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A METHOD OF ANALYZING THE SCALING VIOLATION OF INCLUSIVE SPECTRA
IN HARD PROCESSES

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A B S T R A C T

The analytic solution of the evolution equations in QCD is given in the form of a series of Laguerre polynomials in the variable $y = \ln(1/x)$. The Laguerre series converges very quickly and it can be truncated after few terms with a reasonable accuracy. Also high precision calculations are possible since the coefficients of the series are given by simple recurrence formulae.

The method works both for non-singlet and for singlet spectra and it can be applied to any hard process. The inclusion of higher order effects is immediate. The Laguerre technique provides a natural framework for extracting from the data the input quark and gluon distributions without any prejudices concerning their particular analytic form.

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1. - INTRODUCTION

In the present paper we propose a method of analyzing the scaling violation of inclusive spectra in hard processes in the framework of perturbative QCD. A detailed numerical analysis of the available data making use of the Laguerre method discussed here will be presented elsewhere ¹⁾.

We start with some general remarks concerning the QCD predictions for the scaling violations. Any QCD formula for the inclusive distribution in a hard lepton-hadron interaction has the form of the convolution of a hard lepton-parton cross-section with appropriate parton densities or fragmentation functions. For example, by denoting by $\rho_{lh}^{\ell'}(x, Q^2)$ the inclusive lepton distribution in the process $\ell + h \rightarrow \ell' + \text{anything}$, we have

$$\rho_{lh}^{\ell'}(x, Q^2) = \sum_p \int_0^1 d\hat{x} \int_0^1 d\hat{z} \hat{\rho}_{lp}^{\ell'}(\hat{x}, Q^2) p_h(\hat{z}, Q^2) \delta[x - \hat{x}\hat{z}] \quad (1.1)$$

where $Q^2 = -2\ell\ell'$, $x = Q^2/2\ell h$ and the letter p labels the QCD partons (quarks and gluons) :

$$p = (q_1, q_2, \dots, q_f, \bar{q}_1, \bar{q}_2, \dots, \bar{q}_f, G) = (u, d, \dots, \bar{u}, \bar{d}, \dots, G) \quad (1.2)$$

We present the method on the example of deep inelastic leptonproduction. The application to other hard processes is straightforward.

The lepton-parton cross-sections [as, e.g., $\hat{\rho}$ in Eq. (1.1)] are given by a power series in the running coupling constant $\alpha(Q^2)$:

$$\hat{\rho}(\hat{x}, Q^2) = \hat{\rho}^{(0)}(\hat{x}) + \frac{\alpha(Q^2)}{2\pi} \hat{\rho}^{(1)}(\hat{x}) + \dots \quad (1.3)$$

where

$$\begin{aligned} \frac{\alpha(Q^2)}{2\pi} &= \frac{2}{\beta_0} \frac{1}{\ln Q^2/\Lambda^2} \left[1 - \frac{\beta_1}{\beta_0} \frac{\ln \ln Q^2/\Lambda^2}{\ln Q^2/\Lambda^2} + O\left(\frac{1}{\ln^2 Q^2/\Lambda^2}\right) \right] \\ \beta_0 &= \frac{11}{3} C_G - \frac{4}{3} T_R f \\ \beta_1 &= \frac{34}{3} C_G^2 - \frac{10}{3} C_G f - 2 C_F f \end{aligned} \quad (1.4)$$

f is the number of flavours and the colour factors are defined as follows :

$$C_G = N, \quad C_F = \frac{N^2 - 1}{2N}, \quad T_R = \frac{1}{2} \quad (1.5)$$

where N is the number of colours.

Parton densities $p_h(x, Q^2)$ evolve with Q^2 according to the evolution equations ²⁾ :

$$Q^2 \frac{d}{dQ^2} q_i^{(-)}(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} P_{(-)}(x, \alpha(Q^2)) \otimes q_i^{(-)}(x, Q^2) \quad (1.6a)$$

$$Q^2 \frac{d}{dQ^2} \left[q_i^{(+)}(x, Q^2) - \frac{1}{f} q^{(+)}(x, Q^2) \right] = \frac{\alpha(Q^2)}{2\pi} P_{(+)}(x, \alpha(Q^2)) \otimes \left[q_i^{(+)}(x, Q^2) - \frac{1}{f} q^{(+)}(x, Q^2) \right] \quad (1.6b)$$

$$Q^2 \frac{d}{dQ^2} \begin{bmatrix} q^{(+)}(x, Q^2) \\ G(x, Q^2) \end{bmatrix} = P(x, \alpha(Q^2)) \otimes \begin{bmatrix} q^{(+)}(x, Q^2) \\ G(x, Q^2) \end{bmatrix} = \begin{bmatrix} P_{11}(x, Q^2) & P_{12}(x, Q^2) \\ P_{21}(x, Q^2) & P_{22}(x, Q^2) \end{bmatrix} \otimes \begin{bmatrix} q^{(+)}(x, Q^2) \\ G(x, Q^2) \end{bmatrix} \quad (1.6c)$$

where

$$q_i^{(-)} = q_i - \bar{q}_i, \quad q_i^{(+)} = q_i + \bar{q}_i, \quad q^{(+)} = \sum_{i=1}^f q_i^{(+)} \quad (1.7)$$

and the symbol $\otimes$ denotes the convolution, i.e., $C(x) = A(x) \otimes B(x)$ means

$$C(x) = \int_0^1 dx_1 \int_0^1 dx_2 A(x_1) B(x_2) \delta(x - x_1 x_2) \quad (1.8)$$

The probabilities $P_{(\pm)}(x, \alpha)$, $P(x, \alpha)$ are given by power series in α :

$$P_{(\pm)}(x, \alpha) = P_{(\pm)}^{(0)}(x) + \frac{\alpha}{2\pi} P_{(\pm)}^{(1)}(x) + \left(\frac{\alpha}{2\pi}\right)^2 P_{(\pm)}^{(2)}(x) + \dots \quad (1.9)$$

$$P(x, \alpha) = P^{(0)}(x) + \frac{\alpha}{2\pi} P^{(1)}(x) + \left(\frac{\alpha}{2\pi}\right)^2 P^{(2)}(x) + \dots \quad (1.10)$$

where each term in Eq. (1.10) is a 2×2 matrix. [Since all Eqs. (1.6) are diagonal in h, we will hereafter drop the hadronic label.]

The Eqs. (1.6a,b) will be referred as the non-singlet equations and the Eqs. (1.6c) as the singlet one. The evolution equations for fragmentation functions (timelike region) are formally identical to those for parton densities (spacelike region) given above : the probabilities $P_{(\pm)}^{(i)}(x)$, $P^{(i)}(x)$ are, however, different in both regions for $i > 0$.

At present, we know the functions $P_V^{(0)}(x)$, $P^{(0)}(x)$ (Parisi-Altarelli probabilities ³⁾) as well as the next-to-leading probabilities $P_{(\pm)}^{(1)}(x)$, $P^{(1)}(x)$ both for spacelike and for timelike regions ⁴⁾. Also the terms $\hat{p}^{(0)}$ and $\hat{p}^{(1)}$ have been derived for all typical hard processes like leptonproduction ⁵⁾, one hadron inclusive leptonproduction, e^+e^- annihilation ^{6),7)} and muon pair production (Drell-Yan process) ⁸⁾. [For the presentation of all results quoted above in a unique factorization scheme, see Ref. 2).]

In order to derive the Q^2 dependence for any of these reactions, one should solve the system (1.6) and perform the convolution of the resulting parton densities or fragmentation functions with appropriate hard cross-section $\hat{p}$, as in Eq. (1.1).

From the theoretical point of view, the simplest way of analyzing the scaling violation effects is to look at the Q^2 dependence of moments of inclusive spectra. Using the convention

$$f(n) \equiv \int_0^1 dx x^{n-1} f(x) \quad (1.11)$$

for any function f , we have then $C(n) = A(n)B(n)$ from Eq. (1.8), i.e., all convolutions in Eqs. (1.1), (1.6) are replaced by products of moments and the evolution equations can be solved in a simple way.

The method of moments suffers, however, from serious practical shortcomings. Typically, the large x ($x \rightarrow 1$) [and very small x ($x \rightarrow 0$)] region is measured with small accuracy or is unknown at all. This may seriously affect the "experimental" results for moments, since the scaling violation effects are the strongest in both regions mentioned above.

