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THE $N \rightarrow 0$ LIMIT OF A NON-ABELIAN GAUGE THEORY :
A MODEL FOR SELF-AVOIDING RANDOM SURFACES

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A B S T R A C T

We show that the $N \rightarrow 0$ limit of a non-Abelian gauge theory is described by an ensemble of self-avoiding random surfaces enforcing the connection between gauge and random surface theories.

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The relation between Brownian motion and scalar field theories is already well known¹⁾. The $\lambda\phi^4$ model, for instance, corresponds to a random walk with a repulsive potential which inhibits the intersections of the motion. In this context, there are relevant limits: the $O(N)$ non-linear σ model in the $N \rightarrow 0$ limit describes the self-avoiding random walk without intersections²⁾; the opposite limit $N \rightarrow \infty$, which is the Stanley model³⁾, becomes a free field theory. It has been conjectured that gauge theories are intimately related to the physics of interacting random surfaces⁴⁾, leading to interesting assumptions about lower and upper critical dimensions. The lower critical dimension d_C^L is the dimension below which the only phase transition occurs at zero coupling constant⁵⁾⁻⁷⁾ and if $d_C^L = 4$ confinement is the only phase for $d \leq 4$; the upper one d_L^U is the dimension above which the theory becomes free and mean field approximation holds^{*}).

In a recent work⁹⁾ the $U(N)$ lattice gauge theory was formulated in the $N \rightarrow \infty$ limit in terms of a statistical ensemble of non-interacting random surfaces. We show that the opposite limit $N \rightarrow 0$ leads to the theory of the self-avoiding random surfaces. This new connection between gauge theories and random surfaces emphasizes the similarity of gauge and spin systems^{5),6)} and suggests the possibility of extracting information about gauge models by studying an ensemble of interacting random surfaces.

We begin by recalling the self-avoiding random walk case,^{1),2)} as it is treated in polymer physics, in order to introduce a useful formalism, to be extended later to the random surface problem. A polymer is a flexible chain with a short distance infinite repulsive potential and the main task in polymer physics is the counting of the L step paths joining two lattice sites; this is a problem which is conveniently formulated in terms of the correlation functions of a suited field theory model. Let us consider for this purpose an N component real vector $\vec{S}_x$ defined on the sites x of a d dimensional lattice Λ , normalized as:

$$\vec{S}_x^2 = N \quad (1)$$

The action (energy) of the system

$$A(s) = -\beta \sum_{\langle xx' \rangle} \vec{S}_x \cdot \vec{S}_{x'} \quad (2)$$

where $\langle xx' \rangle$ are nearest neighbour lattice sites, defines the usual non-linear $O(N)$ σ model. Now we shall see that the self-avoiding random walk arises in the $N \rightarrow 0$ limit. In order to show this, we perform a strong coupling expansion of the partition function for low β

^{*}) For a treatment from this point of view, see Refs. 6) and 8).

$$Z = \int d\vec{S} e^{-A} = \int d\vec{S} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\beta \sum_{\langle x x' \rangle} \vec{S}_x \cdot \vec{S}_{x'} \right)^n \quad (3)$$

A general term of this expansion is a product of averages on the lattice sites of the form:

$$\langle S_x^{i_1} S_x^{i_2} \dots S_x^{i_n} \rangle_0 \equiv \int d\vec{S}_x S_x^{i_1} S_x^{i_2} \dots S_x^{i_n} \quad (4)$$

In the $N \rightarrow 0$, as it will be proven, the only non-vanishing term is

$$\langle S_x^i S_x^j \rangle = \delta^{ij} \quad (5)$$

so that

$$Z = 1 + N \sum_{\{\Gamma(L)\}} \beta^L \xrightarrow{N \rightarrow 0} 1 \quad (6)$$

where $\{\Gamma(L)\}$ is the set of all self-avoiding closed paths of length L and the non-trivial two point correlation function gives us the wanted result

$$\langle S_x^i S_{x'}^{i'} \rangle = \delta^{ii'} \sum_{\{\Gamma_{xx'}(L)\}} \beta^L \quad (7)$$

