fields, one is, in fact, interested in the adjoint representation^{6,7)}.

Several recent papers^{5,8-13)} have been devoted to the actual evaluation of (1) appropriate for lattice (QCD) gauge theories, i.e., one considers $G = U(N)$ [or $SU(N)$]. Two very successful methods have been applied. One is the character expansion⁵⁾ of the exponential in (1). The other one is the construction of a partial differential equation which (1) satisfies due to its invariance with respect to left- and right-handed actions of the group on m ¹¹⁾. In Ref. 13), a solution of this one-link Schwinger-Dyson equation of Brower-Nauenberg for $U(N)$ was actually found^{*}) with the result

$$Z(m, m^\dagger) \equiv Z_o(m m^\dagger) = \left(2^{\frac{N(N-1)}{2}} \prod_{k=0}^{N-1} (k!) \right) \frac{\det(z_j^{i-1} I_{i-1}(z_j))}{\det((z_j^2)^{i-1})} \quad (2)$$

^{*}) B.S. wishes to thank R. Brower and P. Rossi for discussions on this result and for informing us about the work in Ref. 13) prior to publication.

where $z_k \equiv 2\sqrt{x_k}$ and x_k are the eigenvalues of the matrix mm^+ .

In the $1/N$ expansion¹⁴⁾ approach to the QCD dynamics [$G = SU(3)$] one argues that N being large is a "good" approximation to $N = 3$. Since $U(N)$ and $SU(N)$ seem to be rather similar in nature for large N , it is justified to study $U(N)$ instead of $SU(N)$. This circumstance simplifies the evaluation of (1) to a large extent since, for $U(N)$, (1) is a function only of mm^+ . In the present short note, we will present an evaluation of $Z(m, m^+)$ for $G = U(N)$, $SU(N)$, where $N = 2, 3$ ^{*}). For the $U(N)$ groups we will verify the expression (2) but, however, we obtain $Z(m, m^+) \equiv Z_0(mm^+)$ in terms of an explicit power expansion in mm^+ . For the $SU(N)$ groups one has, in general, a dependence on more invariants of the matrices m and m^+ . For $N = 3$ we will reveal a drastic change in the nature of the integral (1) when comparing the $SU(3)[U(3)]$ results with the $SU(2)[U(2)]$ expressions.

Concerning the motivation for the study of the invariant, one-link group integral (1), we have, at present, no more to add to the remarks in Refs. 3,5-13). The series expansion of the one-link integral (1) given below can be applied to a strong coupling expansion in a standard manner. Moreover, it is well known^{15,16)} that two-dimensional lattice gauge theories are exactly solvable in that the solution of the theory can be reduced to the evaluation of (1) for a diagonal matrix $m = \beta \mathbf{1}$. Now, for $N = \infty$, the $U(N)$ theory exhibits a third order phase transition¹⁶⁾. For finite N , it should be possible to see a sign of this effect on a lattice and, moreover, one can compare $SU(N)$ with $U(N)$ in an exact manner. Such a study is presently in progress¹⁷⁾.

2. $N = 2$ INTEGRALS

The $SU(2)$ one-link integral can easily be evaluated by using an explicit S^3 parametrization since, topologically, $SU(2) \approx S^3$. Writing the group element as $g = u_0 \mathbf{1} + i \underline{u} \cdot \underline{\sigma}$, where $\underline{\sigma}$ are the three Pauli matrices, one obtains for (1):

$$Z(m, m^+) = \frac{1}{2\pi^3} \int ds \int d^4 u e^{is(u^2-1)} \cdot \exp(u_0 A + u_1 B + u_2 C + u_3 D) \quad (3)$$

^{*}) The expressions we obtain for $N = 2$ seem to be known to some authors (B.S. wishes to thank P. Hasenfratz and P. Rossi for a discussion on this point). We include our method of calculating these integrals as an illustration of the technique.

where $u^2 = u_0^2 + \underline{u}^2$ and $A = 2\text{Re}(m_{11} + m_{22})$, $B = 2\text{Im}(m_{22} - m_{11})$, $C = 2\text{Re}(m_{21} - m_{12})$, $D = -2\text{Im}(m_{12} + m_{21})$. The integration over the u variables in (3) are just Gaussian integrals which can be carried out directly. The final integral over the s parameter is then a Fourier-Mellin transform which can easily be evaluated. Introducing the $SU(2)$ invariant combinations $\text{Tr}(mm^+)$ and $\det m + \det m^+$, the final result is

$$Z(m, m^+) = \sum_{n=0}^{\infty} \frac{(\text{Tr}(mm^+) + \det m + \det m^+)^n}{n!(n+1)!} \quad (4)$$

where we recognize the power series expansion of the first modified Bessel function I_1 . If we treat m and m^+ as independent variables and put $m^+ = 0$, (4) reduces to the known result given by Creutz³⁾. By making use of the expression (4), it is easy to derive an expression for the one-link $U(2)$ generating functional, $Z_0(mm^+)$, by a projection technique. Writing $g = \exp(i\varphi)g'$, where g' belongs to $SU(2)$, we obtain

$$\begin{aligned} Z_0(mm^+) &= \frac{1}{2\pi} \int_0^{2\pi} d\varphi Z(e^{i\varphi}m, e^{-i\varphi}m^+) = \\ &= \sum_{n,r=0}^{\infty} \frac{(\text{Tr}(mm^+))^n (\det(mm^+))^r}{(n+2r+1)! n! (r!)^2} \end{aligned} \quad (5)$$