On the other hand, the analysis of scaling violations in the x variable being much more appropriate from the experimental point of view, is much more complicated since one has to deal then with the system of integro-differential evolution equations. Various methods have been proposed for solving the evolution in the x variable ⁹⁾⁻¹³⁾, but most of them are either simple but only approximate or exact but very complicated.

The problem gets even more serious if one is going to include the next-to-leading effects with reasonable precision : there is no point in studying the $O(\alpha)$ corrections if the error due to the approximate treatment of evolution equations is of the same order of magnitude as the corrections to the leading log result.

The aim of this paper is to propose a new method of solving the evolution equation. The main idea is to expand the solution in a series of Laguerre polynomials in the variable $y = \ln(1/x)$.

The coefficients of this series are given by simple recursive relations so that the method is very appropriate for computer calculations : usually, the series converges quickly and uniformly due to the orthogonality of Laguerre polynomials so that it can be truncated rather quickly with high accuracy. The main virtue of the method is its universality : it works both for non-singlet and for singlet evolutions, it can be applied for all typical hard processes, and it allows us to incorporate in a simple way the higher order effects.

2. - EVOLUTION FUNCTIONS

It will be convenient to use in the following the evolution variable t

$$t = -\frac{2}{\beta_0} \ln \frac{\alpha(Q^2)}{\alpha(Q_0^2)} \quad (2.1)$$

instead of Q^2 . The system (1.6) can be then rewritten in the following form :

$$\frac{d}{dt} q_i^{(-)}(t, x) = \left[P_v^{(0)}(x) + \frac{\alpha(t)}{2\pi} R_{(-)}(x) + \dots \right] \otimes q_i^{(-)}(t, x) \quad (2.2a)$$

$$\frac{d}{dt} \left[q_i^{(+)}(t, x) - \frac{1}{f} q^{(+)}(t, x) \right] = \left[P_v^{(0)}(x) + \frac{\alpha(t)}{2\pi} R_{(+)}(x) + \dots \right] \otimes \left[q_i^{(+)}(t, x) - \frac{1}{f} q^{(+)}(t, x) \right] \quad (2.2b)$$

$$\frac{d}{dt} \begin{bmatrix} q^{(+)}(t, x) \\ g(t, x) \end{bmatrix} = \left[P^{(0)}(x) + \frac{\alpha(t)}{2\pi} R(x) + \dots \right] \otimes \begin{bmatrix} q^{(+)}(t, x) \\ g(t, x) \end{bmatrix} \quad (2.2c)$$

where

$$R_{(\pm)}(x) = P_{(\pm)}^{(1)}(x) - \frac{\beta_1}{2\beta_0} P_v^{(0)}(x) \quad (2.3)$$

$$R(x) = P^{(1)}(x) - \frac{\beta_1}{2\beta_0} P^{(0)}(x) \quad (2.4)$$

The general solution of the system (2.2) can be written in the following way in terms of the evolution functions $E_{(\pm)}(t, x)$, $E(t, x)$ ²⁾ :

$$q_i^{(-)}(t, x) = E_{(-)}(t, x) \otimes \tilde{q}_i^{(-)}(x) \quad (2.5a)$$

$$\begin{aligned} q_i^{(+)}(t, x) = & E_{(+)}(t, x) \otimes \tilde{q}_i^{(+)}(x) + \frac{1}{f} [E_{11}(t, x) - E_{(+)}(t, x)] \otimes \tilde{q}_i^{(+)}(x) \\ & + \frac{1}{f} E_{12}(t, x) \otimes \tilde{G}(x) \end{aligned} \quad (2.5b)$$

$$\begin{bmatrix} q_i^{(+)}(t, x) \\ G(t, x) \end{bmatrix} = E(t, x) \otimes \begin{bmatrix} \tilde{q}_i^{(+)}(x) \\ \tilde{G}(x) \end{bmatrix} \quad (2.5c)$$

where

$$E(t, x) \equiv \begin{bmatrix} E_{11}(t, x) & E_{12}(t, x) \\ E_{21}(t, x) & E_{22}(t, x) \end{bmatrix}$$

The evolution functions $E(t, x)$ satisfy the equations :

$$\frac{d}{dt} E_{(\pm)}(t, x) = \left[P_v^{(0)}(x) + \frac{\alpha(t)}{2\pi} R_{(\pm)}(x) + \dots \right] \otimes E_{(\pm)}(t, x) \quad (2.6a)$$

$$\frac{d}{dt} E(t, x) = \left[P^{(0)}(x) + \frac{\alpha(t)}{2\pi} R(x) + \dots \right] \otimes E(t, x) \quad (2.6b)$$

One can impose the boundary condition :

$$E(0, x) = \delta(x) \cdot \mathbb{1} ; E_{(\pm)}(0, x) = \delta(x) \quad (2.7)$$

by writing the solution of Eqs. (2.6) in the form :

$$E_{[\pm]}(t, x) = E_{[v]}^{(0)}(t, x) + \frac{\alpha^{(0)}}{2\pi} E_{[\pm]}^{(1)}(t, x) + O(\alpha^{(0)2}) \quad (2.8)$$

with

$$E_{[\pm]}^{(1)}(t, x) = \int_0^t d\tau e^{-\frac{\beta_0}{2}\tau} E_{[V]}^{(0)}(t-\tau, x) \otimes R(x) \otimes E_{[V]}^{(0)}(\tau, x) \quad (2.9)$$

The functions $E_V^{(0)}(t, x)$, $E^{(0)}(t, x)$ are the leading log evolution functions ; they are normalized by imposing the boundary conditions

$$E_{(\pm)}^{(0)}(0, x) = \delta(1-x) ; E^{(0)}(0, x) = \delta(1-x) \cdot 1 \quad (2.10)$$

The explicit expressions for $E^{(0)}$ are :

$$E_V^{(0)}(t, x) = e^{P_V^{(0)}(x)t} \equiv \delta(1-x) + t P_V^{(0)}(x) + \frac{t^2}{2!} P_V^{(0)}(x) \otimes P_V^{(0)}(x) + \dots \quad (2.11a)$$

$$E^{(0)}(t, x) = e^{P^{(0)}(x)t} \equiv \delta(1-x) \cdot 1 + t P^{(0)}(x) + \frac{t^2}{2!} P^{(0)}(x) \otimes P^{(0)}(x) + \dots \quad (2.11b)$$

The functions $\tilde{q}^{(\pm)}(x)$, $\tilde{G}(x)$ are arbitrary input distributions. They should be extracted from the data. The formulae (2.5)-(2.11) provide the formal solution of the system (2.2) with next-to-leading corrections included. The expressions (2.10)-(2.11) are, however, not very useful for practical calculations. In the next section we present the Laguerre method which is much more efficient and easier for computer calculations.

3. - THE LAGUERRE METHOD

The main idea is to find a functional basis in which the convolution of two functions is a simple operation and the result can be expressed in a simple way by the functions from the basis. In such a basis the multiple convolutions like those in Eqs. (2.10) and (2.11) could be performed rather quickly.

The simplest trial - the polynomial basis in x variable - is out of the game since it is not closed with respect to the convolution. The convolution of two polynomials in x will also always produce a term $\ln x$.