Following Ref. 2) we sketch the proof of Eq. (5). To this purpose we introduce the generating function of the averages (4)

$$f(\vec{h}) = \int d\vec{S} e^{\vec{h} \cdot \vec{S}} \quad (8)$$

so that

$$\frac{\partial^n}{\partial h^{i_1} \dots \partial h^{i_n}} f(\vec{h}) = \langle S^{i_1} \dots S^{i_n} \rangle_0 \quad (9)$$

The normalization of Eq. (1) implies that f satisfies the equation

$$\sum_{i=1}^N \frac{\partial^2}{\partial h^i \partial h^i} f(\vec{h}) = N f(\vec{h}) \quad (10)$$

Owing to rotational invariance, f depends only on $h = |\vec{h}|$. Equation (9) becomes

$$\frac{d^2}{dh^2} f + \frac{N-1}{h} \frac{d}{dh} f + N f = 0 \quad (11)$$

The $N = 0$ solution of the differential Eq. (11) is of the form $f = a + bh$, so that only $\langle 1 \rangle_0$ and $\langle S_x^i S_x^j \rangle_0$ are different from zero, the boundary conditions $\langle 1 \rangle_0 = 1$ and $\langle S_x^i S_x^j \rangle = \delta^{ij}$ fix a and b giving

$$f(\vec{h}) = 1 + \frac{1}{2} \vec{h}^2 \quad (12)$$

This completes the proof.

We turn to the problem of the self-avoiding surfaces. The obvious candidate for the related field theory model is now a gauge theory, since the plaquette is the building block of a surface. Regarding the choice of the gauge group, there is a certain freedom as for the spin model just analyzed. In fact, we can choose invariance groups of different types, e.g., $O(n)$, $SU(N)$ and $SP(n)$. We believe that this choice does not affect the physics of the $N \rightarrow 0$ limit and therefore we take $SU(N)$ with the Wilson action (10)

$$A(U) = -\beta \sum_P \text{Tr} (U_{e_1} U_{e_2} U_{e_3}^+ U_{e_4}^+) \quad , \quad U_e \in SU(N) \quad (13)$$

where U_e is defined on the links of the lattice Λ and the sum runs over the oriented plaquettes P . Analogously to Eq. (1) we choose for the matrix U the normalization

$$U_e U_e^+ = N \mathbb{1} \quad (14)$$

We now prove, by extending the previous case, that the $N \rightarrow 0$ limit of this gauge model describes the self-avoiding random surfaces. We perform as before a strong coupling expansion of the partition function:

$$Z = \int DU \sum_{n=0}^{\infty} \frac{1}{n!} \left(\beta \sum_p (U_{e_1} U_{e_2} U_{e_3}^+ U_{e_4}^+) \right)^n \quad (15)$$

We could rearrange the expansion (15), expressing each term as a sum of characters of irreducible representations (11), but for our purposes this form turns out to be much more convenient, for instance, both expansions are also possible for the previous spin case. A general term of the expansion is a product of averages on the lattice links of the form:

$$\langle U_e^{\alpha_1 \beta_1} U_e^{\alpha_2 \beta_2} \dots U_e^{\alpha_m \beta_m} \rangle_0 \equiv \int dU_e U_e^{\alpha_1 \beta_1} \dots U_e^{\alpha_m \beta_m} \quad (16)$$

In the $N \rightarrow 0$ limit it will be proven that the only non-zero average is

$$\langle U_e^{\alpha_1 \beta_1} U_e^{\alpha_2 \beta_2} \rangle_0 = \delta^{\alpha_1 \alpha_2} \delta^{\beta_1 \beta_2} \quad (17)$$

so that

$$Z = 1 + N^4 \sum_{\{\Sigma(S)\}} \beta^S \quad (18)$$

where $\{\Sigma(S)\}$ is the set of all the closed self-avoiding random surfaces of area S . The Wilson loop average leads to:

$$\langle \prod_{e \in C} U_e^{\alpha_e \beta_e} \rangle = \prod_e \delta^{\alpha_e \beta_{e+1}} \sum_{\{\Sigma_C(S)\}} \beta^S \quad (19)$$

where $\{\Sigma_C(S)\}$ is the set of all the self-avoiding surfaces of area S and boundary C . In order to prove Eq. (17), we introduce the generating functional of (16)

