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ADDENDUM

SU(N) LATTICE GAUGE THEORY WITH VILLAIN'S ACTION

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Proof of Eq. (2) and, equivalently, of Eq. (42, a-c) has been obtained. Recall that Eq. (2) was proven to hold for $n = 1, 2, 3, 4$ (any N) and for $n \geq 4$ in the $1/N$ expansion up to $O[(\lambda n/N)^4]$. Now a direct proof of Eq. (2) with no limitations has been found, starting from the observation that $\text{tr}(U^n)$ has a simple character expansion, namely

$$\text{tr}(U^n) = \chi_{\square \square \dots \square} - \chi_{\square \square \dots \square} + \dots + (-1)^n \chi_{\square \square \dots \square} \quad (*)$$

$\xleftarrow{n-1} \quad \xrightarrow{n-1} \quad \begin{matrix} \uparrow \\ \square \\ \vdots \\ \square \\ \downarrow \end{matrix} \quad \begin{matrix} \uparrow \\ \square \\ \vdots \\ \square \\ \downarrow \end{matrix}$

The expectation value $\langle \text{tr} U^n \rangle$ can be easily obtained from the character expansion of e^{-A} (Eq. 58). Equation (*) allows us to calculate $\langle \text{tr} U^n \rangle$ also for the heat kernel action where the orthogonal polynomial method is inapplicable. The result is the following

$$\langle \text{tr}(U^n) \rangle_{\text{H.K.}} = q^{n(N-n+1)} {}_2F_1(N+1, 1-n; 2; 1-q^{2n}) \quad (**)$$

As $N \rightarrow \infty$

$$\langle \text{tr}(U^n) \rangle_{\text{H.K.}} \sim e^{-n\lambda/4} {}_1F_1(1-n; 2; n\lambda/2) \quad (***)$$

which proves a conjecture of P. Rossi⁽¹⁾. (All the results refer to $U(N)$; a factor $q^{-n^2/N}$ must be added for $SU(N)$)

1) P. Rossi, Annals of Physics, 132 (1981) 463-481.

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SU(N) LATTICE GAUGE THEORY WITH VILLAIN'S ACTION

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ABSTRACT

We consider the pure gauge lattice theory with Villain's action $e^{-A(U)} = \prod_{j=1}^N \sum_{n=-\infty}^{+\infty} e^{-N/\lambda(\theta_j + 2n\pi)^2}$, where $\theta_1, \dots, \theta_N$ are the invariant angles of $U \in U(N)$ or $SU(N)$. For the two dimensional lattice we calculate the partition function, $Z(\lambda, N)$, the specific heat, the level density $\rho_N(\theta)$ and Wilson's loops $W_n = 1/N \langle \text{tr}(U^n) \rangle$ ($n = 1, 2, 3, \dots$). The $1/N$ expansion of Z and W_n is convergent for sufficiently small $|\lambda/N|$ and its coefficients are analytic in λ near the real axis (no "Gross-Witten" singularity to all orders in $1/N$), but it is still not possible to commute the strong coupling limit with the planar limit ($\lambda \rightarrow \infty, N \rightarrow \infty$). We also calculate the character expansion which is needed for strong coupling calculations in four dimensions. A comparison with Monte Carlo data ($N = 2$) and a preliminary discussion of the large N limit is given.

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INTRODUCTION

The large N limit of $U(N)$ lattice gauge theory in two dimensions has been studied in detail by various authors^{1),2),3)}. Even if the two dimensional case is dynamically trivial, its analytic properties in the (g^2, N) plane raised some interest due to the observation¹⁾ of a singularity in the large N limit at $\lambda = g^2 N$ fixed. This "Gross-Witten singularity" forbids the interchange of the two limits $\lambda \rightarrow \infty$ (strong coupling) with $N \rightarrow \infty$ (planar limit). It was noticed that a different choice of the action (formally giving the same continuum limit) led to a singularity free planar limit. This is the case for both Manton's action⁴⁾ and for the "heat kernel" action⁵⁾. We will show in this paper that even in this case it is not possible, in general, to commute the strong coupling limit with the planar limit. This will be shown by a detailed study of the generalized Villain's action

$$e^{-A(U)} = \prod_{i=1}^N \sum_{\ell_i=-\infty}^{+\infty} e^{-\frac{N}{\lambda}(\theta_i + 2\pi\ell_i)^2} \quad (1)$$