The basis of polynomials in $y = \ln 1/x$ satisfies our criteria :

$$\int_0^1 dx_1 \frac{1}{n!} \left(\ln \frac{1}{x_1}\right)^n \int_0^1 dx_2 \frac{1}{m!} \left(\ln \frac{1}{x_2}\right)^m \delta(x-x_1 x_2) = \frac{1}{(n+m+1)!} \left(\ln \frac{1}{x}\right)^{n+m+1} \quad (3.1)$$

Therefore, from now on we will always use the variable y instead of x in all formulae. To make contact with the previous notation we adopt the simple convention $x F(x) \equiv F(y)$. It will not create any confusion since the original function $F(x)$ will never be used. For example, we will write $E(t,y)$ instead of $x E(t,x)$, $P(y)$ instead of $x P(x)$, etc. (The origin of an extra x factor in the above convention will be explained later.) Changing the integration variables $x_1 \rightarrow y_1 = \ln 1/x_1$, $x_2 \rightarrow y_2 = \ln 1/x_2$ in Eq. (1.8), the formula for the convolution in y variable takes the form :

$$\begin{aligned} C(y) &= A(y) \otimes B(y) = \\ &= \int_0^y dy_1 \int_0^{y_1} dy_2 A(y_1) B(y_2) \delta(y - y_1 - y_2) = \\ &= \int_0^y d\eta A(\eta) B(y - \eta) \end{aligned} \quad (3.2)$$

The rule (3.1) for convoluting polynomials in y is now completely trivial :

$$\begin{aligned} \int_0^y d\eta \frac{\eta^n}{\Gamma(n+1)} \frac{(y-\eta)^m}{\Gamma(m+1)} &= \frac{y^{n+m+1}}{\Gamma(n+1) \Gamma(m+1)} \int_0^1 dz z^n (1-z)^m = \\ &= \frac{y^{n+m+1}}{\Gamma(n+m+2)} \end{aligned} \quad (3.3)$$

Since we want to construct a polynomial approximation, the optimal basis is the one given by orthogonal polynomials (see Section 4 for the discussion of this point). The corresponding polynomials in the interval $0 < y < \infty$ are Laguerre polynomials $L_n(y)$:

$$L_n(y) = \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{y^k}{k!} \quad (3.4)$$

Let us recall at this point a few basic properties of Laguerre polynomials :

a) orthogonality :

$$\int_0^{\infty} dy e^{-y} L_n(y) L_m(y) = \delta_{nm} \quad (3.5)$$

b) recurrence relations :

$$L_0(y) = 1, \quad L_1(y) = 1 - y$$

$$(n+1) L_{n+1}(y) = (2n+1-y) L_n(y) - n L_{n-1}(y) \quad (3.6)$$

c) generating function :

$$\frac{1}{1-z} e^{-y \frac{z}{1-z}} = \sum_{n=0}^{\infty} L_n(y) z^n \quad (3.7)$$

d) convolution rule :

$$L_n(y) \otimes L_m(y) = L_{n+m}(y) - L_{n+m+1}(y) \quad (3.8)$$

e) Laguerre series of arbitrary function $F(y)$

$$F(y) = \sum_{n=0}^{\infty} f_n L_n(y) \quad (3.9)$$

where

$$f_n = \int_0^{\infty} dy e^{-y} L_n(y) F(y) \quad (3.10)$$

Using the formula (3.8) (in fact the basic one for the method proposed in this paper), one can immediately derive the convolution $C(y)$ of two functions $A(y)$, $B(y)$, once their Laguerre series are given. If

$$A(y) = \sum_{n=0}^{\infty} A_n L_n(y) \quad B(y) = \sum_{n=0}^{\infty} B_n L_n(y)$$

$$C(y) = A(y) \otimes B(y) = \sum_{n=0}^{\infty} C_n L_n(y) \quad (3.11)$$

then

$$C_n = \sum_{i=0}^n A_i b_{n-i} = \sum_{i=0}^n B_i a_{n-i} \quad (3.12)$$

where

$$\begin{aligned} a_i &= A_i - A_{i-1} & (A_{-1} \equiv 0) \\ b_i &= B_i - B_{i-1} & (B_{-1} \equiv 0) \end{aligned} \quad (3.13)$$

The Laguerre series exists for any function $F(y)$ for which the integrals (3.10) are convergent. It means that the singularity of $F(y)$ for $y \rightarrow \infty$ ($x \rightarrow 0$) must be of the form

$$|F(y)| \underset{y \rightarrow \infty}{<} e^{(1-\delta)y} = \frac{1}{x^{1-\delta}} \quad (\delta > 0) \quad (3.14)$$

In order to remove dx/x singularities in the probabilities P and evolution functions E , we first multiply all functions by x and then write the Laguerre series for them.

The coefficients F_n in Eq. (3.10) can also be derived from the analytic expression for the moments of $F(y)$. Indeed, if

$$\hat{F}(s) = \int_0^1 dx x^{s-1} F\left(\ln \frac{1}{x}\right) = \int_0^\infty dy e^{-ys} F(y) \quad (3.15)$$

then, after simple manipulation with the generating function (3.7) we get

$$\frac{1}{1-z} \hat{F}\left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} F_n z^n \quad (3.16)$$

Thus the analytic formula for moments of $F(y)$ provides a generating function for Laguerre coefficients F_n .

3.1. - Non-singlet case

Let us return to the evolution equation. We start with the non-singlet equation (2.6a) in the leading log approximation :

$$\frac{d}{dt} E_v^{(0)}(t, y) = P_v^{(0)}(y) \otimes E_v^{(0)}(t, y) \quad (3.17)$$

We look for the solution of Eq. (3.17) in the form of a Laguerre series

$$E_v^{(0)}(t, y) = \sum_{n=0}^{\infty} E_n^{(0)}(t) L_n(y) \quad (3.18)$$

Expanding the probability $P_v^{(0)}(y)$

$$P_v^{(0)} = \sum_{n=0}^{\infty} P_n^{(0)} L_n(y) \quad (3.19)$$

inserting the series (3.18) and (3.19) into Eq. (3.17) and using the formula (3.12) we get the system of differential equations for the coefficients $E_n^{(0)}(t)$:

$$\frac{d}{dt} E_n^{(0)}(t) = \sum_{i=0}^n P_{n-i}^{(0)} E_i^{(0)}(t) \quad (3.20)$$

where

$$P_i^{(0)} = P_i^{(0)} - P_{i-1}^{(0)} \quad (P_{-1}^{(0)} \equiv 0)$$

The general functional form of $E_n^{(0)}(t)$ can be derived from Eq. (3.16). We have

$$E_n^{(0)}(t) = \frac{1}{n!} \frac{d^n}{dz^n} \left[e^{t \hat{P}_v^{(0)} \left(\frac{1}{1-z} \right)} \cdot \frac{1}{1-z} \right]_{z=0} \quad (3.21)$$

and therefore

$$E_n^{(0)}(t) = e^{P_0^{(0)} t} \sum_{k=0}^n A_n^{(k)} \frac{t^k}{k!} \quad (3.22)$$

Inserting the expression (3.22) into Eq. (3.20), we get the following recurrence relations for $A_n^{(k)}$:

$$A_n^{(0)} = 1 \quad [\text{from Eq. (2.10)}]$$

$$A_n^{(k+1)} = \sum_{i=k}^{n-1} P_{n-i}^{(0)} A_i^{(k)} \quad (k=0,1,\dots,n-1) \quad (3.23)$$

Thus, we have found the solution of Eq. (3.17) in the form :

$$E_v^{(0)}(t,y) = \sum_{n=0}^{\infty} E_n^{(0)}(t) L_n(y) \quad (3.24)$$

where

$$E_n^{(0)}(t) = e^{\Gamma_0^{(0)} t} \sum_{k=0}^n A_n^{(k)} \frac{t^k}{k!}$$

[with $A_n^{(k)}$ defined by Eqs. (3.23)]

The inclusion of $O(\alpha)$ corrections for the non-singlet case is trivial : one should just perform the convolution of the leading log evolution function $E_v^{(0)}(t,y)$ with $R(y)$ [see Eq. (2.6a)]. Expanding

$$R(y) = \sum_{n=0}^{\infty} R_n L_n(y) \quad (3.25)$$

[for simplicity we suppress $(\pm)$ indices] and using the convolution rule (3.12) we get

$$E(t,y) = E_v^{(0)}(t,y) - \frac{\beta_0}{2} \cdot \frac{d(t) - d(0)}{2\pi} \cdot E^{(0)}(t,y) \equiv \sum_{n=0}^{\infty} E_n(t) L_n(y)$$

$$= \sum_{n=0}^{\infty} \left\{ E_n^{(0)}(t) - \left(\frac{\beta_0}{2}\right) \frac{d(t) - d(0)}{2\pi} \cdot E_n^{(1)}(t) \right\} L_n(y) \quad (3.26)$$

where

$$E_n^{(1)}(t) = \sum_{i=0}^n r_{n-i} E_i^{(0)}(t)$$

$$r_i \equiv R_i - R_{i-1}; \quad R_{-1} \equiv 0 \quad (3.27)$$

In a similar way one can perform the convolution of $E(t,y)$ with the hard cross-section $\hat{\sigma}(y)$.

3.2. - Singlet case

We proceed in a similar way as in the non-singlet case. The leading log evolution matrix $E^{(o)}(t, y)$ is expanded as before

$$E^{(o)}(t, y) = \sum_{n=0}^{\infty} E_n^{(o)}(t) L_n(y) \quad (3.28)$$

where $E_n^{(o)}(t)$ are now 2×2 matrices.