$$f(K, K^+) = \int dU e^{T_K(KU^+ + K^+U)} \quad (20)$$

Owing to the invariance of the Haar measure f is invariant for $K \rightarrow SKR$, $K^+ \rightarrow R^+K^+S^+$, with $SS^+ = RR^+ = 1$, so that f depends only on invariant quantities as $\text{tr}(KK^+)$, $\text{tr}(KK^+)^n$, $\text{dek}(K)$, etc... Using Eq. (14) we have

$$\sum_{\alpha\beta} \frac{\partial^2}{\partial K^{\alpha\beta} \partial K^{\alpha\beta}} f(K, K^+) = N^2 f(K, K^+) \quad (21)$$

In the limit $N \rightarrow 0$ Eq. (21) greatly simplifies: in fact, if we expand f in the above invariants only 1 and $\text{tr}(KK^+)$ can appear, because

$$\frac{\partial^2}{\partial K \partial K^+} 1 = 0 \quad \text{and} \quad \frac{\partial^2}{\partial K \partial K^+} T_K(KK^+) = N^2$$

whereas the second derivative of the other invariants always contains K dependent terms which do not vanish as $N \rightarrow 0$. Imposing boundary conditions we have

$$f(KK^+) = 1 + T_K(KK^+) \quad (22)$$

which leads to the vanishing of the averages in Eq. (16) with $n > 2$, and this completes our proof. Alternatively one can also proceed by induction. For example, the UUU^+U^+ average is

$$\begin{aligned} \langle U^{\alpha_1\beta_1} U^{\alpha_2\beta_2} U^{\alpha_3\beta_3} U^{\alpha_4\beta_4} \rangle_0 = & \\ & \frac{N^2}{N^2-1} (\delta^{\alpha_1\alpha_3} \delta^{\beta_1\beta_3} \delta^{\alpha_2\alpha_4} \delta^{\beta_2\beta_4} + \delta^{\alpha_1\alpha_4} \delta^{\beta_1\beta_4} \delta^{\alpha_2\alpha_3} \delta^{\beta_2\beta_3}) + \\ & + \frac{N}{1-N^2} (\delta^{\alpha_1\alpha_3} \delta^{\beta_1\beta_4} \delta^{\alpha_2\alpha_4} \delta^{\beta_2\beta_3} + \delta^{\alpha_1\alpha_4} \delta^{\beta_1\beta_3} \delta^{\alpha_2\alpha_3} \delta^{\beta_2\beta_4}) \end{aligned} \quad (23)$$

which vanishes in the $N \rightarrow 0$ limit. It is easy to see that the next term $UUUU^+U^+U^+$ vanishes at least linearly in N . The remaining possibility in $SU(N)$, i.e., $\langle U^{\alpha_1 \beta_1}, \dots, U^{\alpha_N \beta_N} \rangle_0 \propto \epsilon^{\alpha_1, \dots, \alpha_N} \epsilon^{\beta_1, \dots, \beta_N}$, which corresponds to N plaquettes sharing the same oriented link, does not contain any plaquette in the limit and therefore does not contribute. Furthermore, we realize that it is possible to describe the physics of the random surfaces by means of the action (13) where this time the $N \times N$ matrix U satisfies the relaxed constraint

$$T_K UU^T = N^2 \quad (24)$$

and does not belong to any group, but rather is an element of the sphere S_{N^2-1} . If we take again the $N \rightarrow 0$ limit, the proof that the model of Eqs. (13) and (24) describes the self-avoiding random surfaces is just the same as in the spin case.

In fact, owing to S_{N^2-1} rotational invariance the functional generator $f(K) = \int_{S_{N^2-1}} dU e^{\text{tr} KU^T}$ depends only on the modulus $(\text{tr} KK^T)^{1/2}$. The theory, so obtained, is again invariant under local $O(N)$ rotations but now we have more degrees of freedom than in the standard theory where $U \in SO(N)$ or $O(N)$. It is tempting to speculate that the theory of the self-avoiding random surfaces is independent of the starting group.

The similarities between the random walk and the random surface stress once again the strict link between the spin and gauge system (5) and it is hoped that further analogies may be discussed in the near future.

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