$$\left(\sum_i \ell_i = 0 \text{ for } SU(N) \right)$$

$[\theta_i]$ are the invariant angles of $U \in U(N)$ or $SU(N)$. The method of orthogonal polynomials³⁾ will be used to derive the relevant objects of the theory in closed form, namely the partition function $Z(\lambda, N)$, the specific heat $C(\lambda, N)$, the loop variables $W_n(\lambda, N) = 1/N \langle \text{tr } U^n \rangle$ and the level density $\rho_N(\theta) = \sum_{-\infty}^{+\infty} W_n e^{in\theta}$. The general form of W_n turns out to be

$$W_n(\lambda, N) = \sum_{p=1}^n \frac{1 - e^{-\lambda p/2}}{\lambda p/2} \binom{n}{p} \binom{n+p-1}{n} (-)^{n+p} \frac{\prod_{k=n-p+1}^{n+p-1} f\left(\frac{\lambda k}{4N}\right)}{\prod_{k=1}^p f\left(\frac{\lambda k}{4N}\right)} \quad (2)$$

where $f(x) = \text{sh}x/x$ [for $SU(N)$ there is an extra factor $\exp\{n^2\lambda/4N^2\}$]. The planar limit is then given by

$$W_n(\lambda, \infty) = -2 \frac{(-)^n}{n\lambda} {}_2F_1(n, -n; 1; e^{-\lambda/2}) \quad (3)$$

Since the $1/N$ series is in fact a series in powers of λ/N , it is obvious that we cannot interchange $\lambda \rightarrow \infty$ with $N \rightarrow \infty$. From a mathematical viewpoint, our analysis leads to the introduction of the so-called Rogers-Szegö

polynomials^{6),7)}; moreover, Eq. (2) can be reformulated in terms of a "basic hypergeometric function" ϕ_{21} [See Eq. (42)]. We shall avoid the theory of these special functions in this paper, but it could be that they will play some role in lattice gauge theory [see Ref. 8) for an application to dual resonance models].

Another point that we may easily check using our result [Eq. (2)] is the following: the planar limit is usually very good for the Wilson loop W_1 up to non-negligible values of λ (error less than $\lambda^2/96N^2$, i.e., 1% error at $g^2 = 1$ but higher order corrections become more and more important for W_n , $n \geq 2$ [error of order $O((\lambda n/4N)^2)$]). As a consequence the level density $\rho_N(\theta)$ is poorly represented by its leading $N \rightarrow \infty$ limit $\rho_\infty(\theta)$, in a way which is quite similar to a WKB approximation to the N fermion collective wave function in ordinary quantum mechanics (ρ_∞ is vanishing outside a compact interval $|\theta| < \theta_c$ and its $1/N$ corrections are singular at the end points).

A potentially important property of the level density which emerges from our analysis is an approximate scaling property in the weak coupling region $\lambda \lesssim 1$ [Eq. (53)]. Since this property holds in the weak coupling region, it could be a general (universal) property of the level density. We will also show that in the $\lambda \rightarrow 0$ (or $N \rightarrow \infty$) limit everything can be described in terms of ordinary Hermite polynomials, in particular the level density at $N \rightarrow \infty$ converges to a "Wigner's law" in the scaled variable $x = \sqrt{N/\lambda} \theta$. This is shown not only for consistency, but also to suggest that most of the properties proved for Villain's action should be common to all other actions in a suitable range of λ (say $\lambda < 2$). In four dimensions the main tool is given by strong coupling expansions, for which one needs the character expansion of e^{-A} . While it is not as simple as for the heat kernel⁵⁾, the character expansion of Villain's action can be given in closed form for any N for both $SU(N)$ and $U(N)$; the behaviour of the coefficient as $N \rightarrow \infty$ is much easier to analyze than in Wilson's action. For $N = 2$ good agreement is found with Monte Carlo data⁹⁾ up to the tangency point to asymptotic freedom ($\beta = \lambda/8 \sim 1.55$). Notice that the usual difficulties in dealing with $SU(N)$ due to the constraint $\sum_j \theta_j = 0$ are not encountered with Villain's action because of an effective factorization of the action (analogous to the heat kernel case).