We have verified explicitly that (5) actually agrees with the expression (2) for $N = 2$. It is amusing to notice that the $SU(2)$ result (4) is much simpler in structure than the $U(2)$ form (5). This is due to the simplicity of the $SU(2)$ group and for $SU(3)$ the result, which will be given below, is quite complicated in terms of $SU(3)$ invariants.

For the sake of completeness, we also mention here the generating functional (1) in the case of the adjoint representation of $SU(2)$. m is then a 3×3 matrix and the g 's can be represented by the rotation matrices $R_{ij} = \frac{1}{2} \cdot \text{Tr}(\sigma_i g \sigma_j g^+)$, where $g = u_0 \mathbb{1} + i \underline{u} \cdot \underline{\sigma}$. Since R_{ij} is real, only the real part of m contributes in (1). Writing $M = 2\text{Re}(m)$ we obtain^{*)}

$$Z(m, m^+) = Z(M) = \sum_{j,k,l=0}^{\infty} \frac{(2(j+k+l)-1)!!}{j! k! l! (1+2j+3k+4l)!} x^j y^k z^l \quad (6)$$

where

$$\begin{cases} x = \text{Tr}(MM^{\sim}) \\ y = 4 \det M \\ z = \frac{1}{2}(\text{Tr}(MM^{\sim}))^2 - \text{Tr}(MM^{\sim}MM^{\sim}) \end{cases} \quad (7)$$

Since $SU(2)/Z_2 \simeq SO(3)$, the generating functional (6) for the $SU(2)$ group in adjoint representation gives the $SO(3)$ group generating functional.

^{*)} We refer the reader to Refs. 6) and 7) for details, where an integral representation of (6) can be found, and where some physical applications are discussed.

3. N = 3 INTEGRALS

In evaluating the $SU(3)$ one-link integral, we have found it very useful to parametrize⁸⁾ the $SU(3)$ group manifold in terms of two normalized, complex three-vectors $\underline{u}$ and $\underline{v}$. Each g in $SU(3)$ can then be written in the form:

$$g = \begin{pmatrix} u_1^* & u_2^* & u_3^* \\ v_1^* & v_2^* & v_3^* \\ w_1 & w_2 & w_3 \end{pmatrix} \quad (8)$$

where $w_i = \epsilon_{ijk} u_j^* v_k$ and $u^* \cdot \underline{v} = 0$, which leads to eight independent variables. An integration over $SU(3)$ can be re-expressed in terms of the $\underline{u}, \underline{v}$ variables in (8). For the case $G = SU(3)$ in (1) we obtain

$$\begin{aligned} \int_{SU(3)} dg e^{\text{Tr}(g m^\dagger + g^\dagger m)} &= \frac{1}{2\pi^2} \int_{-\infty}^{\infty} ds e^{is} \int_{-\infty}^{\infty} dt e^{it} \cdot \\ &\cdot \int d^2 z \int d^6 u \int d^6 v e^{-i(s \underline{u}^* \cdot \underline{u} + t \underline{v}^* \cdot \underline{v})} e^{iz(\underline{u}^* \cdot \underline{v} + \underline{v}^* \cdot \underline{u})} \cdot (9) \\ &\cdot e^{\text{Tr}(g m^\dagger + g^\dagger m)} \end{aligned}$$

By making use of (8) we see that the argument in the exponential of (9) is a quadratic function in $\underline{u}, \underline{v}, \underline{u}^*, \underline{v}^*$. The complex Gaussian $\underline{u}$ and $\underline{v}$ integrals can be performed in a straightforward manner^{*)}. All except one of the Fourier-Mellin transformations can then be explicitly performed. As a result we obtain

$$Z(m, m^\dagger) = \frac{i}{\pi} \int_{\gamma} \frac{dp}{p} \cdot \frac{1}{\sqrt{T(p)}} J_1\left(\frac{2}{p} \sqrt{T(p)}\right) e^{-\Delta/p} \quad (10)$$

*) We observe that we can always write $m = g_1 D g_2$, $g_1, g_2 \in SU(3)$, where D is a diagonal matrix whose matrix elements have a common phase.

where

$$T(p) = p^3 - Xp^2 + Yp + Z \equiv -\det(mm^\dagger - p\mathbb{1}) \quad (11)$$

and X, Y, Z, Δ are the $SU(3)$ invariant combinations

$$\begin{cases} X = \text{Tr}(mm^\dagger) \\ Y = \frac{1}{2} (\text{Tr}(mm^\dagger))^2 - \text{Tr}((mm^\dagger)^2) \\ Z = \det(mm^\dagger) \\ \Delta = \det m + \det m^\dagger \end{cases} \quad (12)$$