The general form of $E_n^{(o)}(t)$ can be found again by considering the analytic expression for moments. We have

$$\hat{E}^{(o)}(t, s) = e_1(s) e^{\lambda_1(s)t} + e_2(s) e^{\lambda_2(s)t} \quad (3.29)$$

where $e_1(s), e_2(s)$ are projecting matrices :

$$e_1(s) + e_2(s) = \mathbb{1} ; \quad \lambda_1(s) e_1(s) + \lambda_2(s) e_2(s) = \hat{P}^{(o)}(s)$$

$$e_i(s) e_j(s) = \delta_{ij} e_i(s) \quad (3.30)$$

and $\lambda_i(s)$ ($i=1,2$) are the eigenvalues of the matrix $\hat{P}^{(o)}(s)$. We have

$$\lambda_1(1) \equiv \lambda = - \left(\frac{4}{3} C_F + \frac{2}{3} \beta T_R \right)$$

$$\lambda_2(1) = 0 \quad (\text{momentum conservation}) \quad (3.31)$$

and therefore we conclude from Eq. (3.16) that $E_n^{(o)}(t)$ must be of the form :

$$E_n^{(o)}(t) = \sum_{k=0}^n \frac{(\eta t)^k}{k!} [A_n^{(k)} + B_n^{(k)} e^{\lambda t}] \quad (3.32)$$

with η an arbitrary parameter and $A_n^{(k)}, B_n^{(k)}$ 2×2 matrices. They can be derived as in the non-singlet case. The resulting recurrence relations for $A_n^{(k)}, B_n^{(k)}$ can be written down in a compact matrix form by means of the projecting matrices :

$$e_1 \equiv e_1(1) , \quad e_2 \equiv e_2(1) \quad (3.33)$$

We then have

$$A_0^{(0)} = e_2 \quad ; \quad B_0^{(0)} = e_1 \quad (3.34)$$

$$\begin{aligned} A_m^{(k+1)} &= \frac{\lambda}{\eta} e_1 A_m^{(k)} + \frac{1}{\eta} \sum_{i=k}^{n-1} p_{m-i}^{(0)} A_i^{(k)} \\ B_m^{(k+1)} &= -\frac{\lambda}{\eta} e_2 B_m^{(k)} + \frac{1}{\eta} \sum_{i=k}^{n-1} p_{m-i}^{(0)} B_i^{(k)} \end{aligned} \quad \begin{array}{l} m > 0 \\ k = 0, 1, \dots, n-1 \end{array} \quad (3.35)$$

In order to start the recurrences given by Eqs. (3.35) one needs the values of $A_n^{(0)}$, $B_n^{(0)}$. They can be derived as follows :

$$\begin{aligned} A_n^{(0)} &= e_2 - \left(\frac{\eta}{\lambda}\right)^n [e_1 a_n^{(n)} - (-1)^n e_2 b_n^{(n)}] \\ B_n^{(0)} &= e_1 + \left(\frac{\eta}{\lambda}\right)^n [e_1 a_n^{(n)} - (-1)^n e_2 b_n^{(n)}] \end{aligned} \quad (3.36)$$

where $a_n^{(n)}$, $b_n^{(n)}$ should be derived from the relations :

$$\begin{aligned} a_m^{(0)} &= 0 \\ a_m^{(k+1)} &= \frac{\lambda}{\eta} e_1 a_m^{(k)} + \frac{1}{\eta} \sum_{i=k}^{n-1} p_{m-i}^{(0)} A_i^{(k)} \\ b_m^{(0)} &= 0 \\ b_m^{(k+1)} &= -\frac{\lambda}{\eta} e_2 b_m^{(k)} + \frac{1}{\eta} \sum_{i=k}^{n-1} p_{m-i}^{(0)} B_i^{(k)} \end{aligned} \quad (3.37)$$

In order to find the $O(\alpha)$ correction to the singlet evolution matrix, we have to perform the convolutions in Eq. (2.9) :

$$E^{(1)}(t, y) = \sum_{n=0}^{\infty} E_n^{(1)}(t) L_n(y) \quad (3.38)$$

where

$$E_n^{(1)}(t) = \tilde{E}_n^{(1)}(t) - 2\tilde{E}_{n-1}^{(1)}(t) + \tilde{E}_{n-2}^{(1)}(t) \quad (3.39)$$

with

$$\tilde{E}_n^{(1)}(t) = \int_0^t d\tau e^{-\frac{\beta_0}{2}\tau} \sum_{i,j,k} E_i^{(0)}(t-\tau) R_j E_k^{(0)}(\tau) \delta[n-i-j-k] \quad (3.40)$$

3.3 - Convolution with the input distribution

The final step in deriving the Q^2 dependent parton densities or fragmentation functions is to convolute the evolution functions E with some input distributions [Eqs. (2.5)]. If the input is given by an analytic function like, e.g. :

$$F(y) = x^\beta (1-x)^\alpha \quad (3.41)$$

then we can expand it into Laguerre polynomials and calculate the convolution as usual :

$$F(t, y) = \sum_{n=0}^{\infty} F_n(t) L_n(y) = E(t, y) \otimes F(y) \quad (3.42)$$

where

$$F_n(t) = \sum_{i=0}^n F_i e_{n-i}(t) \quad (3.43)$$

$$e_i(t) = E_i(t) - E_{i-1}(t), \quad (E_{-1} \equiv 0)$$

However, a typical situation is when the input is not known but we have just to extract it from the data. In this case one can fit Eqs. (3.42)-(3.43) to the data, treating the Laguerre coefficients F_n , $n = 0, 1, \dots, N$ of the input as free parameters. A more detailed discussion of this point is given in Section 5.

4. - CONVERGENCE OF THE LAGUERRE SERIES

In practical calculations we have to truncate the Laguerre series. The question we are going to answer now is how well the series, truncated at some value of $n = N$, approximates the function.

For this reason, let us consider the following problem : given the function $F(y)$, find the polynomial approximation $\tilde{F}(y)$ to it :

$$\tilde{F}(y) = \sum_{n=0}^N \tilde{F}_n y^n \quad (4.1)$$

which is optimal in the sense that the mean squared deviation is as small as possible :

$$\int_0^{\infty} dy e^{-y} |F(y) - \tilde{F}(y)|^2 = \int_0^1 dx |F(\ln \frac{1}{x}) - \tilde{F}(\ln \frac{1}{x})|^2 = \text{minimum} \quad (4.2)$$

The polynomial $\tilde{F}(y)$ can be expressed in terms of Laguerre polynomials :

$$\tilde{F}(y) = \sum_{n=0}^N \tilde{F}_n L_n(y) \quad (4.3)$$

Inserting the expression (4.3) into (4.2) we immediately find, after differentiating over $\tilde{F}_n$ and using the orthogonality of Laguerre polynomials [Eq. (3.5)], that the minimum is obtained if

$$\tilde{F}_n = F_n, \quad n = 0, 1, 2, \dots \quad (4.4)$$

where F_n are the Laguerre coefficients of the function $F(y)$ [Eq. (3.10)]. In other words, the truncated Laguerre series

$$\tilde{F}(y) = \sum_{n=0}^N F_n L_n(y) \quad (4.5)$$

gives the best (among all polynomials of order N in y) least squared fit to the function

$$F(y) = \sum_{n=0}^{\infty} F_n L_n(y) \quad (4.6)$$

Consequently, if the function $F(y)$ does not oscillate very rapidly, its Laguerre series converges quickly and uniformly in y . If the function has strong variations, the convergence is fast in the region where the function is large and slower in the region where the function is close to zero [since the latter region gives a small contribution to the integral (4.2)].

In practice, one should always check the convergence by observing the behaviour of $\tilde{F}$ with growing N . Typically, the successive approximations oscillate around the exact value with decreasing amplitude so that it is relatively easy to decide when the series can be truncated.

The above remarks are valid for sufficiently regular functions. The Laguerre series for the evolution functions $E^{(0)}(t, y)$ should be treated with care since $E^{(0)}(t, y)$ is singular for $t = 0$ due to the initial condition ; e.g. :

$$E_v^{(0)}(t, y)_{|t=0} = \delta(y) = \sum_{n=0}^{\infty} L_n(y) \quad (4.7)$$

i.e., $E_n^{(0)}(0) = 1$ for any n . The convergence is now very poor for obvious reasons : we are trying to construct a polynomial approximation to the δ function. On the other hand, after convoluting E with some regular function F , as, e.g., in Eq. (3.42), only the differences $e_n(t)$ enter into the final result. Now $e_n(0) = \delta_{n0}$ and we get from Eq. (3.43) $F_n(0) = F_n$ as it should be. Thus there is no problem with convergence after the Laguerre series of the singular function has been convoluted with a regular function. The same phenomenon occurs for other singular functions like $(1/1-x)_+$, $(\ln(1-x)/(1-x))_+$. If one insists on calculating the evolution function $E(t, y)$ itself, it can be done after smearing it over a small interval in y :

$$\int_{y_1}^{y_2} dy E(t, y) = \sum_{n=0}^{\infty} e_n(t) [L_n(y_2) - L_n(y_1)] \quad (4.8)$$

[we have used Eq. (3.8) for $m = 0$] so that again only the differences $e_n(t)$ enter. If the function $F(y)$ vanishes for $y \rightarrow 0$, $(x \rightarrow 1)$, the convergence in this region may be poor, as discussed above. If the asymptotic behaviour is like

$$F(y) \underset{(y \rightarrow 0)}{\sim} y^\alpha \sim (1-x)^\alpha \quad (4.9)$$

one can improve the convergence by extracting the term y^α explicitly.

We then force $F(y)$ to be of the form

$$F(y) = \sum_{n=0}^{\infty} F_n L_n(y) = \frac{y^{\alpha-1}}{\Gamma(\alpha)} \otimes \sum_{n=0}^{\infty} F_n^{(\alpha)} L_n(y) \quad (4.10)$$

and, using the convolution rule (3.12) we derive the coefficients $F_n^{(\alpha)}$ in terms of F_n as follows :

$$\begin{aligned} F_0^{(\alpha)} &= F_0 \\ F_n^{(\alpha)} &= F_n - \sum_{i=0}^{n-1} F_i^{(\alpha)} \alpha_{n-i} \end{aligned} \quad (4.11)$$

where

$$\alpha_0 = 1, \quad \alpha_1 = -\alpha$$

$$\alpha_{n+1} = \frac{n-\alpha}{n+1} \alpha_n$$

The convolution in Eq. (4.10) can be performed analytically and we get

$$F(y) = \frac{y^\alpha}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} F_n^{(\alpha)} \mathcal{L}_n^{(\alpha)}(y) \quad (4.12)$$

where the polynomials $\mathcal{L}_n^{(\alpha)}(y)$ are related in a simple way to the generalized Laguerre polynomials $L_n^{(\alpha)}(y)$ ¹⁴⁾:

$$\mathcal{L}_n^{(\alpha)}(y) = \frac{\Gamma(\alpha+1) \Gamma(n+1)}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(y) \quad (4.13)$$

and are given by the recurrence relations

$$\mathcal{L}_0^{(\alpha)}(y) = 1, \quad \mathcal{L}_1^{(\alpha)}(y) = 1 - \frac{y}{\alpha+1}$$

$$(n+1) \mathcal{L}_{n+1}^{(\alpha)}(y) = (2n+\alpha+1-y) \mathcal{L}_n^{(\alpha)}(y) - n \mathcal{L}_{n-1}^{(\alpha)}(y) \quad (4.14)$$

The above method also provides a very efficient way of improving the convergence of $E(t,y)$ for small t . We know ¹⁵⁾ that the diagonal evolution functions $E_v^{(0)}, E_{11}^{(0)}, E_{22}^{(0)}$ are singular for small t and $y \rightarrow 0$:

$$\begin{aligned} E_v^{(0)}(t,y) &= E_{11}^{(0)}(t,y) = \frac{y^{2c_F t - 1}}{\Gamma(2c_F t)} e^{a_F t} [1 + O(y)] \\ E_{22}^{(0)}(t,y) &= \frac{y^{2c_G t - 1}}{\Gamma(2c_G t)} e^{a_G t} [1 + O(y)] \end{aligned} \quad (4.15)$$

Using the expansion (4.12), we can extract the singular part $y^{\alpha-1}$ in the above equations as follows:

$$E(t,y) = \frac{y^{\alpha-1}}{\Gamma(\alpha)} e^{at} + \frac{y^\alpha}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} [E_n^{(\alpha)}(t) - e^{at}] \mathcal{L}_n^{(\alpha)}(y) \quad (4.16)$$

where $\alpha = 2C_F t$, $a = a_F$ or $\alpha = 2C_G t$, $a = a_G$. The convergence of the series in Eq. (4.16) is extremely fast for all values of t and y relevant in practical applications.

5. - FITTING PROCEDURE

Here we discuss the method of analyzing the deep inelastic data using the Laguerre technique developed in this paper.

The QCD prediction for any structure function $F(t,y)$ in any one arm hard process like $l + h \rightarrow l' + x$, $e^+e^- \rightarrow h + x$, etc., can be written in the following generic form [see Eq. (3.44)] :

$$F(t,y) = \sum_{n=0}^{\infty} F_n(t) L_n(y) \quad (5.1)$$

where

$$F_n(t) = \sum_{i=0}^n F_i e_{n-i}(t) \quad (5.2)$$

Here F_i are the Laguerre coefficients of the input distributions and $e_i(t)$ are the Laguerre coefficients of appropriate combinations of evolution functions and hard cross-sections.

The Laguerre series in Eq. (5.1) can be rewritten as follows :

$$F(t,y) = \sum_{n=0}^N F_n \psi_n(t,y) \quad (5.3)$$

where

$$\psi_n(t,y) = \sum_{k=0}^K e_k(t) L_{n+k}(y) \quad (5.4)$$

As seen, in Eqs. (5.3), (5.4), both series have been truncated. The function $\psi_n(t,y)$, describing the Q^2 evolution of a given Laguerre coefficient F_n is uniquely determined by perturbative QCD. Typically, it is enough to take $K \sim 20$ in order to reproduce the functions $\psi_n(t,y)$ with an accuracy better than 0.5%.

The choice of the parameter N in Eq. (5.3) is more subtle and will be discussed in the following.

The relation of the coefficients F_n to the Laguerre coefficients of the functions $\tilde{q}^{(-)}$, $\tilde{q}^{(+)}$, and $\tilde{G}$ in Eq. (2.5) depends on a particular structure function. For example, for $F_3^{VI} + F_3^{\bar{VI}}$ (I = isoscalar) we have

$$\vec{F} \equiv (F_0, F_1, \dots, F_N) = (\tilde{q}_0^{(-)}, \tilde{q}_1^{(-)}, \dots, \tilde{q}_N^{(-)}) \quad (5.5)$$

and for $F_2^{VI} + F_2^{\bar{VI}}$ we have

$$\vec{F} \equiv (F_0, F_1, \dots, F_N) = (\tilde{q}_0^{(+)}, \tilde{q}_1^{(+)}, \dots, \tilde{q}_M^{(+)}, \tilde{G}_0, \tilde{G}_1, \dots, \tilde{G}_{N-M}) \quad (5.6)$$

We are going to fit the formula (5.3) to the data, treating Λ [see Eq. (1.4)] and the coefficients F_n of the input as free parameters.

From the experiment, we have the values of $F(t, y)$ for a set of points t_i, y_i . We will use a short notation

$$F(t_i, y_i) \equiv F(i) ; \quad \psi_n(t_i, y_i) \equiv \psi_n(i)$$

Typically, the data points are the averaged values of $F(t, y)$ over some intervals in y and Q^2 . The averages in y of Eq. (5.3) can also be derived analytically :

$$\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} dx F(t, y) = \frac{1}{x_2 - x_1} \sum_{n=0}^{\infty} \{ x_1 p_{n+1}(t) L_n(y_2) - x_2 p_{n+1}(t) L_n(y_1) \} \\ \text{with } p_i \equiv F_i - F_{i-1} \quad (F_{-1} \equiv 0) \quad (5.7)$$

The averages over Q^2 must be done numerically.