The contour γ in (10) encloses the pole $p = 0$. By expanding the Bessel function in (10) and performing the contour integral, we obtain

$$Z(m, m^\dagger) = 2 \sum_{j,k,l,n=0}^{\infty} \frac{1}{(j+2k+3l+n+2)!(k+2l+n+1)!} \cdot \frac{X^j}{j!} \cdot \frac{Y^k}{k!} \cdot \frac{Z^l}{l!} \cdot \frac{\Delta^n}{n!} \quad (13)$$

For $m^\dagger = 0$ this result agrees with that of Creutz³⁾.

The $U(3)$ one-link integral can now be obtained by making use of the method analogous to the one used in obtaining (5). The final $U(3)$ integral can be written in the following forms

$$\begin{aligned}
 Z_o(m m^+) &= \frac{i}{\pi} \int \frac{dp}{p \sqrt{T(p)}} I_o\left(2 \frac{\sqrt{Z}}{p}\right) J_1\left(\frac{2}{p} \sqrt{T(p)}\right) = \\
 &= 2 \sum_{j,k,l=0}^{\infty} \frac{(4j+2k+l+2)!}{[(3j+2k+l+2)!]^2} \\
 &\cdot \frac{X^j}{j!} \cdot \frac{Y^k}{k!} \cdot \frac{Z^l}{l!}
 \end{aligned} \tag{14}$$

One can verify in a straightforward manner that (14) satisfies the Brower-Nauenberg equation, which, in terms of the invariants (12) reads for $U(3)$

$$\begin{aligned}
 &(X^2 - 2Y) \partial_X^2 + 2(Y^2 - XZ) \partial_Y^2 + 3Z^2 \partial_Z^2 + \\
 &+ 2(XY - 3Z) \partial_X \partial_Y + 2XZ \partial_X \partial_Z + 4YZ \partial_Y \partial_Z + \\
 &+ 3X \partial_X + 4Y \partial_Y + 3Z \partial_Z] Z_o(m m^+) = X Z_o(m m^+)
 \end{aligned} \tag{15}$$

We have not shown explicitly the equivalence of (14) and (2). They are, however, equal, since they satisfy the same differential equation and boundary condition¹³⁾. In the same way, (13) satisfies the $SU(3)$ Brower-Nauenberg equation. If we write¹¹⁾ $\Delta \equiv 2 \cos 3\theta \sqrt{Z}$, an additional term $-1/12 \partial^2 / \partial \theta^2 Z(m, m^+)$ will be present on the right-hand side of Eq. (15).

Finally, we remark that in the case of a matrix $m = \beta \mathbb{1}$, one can evaluate (10) directly [for $U(3)$ and $SU(3)$] instead of using (13) and (14). (1) can then be expressed as a power expansion^{*)} in β .

*) See Ref. 17) for details and for an application in two-dimensional (lattice) QCD.

4. CONCLUSIONS AND REMARKS

We have given a closed expression for the $SU(3)$ [and $U(3)$] one-link integrals in terms of $SU(3)$ [$U(3)$] invariants. Bars has given⁵⁾ a perspective of how to make use of the one-link integrals in lattice gauge theories. A strong coupling expansion can, for example, be done for $SU(3)$ by making use of the generating functional (13). Corrections to the Bars-Greens truncation procedure¹⁰⁾ can furthermore be studied systematically for $SU(3)$ [and $U(3)$]. Here we only notice that in the strong coupling limit, i.e., [β is small $\beta \lesssim 1/4(d-1)$ where d is the dimension of the lattice] in $m = \beta J$, we obtain $Z(m, m^+) \approx 3\beta^2 \text{Tr} J J^+$. As shown by Bars and Green¹⁰⁾, one can then find an approximative solution of the lattice gauge theory. It would be interesting to apply the expression (13) or (14) to a study of corrections to the Bars-Greens approximation. Polyakov¹⁹⁾ has furthermore shown that the one-link integral (1) enters into the mean field approximation of lattice gauge theories. The one-link integral appropriate for this approximation in the large N limit has recently been discussed by Brezin and Gross²⁰⁾, where it was shown that (1) exhibits a two-phase structure as in the two-dimensional $U(\infty)$ theory. It has been conjectured¹⁶⁾ that for sufficiently large N one should see a sign of this two-phase structure. In two dimensions this is what actually happens¹⁷⁾. The results of the present paper can be used to study this issue in higher (space-time) dimensions.

As a final remark we notice that the steepest descent method can be applied to the integral representation (10) [and to the integral representation for $U(3)$ easily derivable from (10)] to study the weak coupling limit (β being large). As suggested by Brower and Nauenberg¹¹⁾, one could also study this limit by considering stationary points of (1) directly. We will not, however, develop these considerations in the present note.

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