In order to fit the expression (5.3) to the data we have to minimize χ^2 , constructed as follows :

$$\chi^2(\Lambda, \vec{F}) = \sum_{i=1}^I \frac{1}{\sigma_i^2} \left[\sum_{n=0}^N F_n \psi_n(\Lambda, i) - D_i \right]^2 \quad (5.8)$$

where D_i are experimental values, $\sigma_i^2 = \langle D_i^2 \rangle - \langle D_i \rangle^2$ and I is the number of data points. Equation (5.8) can be rewritten as follows :

$$\chi^2(\lambda, \vec{F}) = \frac{1}{\sigma_0^2} [F^T A F - 2 F^T B + \chi_0^2] \quad (5.9)$$

where

$$\begin{aligned} A_{nm}(\lambda) &= \sum_{i=0}^I \omega_i \psi_n(\lambda, i) \psi_m(\lambda, i) \\ B_n(\lambda) &= \sum_{i=0}^I \omega_i \psi_n(\lambda, i) D_i \\ \chi_0^2 &= \sum_{i=0}^I \omega_i D_i^2 \\ \omega_i &= \frac{\sigma_0^2}{\sigma_i^2} \end{aligned} \quad (5.10)$$

and σ_0^2 is some reference error (e.g., the smallest one). In Eq. (5.9) the matrix notation is implicit ; T denotes the transposition.

The condition for the minimum of χ^2 reads :

$$\frac{d\chi^2}{d\lambda} = F^T \left(\frac{d}{d\lambda} A(\lambda) \right) F - 2 F^T \frac{d}{d\lambda} B(\lambda) = 0 \quad (5.11)$$

$$\frac{d\chi^2}{dF_n} = 2 (AF - B)_n = 0 \quad (5.12)$$

Since χ^2 is quadratic in $\vec{F}$, there exists a minimum of χ^2 with respect to $\vec{F}$ for any value of λ , if only $\det A \neq 0$.

Therefore, in order to find the global minimum, we can first minimize χ^2 with respect to $\vec{F}$, i.e., to take $\vec{F} = \vec{F}(\lambda)$, where

$$\vec{F}(\lambda) = \bar{A}^{-1}(\lambda) B(\lambda) \quad (5.13)$$

and then find the best value of λ , minimizing

$$\chi^2(\lambda) = \frac{1}{\sigma_0^2} [F^T(\lambda) A(\lambda) F(\lambda) - 2 F^T(\lambda) B(\lambda) + \chi_0^2] \quad (5.14)$$

Therefore, our multiparameter fit can be reduced to solving a system of linear equations (5.12) and to one parameter minimization.

The value of N in Eq. (5.3) should be, in principle, as large as possible if we are going to determine the input distribution with high precision. However, the solution of the system (5.12) may exist only if $N \leq I$; otherwise $\det A = 0$. It provides a natural upper bound on N . In practice, it happens that if N is close to I , the matrix A is already "almost" singular which creates numerical difficulties. The comfortable situation is when

$$1 \ll N \ll I \quad (5.15)$$

which is only possible when we have a large number of data points.

However, there is also another, more subtle, upper bound on N . Imagine that we proceed as follows : we solve Eq. (5.12) for $N = 1, 2, 3, \dots$ and observe the N dependence of the reduced χ^2 :

$$\bar{\chi}^2(\Lambda, N) = \frac{1}{I - N} \chi^2(\Lambda, N) \quad (5.16)$$

It should first decrease rather quickly with N and then become more or less stable. Further increasing of N is pointless since then our fitted input distribution would become sensitive to the fluctuations of the individual data points.

Therefore, a reasonable procedure is to solve Eq. (5.12) iteratively as described above and to stop where $\bar{\chi}^2(\Lambda, N)$ begins to stabilize.

An efficient way of inverting the matrix A is then the following :

a) $N = 1$

$$(A^{-1})_{11} \equiv \bar{\chi}_{11}^{\text{OLD}} = \frac{1}{A_{11}}, \quad b_1 \equiv A_{11} \quad (5.17)$$

b) $N = n+1$; ($n = 1, 2, \dots$)

the $n \times n$ portion of the matrix is already inverted and is denoted by z_{ik}^{old} , $i, k \leq n$. We calculate the inverse of the $(n+1) \times (n+1)$ portion of A (denoted as z_{ik}^{new} ; $i, k \leq n+1$) as follows :

$$\begin{aligned}
 \Lambda_k &= \sum_{i=1}^N \tilde{Z}_{ki}^{\text{OLD}} A_{i,n+1} \quad (k=1,2,\dots,n) \\
 \eta_{n+1} &= A_{n+1,n+1} - \sum_{i=1}^n A_{n+1,i} \sigma_i \\
 \tilde{Z}_{n+1,n+1}^{\text{NEW}} &= 1/\eta_{n+1} \\
 \tilde{Z}_{i,n+1}^{\text{NEW}} &= \tilde{Z}_{n+1,i}^{\text{NEW}} = (-1/\eta_{n+1}) \sigma_i \\
 \tilde{Z}_{ik}^{\text{NEW}} &= \tilde{Z}_{ik}^{\text{OLD}} - \sigma_i \tilde{Z}_{n+1,k}^{\text{NEW}}
 \end{aligned} \tag{5.18}$$

Now we calculate $\bar{\chi}^2(\Lambda, n+1)$ and decide whether it has been decreased significantly with respect to $\bar{\chi}^2(\Lambda, n)$. If so, we increase n by 1 and repeat point b). If not, we stop the iteration. Now we change Λ and repeat the full inversion. After doing it for some values of Λ , we find the global minimum of $\bar{\chi}^2(\Lambda)$, e.g., by fitting the second order polynomial in Λ to derived values of $\bar{\chi}^2(\Lambda)$.

At each step of iteration one can verify whether we are still in the safe region $\det A \neq 0$, since

$$(\det A)_{n \times n \text{ portion}} = \prod_{i=1}^n \eta_i \tag{5.19}$$

After finding the minimum value $\Lambda = \Lambda_0$ we calculate the covariance matrix C as follows :

$$\begin{aligned}
 \sigma^2(\Lambda) &= \langle \Lambda^2 \rangle - \langle \Lambda \rangle^2 = \left[\frac{d^2 \bar{\chi}^2(\Lambda)}{d\Lambda^2} \right]^{-1} \\
 C(F_n, \Lambda) &= \langle F_n \Lambda \rangle - \langle F_n \rangle \langle \Lambda \rangle = \frac{dF_n(\Lambda)}{d\Lambda} \cdot \sigma^2(\Lambda) \\
 C(F_n, F_m) &= \langle F_n F_m \rangle - \langle F_n \rangle \langle F_m \rangle = \sigma_0^2 \tilde{Z}_{nm} + \sigma^2(\Lambda) \left(\frac{dF_n(\Lambda)}{d\Lambda} \right) \left(\frac{dF_m(\Lambda)}{d\Lambda} \right)
 \end{aligned} \tag{5.20}$$

The derivatives on the right-hand side of Eqs. (5.20) should be calculated for $\Lambda = \Lambda_0$. They can be calculated numerically during the search for χ_0 .

The final results of the fit are :

$$\Lambda = \Lambda_0 \pm \sigma(\Lambda)$$

$$F(y) = \sum_{n=0}^N F_n(\Lambda_0) L_n(y) \pm \Delta F(y) \quad (5.21)$$

where

$$\Delta F(y) = \left\{ \sum_{n=0}^N \sum_{m=0}^N L_n(y) C(F_n, F_m) L_m(y) \right\}^{1/2} \quad (5.22)$$

Now the value of $F(t, y)$ can be predicted for any value of t and y as follows :

$$F(t, y) = \sum_{n=0}^N F_n(\Lambda_0) \psi_n(t, y) \pm \Delta F(t, y) \quad (5.23)$$

where

$$\Delta F(t, y) = \left\{ \sigma_{FF}^2 + \sigma_{F\Lambda}^2 + \sigma_{\Lambda\Lambda}^2 \right\}^{1/2}$$

$$\sigma_{FF}^2 = \sum_{n,m} \psi_n(t, y) C(F_n, F_m) \psi_m(t, y)$$

$$\sigma_{F\Lambda}^2 = 2 \sum_{n,m} \psi_n(t, y) C(F_n, \Lambda) F_m(\Lambda_0) \left[\frac{d}{d\Lambda} \psi_m(t, y) \right]_{\Lambda=\Lambda_0} \quad (5.24)$$

$$\sigma_{\Lambda\Lambda}^2 = \sum_{n,m} \left(\frac{d}{d\Lambda} \psi_n(t, y) \right)_{\Lambda=\Lambda_0} \cdot \left(\frac{d}{d\Lambda} \psi_m(t, y) \right)_{\Lambda=\Lambda_0} \sigma^2(\Lambda)$$

If the structure function contains the gluon input distribution [Eq. (5.6)], one can test the significance of the gluon term by varying M in Eq. (5.6). A simple check consists of the comparison of two values of $\bar{\chi}^2(\Lambda_0, N)$: one derived for $M = N/2$ and the other one for $M = N$. Both fits have the same number of free parameters ; nevertheless, if QCD is right, the first fit should give a definitely lower value of $\bar{\chi}^2$.

APPENDIX - LAGUERRE COEFFICIENTS OF PROBABILITIES

The coefficients

$$P_m = \int_0^\infty dy e^{-y} L_m(y) P(y) \quad (A.1)$$

can be derived analytically using the explicit expression for Laguerre polynomials [(3.4)]. We have

$$P_m = \sum_{k=0}^m \binom{m}{k} (-1)^k \hat{P}_k, \quad \hat{P}_k = \int_0^\infty dy e^{-y} \frac{y^k}{k!} P(y). \quad (A.2)$$

In this Appendix, we present the formulae for the coefficients $\hat{P}_k$ for all functions appearing up to a two-loop level. Let us introduce the following notation :

$$\begin{aligned} F_j^{(+)}(x) &= \frac{1}{1+x} f_j(x) \\ F_j^{(-)}(x) &= \frac{1}{1-x} f_j(x) \\ F_j^{(r)}(x) &= x^r f_j(x) \quad (r=0,1,2,3) \end{aligned} \quad (A.3)$$

Now, any function obtained from two-loop calculations in perturbative QCD can be written as a linear combination of functions $F_j^{(+)}$, $F_j^{(-)}$, $F_j^{(r)}$, where $f_j(x)$ denotes one of the following functions :

$$\begin{aligned} f_j(x) &= \{ \delta(1-x), 1, \ln(1-x), \ln^2(1-x), \ln x, \ln^2 x, \ln x \ln(1-x), \Phi_1(x), \Phi_2(x) \} \\ \text{e.g. } f_3(x) &= \ln(1-x), f_9(x) = \Phi_2(x) \end{aligned} \quad (A.4)$$

$\Phi_1(x)$, $\Phi_2(x)$ are the Spence functions :

$$\begin{aligned} \Phi_1(x) &= \int_x^{1/x} \frac{dz}{1-z} \ln z = -\frac{1}{2} \ln^2 x - \frac{\pi^2}{3} + 2 \ln x \ln(1-x) + 2 \text{Li}_2(x) \\ \Phi_2(x) &= \int_x^{1/x} \frac{dz}{1+z} \ln z = \frac{1}{2} \ln^2 x - \frac{\pi^2}{6} - 2 \ln x \ln(1+x) - 2 \text{Li}_2(-x) \end{aligned} \quad (A.5)$$

and

$$\text{Li}_2(x) = - \int_0^x \frac{dz}{z} \ln(1-z) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad (\text{A.6})$$

The Laguerre coefficients of all functions in Eq. (A.3) can be expressed by the following sums :

$$\begin{aligned} \zeta_k &= \sum_{i=1}^{\infty} \frac{1}{i^k} && (\text{Rieman zeta function}) \\ \eta_k &= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^k} = \alpha_k \zeta_k && \alpha_k = 1 - \frac{1}{2^{k-1}} \\ \zeta_k(\tau) &= \sum_{i=1}^{\tau} \frac{1}{i^k} \\ \eta_k(\tau) &= \sum_{i=1}^{\tau} \frac{(-1)^{i+1}}{i^k} \\ \zeta_k^{(\alpha)} &= \sum_{i=1}^{\infty} \frac{1}{i^k} \zeta_{\alpha}(i) \\ \eta_k^{(\alpha)} &= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^k} \zeta_{\alpha}(i) \\ \sigma_1(k, \tau) &= \sum_{i=1}^{\infty} \frac{1}{i(i+\tau+1)^{k+1}} \\ \sigma_2(k, \tau) &= \sum_{i=1}^{\infty} \frac{(-1)^i}{i(i+\tau+1)^{k+1}} \\ \sigma_3(k, \tau) &= \sum_{i=1}^{\infty} \frac{1}{i^2(i+\tau+1)^{k+1}} \\ \sigma_4(k, \tau) &= \sum_{i=1}^{\infty} \frac{(-1)^i}{i^2(i+\tau+1)^{k+1}} \\ \sigma_5(k, \tau) &= \sum_{i=1}^{\infty} \frac{\zeta_1(i)}{i(i+\tau+1)^{k+1}} \end{aligned}$$

(A.7)

The functions $\sigma_i(k, t)$ can in fact be derived by using the recurrence relations, without resumming the infinite series given above. The relations are the following :

$$\begin{aligned}
 \sigma_1(0, \tau) &= \frac{1}{\tau+1} \bar{z}_1(\tau+1) \\
 \sigma_1(k+1, \tau) &= \frac{1}{\tau+1} [\sigma_1(k, \tau) - \{\bar{z}_{k+2} - \bar{z}_{k+2}(\tau)\}] \\
 \sigma_2(0, \tau) &= -\frac{1}{\tau+1} [\eta_1 + (-1)^\tau \{\eta_1 - \eta_1(\tau+1)\}] \\
 \sigma_2(k+1, \tau) &= \frac{1}{\tau+1} [\sigma_2(k, \tau) + (-1)^{\tau+1} \{\eta_{k+2} - \eta_{k+2}(\tau+1)\}] \\
 \sigma_3(0, \tau) &= \frac{1}{\tau+1} [\bar{z}_2 - \frac{1}{\tau+1} \bar{z}_1(\tau+1)] \\
 \sigma_3(k+1, \tau) &= \frac{1}{\tau+1} [\sigma_3(k, \tau) - \sigma_1(k+1, \tau)] \\
 \sigma_4(0, \tau) &= -\frac{1}{\tau+1} [\eta_2 - \frac{1}{\tau+1} \{\eta_1 + (-1)^\tau [\eta_1 - \eta_1(\tau+1)]\}] \\
 \sigma_4(k+1, \tau) &= \frac{1}{\tau+1} [\sigma_4(k, \tau) - \sigma_2(k+1, \tau)] \\
 \sigma_5(0, \tau) &= \frac{1}{2(\tau+1)} [\bar{z}_1^2(\tau) + \bar{z}_2(\tau) + 2\bar{z}_2] \\
 \sigma_5(k+1, \tau) &= \frac{1}{\tau+1} [\sigma_5(k, \tau) - \frac{1}{2}(k+2) \{\bar{z}_{k+3} - \bar{z}_{k+3}(\tau)\} \\
 &\quad - \bar{z}_1(\tau) \{\bar{z}_{k+2} - \bar{z}_{k+2}(\tau)\} + \\
 &\quad + \frac{1}{k+1} \sum_{i=1}^k i \{\bar{z}_{i+1} - \bar{z}_{i+1}(\tau)\} \{\bar{z}_{k-i+2} - \bar{z}_{k-i+2}(\tau)\}]
 \end{aligned}$$

(A.8)

Let us introduce the notation

$$\begin{aligned}
 F_j^{(+)}(k) &= \int_0^\infty dy e^{-y} \frac{y^k}{k!} F_j^{(+)}(e^{-y}), \\
 F_j^{(-)}(k) &= \int_0^\infty dy e^{-y} \frac{y^k}{k!} F_j^{(-)}(e^{-y}).
 \end{aligned}$$

(A.9)

The coefficients $F_j^{(\pm, r)}(k)$ are given by the following expressions :

$$F_1^{(+)}(k) = \delta_{k0}$$

$$F_2^{(+)}(k) = \left(\frac{1}{n+1}\right)^{k+1}$$

$$F_3^{(+)}(k) = -\delta_1(k, r)$$

$$F_4^{(+)}(k) = 2[\delta_5(k, r) - \delta_3(k, r)]$$

$$F_5^{(+)}(k) = -(k+1) \left(\frac{1}{n+1}\right)^{k+2}$$

$$F_6^{(+)}(k) = (k+1)(k+2) \left(\frac{1}{n+1}\right)^{k+3}$$

$$F_7^{(+)}(k) = (k+1) \delta_1(k+1, r)$$

$$F_8^{(+)}(k) = 2[(k+1) \delta_1(k+1, r) + \delta_3(k, r)] - 2\zeta_2 \left(\frac{1}{n+1}\right)^{k+1} - \frac{1}{2} (k+1)(k+2) \left(\frac{1}{n+1}\right)^{k+3}$$

$$F_9^{(+)}(k) = -2[(k+1) \delta_2(k+1, r) + \delta_4(k, r)] - \zeta_2 \left(\frac{1}{n+1}\right)^{k+1} + \frac{1}{2} (k+1)(k+2) \left(\frac{1}{n+1}\right)^{k+3}$$

$$F_2^{(-)}(k) = \theta(k) \zeta_{k+1}$$

$$F_3^{(-)}(k) = \theta(k) [\zeta_{k+2} - \zeta_{k+1}^{(1)}]$$

$$F_5^{(-)}(k) = -(k+1) \zeta_{k+2}$$

$$F_6^{(-)}(k) = (k+1)(k+2) \zeta_{k+3}$$

$$F_7^{(-)}(k) = (k+1) [\zeta_{k+2}^{(1)} - \zeta_{k+3}]$$

$$F_2^{(+)}(k) = \eta_{k+1}$$

$$F_5^{(+)}(k) = -(k+1) \eta_{k+2}$$

$$F_6^{(+)}(k) = (k+1)(k+2) \eta_{k+3}$$

$$F_9^{(+)}(k) = \frac{1}{2} (k+1)(k+2) \eta_{k+3} - \zeta_2 \eta_{k+1} - 2(k+1) [\eta_{k+2}^{(1)} - \eta_{k+3}] - 2\eta_{k+1}^{(2)} + 2\eta_{k+3}$$

$$\theta(k) = \begin{cases} 0 & \text{if } k=0 \\ 1 & \text{if } k>0 \end{cases}$$

(A.10)

The functions absent in the above list (e.g., $F_3^{(+)}$, etc.) do not appear in the QCD results in one or two-loop calculations.

After applying the recurrence relations for $\sigma_i(k,t)$, all the coefficients $F_j^{(\pm,r)}(k)$ can be derived as finite sums, containing the numbers $\zeta_k, \eta_k, \zeta_k(r), \eta_k(r), \zeta_k^{(\alpha)}, \eta_k^{(\alpha)}$, $\alpha = 1, 2$. The values of ζ_k are tabulated [see, e.g., Ref. 16] and the constants η_k can be expressed by ζ_k :

$$\eta_1 = \ln 2, \quad \eta_k = \alpha_k \zeta_k \quad (k > 1)$$

where

$$\alpha_k = 1 - \frac{1}{2^{k-1}} \quad (A.11)$$

The numbers $\zeta_k(r)$ and $\eta_k(r)$ are given by finite sums. The constants $\zeta_k^{(1)}$ can be derived from the recurrence relations:

$$\begin{aligned} \zeta_2^{(1)} &= 2\zeta_3 \\ \zeta_k^{(1)} &= (1 + \frac{1}{2}k) \zeta_{k+1} - \frac{1}{k-1} \sum_{i=1}^{k-2} i \zeta_{i+1} \zeta_{k-1} \quad (k > 2) \end{aligned} \quad (A.12)$$

We are not able to express the constants $\eta_k^{(1)}, \eta_k^{(2)}, \zeta_k^{(2)}$ by finite sums. The convergence of the series expansions for them given above can be, however, substantially improved by means of the Euler-Maclaurin summation formula. We get

$$\begin{aligned} \eta_k^{(1)} &= \alpha_k \zeta_k^{(1)} + \sum_{i=1}^{\infty} \frac{1}{i^k} \left[(-1)^{i+1} - \alpha_k \right] \left[\zeta_1(i) - \left\{ \ln i + \gamma + \frac{1}{2i} - \sum_{n=1}^R \frac{B_{2n}}{2n i^{2n}} \right\} \right] \\ &\quad + \frac{1}{2} [\eta_{k+1} - \alpha_k \zeta_{k+1}] - \sum_{n=1}^R \frac{B_{2n}}{2n} [\eta_{2n+k} - \alpha_k \zeta_{2n+k}] - \frac{\ln 2}{2^{k-1}} \zeta_k \quad (k > 1) \\ \eta_k^{(2)} &= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^k} \left[\zeta_2(i) - \left\{ \zeta_2 - \frac{1}{i} + \frac{1}{2i^2} - \sum_{n=1}^R \frac{B_{2n}}{i^{2n+1}} \right\} \right] \\ &\quad + \zeta_2 \eta_k - \eta_{k+1} + \frac{1}{2} \eta_{k+2} - \sum_{n=1}^R B_{2n} \eta_{k+2n+1} \\ \zeta_k^{(2)} &= \sum_{i=1}^{\infty} \frac{1}{i^k} \left[\zeta_2(i) - \left\{ \zeta_2 - \frac{1}{i} + \frac{1}{2i^2} - \sum_{n=1}^R \frac{B_{2n}}{i^{2n+1}} \right\} \right] \\ &\quad + \zeta_2 \zeta_k - \zeta_{k+1} + \frac{1}{2} \zeta_{k+2} - \sum_{n=1}^R B_{2n} \zeta_{k+2n+1} \\ \eta_1^{(1)} &= \eta_2 - \frac{1}{2} \ln^2 2, \quad \eta_1^{(2)} = \zeta_3 - \frac{1}{2} \zeta_2 \ln 2. \end{aligned} \quad (A.13)$$

The constants B_{2r} are Bernoulli numbers

$$B_{2r} = 2\zeta_{2r}(-1) \frac{(2r)!}{(2\pi)^{2r}} \quad (\text{A.14})$$

Taking $R = 7$ and truncating the series for $i = 20$, one obtains 15 digits accuracy for any k .

The final step is to calculate the sum in Eq. (A.2), using the values of $\hat{P}_k$, derived from the formulae given above. Here one should be careful when n is large. Since P_n is then given by an oscillating sum of large numbers, the rounding errors start to be essential.

In order to reduce the rounding errors, one can calculate the sum in (A.2) as follows : if we define

$$P(m, i) = \sum_{k=0}^m \binom{m}{k} (-1)^k \hat{P}_{k+i} \quad (\text{A.15})$$

then

$$\begin{aligned} \text{a) } P_m &= P(m, 0) \\ \text{b) } P(m, i) &\text{ can be derived in a recursive way as follows :} \quad (\text{A.16}) \\ 1) \quad P(0, i) &= \hat{P}_i, \quad i=0, 1, \dots, n \\ 2) \quad P(m+1, i) &= P(m, i) - P(m, i+1). \end{aligned}$$

In this way one avoids the subtraction of large numbers.

For some coefficients $F_j^{(\pm, r)}(k)$, the sum in Eq. (A.15) can be performed in a trivial way. For example, the Laguerre coefficients of the leading log probabilities can be expressed by one non-trivial sum :

$$\mathcal{F}(n) = \theta(n) \sum_{k=1}^n \binom{n}{k} (-1)^k \zeta_{k+1}$$

in the following way :

$$(P_{11})_n = -\frac{4}{3}C_F\delta_{m0} + \Theta(n) \left[2C_F \left\{ Z(n) + \frac{1}{4} \left(3 - \left(\frac{1}{2} \right)^n - \left(\frac{2}{3} \right)^{n+1} \right) \right\} \right]$$

$$(P_{12})_n = 2f_{TR} \left[\left(\frac{1}{2} \right)^{n+1} - \left(\frac{2}{3} \right)^{n+1} + \frac{2}{3} \left(\frac{3}{4} \right)^{n+1} \right]$$

$$(P_{21})_n = 2C_F \left[-\left(\frac{1}{2} \right)^{n+1} + \frac{1}{4} \left(\frac{2}{3} \right)^{n+1} \right]$$

$$(P_{22})_n = -\frac{2}{3}f_{TR}\delta_{m0} + \Theta(n) \left[2C_G \left\{ Z(n) - \left(\frac{1}{2} \right)^n + \frac{1}{2} \left(\frac{2}{3} \right)^{n+1} - \frac{1}{3} \left(\frac{3}{4} \right)^{n+1} \right\} + \frac{11}{6}C_G - \frac{2}{3}f_{TR} \right].$$

In general, however, the way of presentation of the results collected in this Appendix is more convenient for computer calculations.

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