The paper is organized as follows: in Section 1 we set the notation, calculate the partition function and the specific heat both for $SU(N)$ and $U(N)$. In Section 2 we calculate the loop variables W_n . Equation (2) is obtained by first examining the $1/N$ expansion and comparing it with the closed form obtained for small n ; it is then conjectured that Eq. (2) gives the right answer for any n .

The level density is considered in Section 3. Section 4 contains the character expansion and a comparison with Monte Carlo data for $N = 2$. Some concluding remarks are given in Section 5. Most of the mathematical details are given in the Appendices.

1. THE PARTITION FUNCTION AND THE SPECIFIC HEAT IN TWO DIMENSIONS

It is known that the two dimensional gauge theory on the lattice reduces to a one plaquette problem. The relevant objects of the theory are simply given by certain integrals over the group, namely [for the moment we consider $U(N)$]:

a) the partition function

$$Z(\lambda, N) = \frac{1}{(2\pi)^N N!} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} d\theta_1 \dots d\theta_N |\Delta(e^{i\theta_j})|^2 \prod_{j=1}^N e^{-A(\theta_j)} \quad (4)$$

where $(\theta_1, \dots, \theta_N)$ are the invariant angles, $\Delta(x_j) = \prod_{i>j} (x_i - x_j)$ is Vandermonde's determinant and $\exp\{-A(U)\}$ is Villain's action (Jacobi's θ_3 function)

$$e^{-A(\theta)} = \sum_{n=-\infty}^{+\infty} e^{-\frac{N}{\lambda}(\theta + 2n\pi)^2} = \left(\frac{\lambda}{4\pi N}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{+\infty} e^{-\frac{\lambda}{4N}n^2} e^{in\theta} \quad (5)$$

b) the specific heat^{*})

$$C(\lambda, N) = \frac{\partial}{\partial \lambda} \lambda^2 \frac{\partial}{\partial \lambda} \ln Z(\lambda, N) \quad (6)$$

c) the loop variables

$$W_n(\lambda, N) = \left\langle \frac{1}{N} \text{tr} U^n \right\rangle = \frac{1}{Z} \int e^{-A(U)} \frac{1}{N} \text{tr}(U^n) [dU] \quad (7)$$

d) the generating function of the loop variables

$$\Phi(z) = \left\langle \frac{1}{N} \text{tr} (1 - zU)^{-1} \right\rangle \quad (8)$$

and

e) the level density

^{*}) Since the action is not proportional to $\beta = \lambda^{-1}N$ this is a (small) abuse of terminology.

$$\rho_N(\theta) = 2 \operatorname{Re} \{ \Phi(e^{i\theta}) \} - 1 = \sum_{-\infty}^{+\infty} W_n(\lambda, N) e^{in\theta} \quad (9)$$

Because of the determinant $|\Delta|^2$ appearing in Eq. (4), $Z(\lambda, N)$ is formally the normalization constant of a Slater-Fock determinant

$$\Psi(\theta_1, \dots, \theta_N) = \frac{1}{\sqrt{N!}} \det \left\| e^{-\frac{1}{2}A(\theta_j)} P_{k-1}(e^{i\theta_j}) \right\|_{j,k=1, \dots, N} \quad (10)$$

where $P_j(z)$ is a polynomial of degree j with leading coefficient equal to one. If we choose the polynomials $P_j(z)$ to be orthogonal on the unit circle with respect to the measure $e^{-A(\theta)}$, then the integral Eq. (4) is given simply by

$$Z(\lambda, N) = \prod_{j=0}^{N-1} h_j \quad (11)$$

where

$$\begin{aligned} h_j &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_j(e^{i\theta})|^2 e^{-A(\theta)} d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |P_j(e^{i\theta})|^2 e^{-N\theta^2/\lambda} d\theta \end{aligned} \quad (12)$$

From the theory of orthogonal polynomials on the unit circle¹⁰⁾ it is well known that $h_j = D_j/D_{j-1}$ where $D_j(\lambda, N)$ is a determinant built up with the Fourier components of e^{-A} , namely

$$e^{-A(\theta)} = \sum_{-\infty}^{+\infty} c_n e^{in\theta} \quad (13)$$

$$D_n(\lambda, N) = \det \| c_{j-k} \|_{j,k=0,1, \dots, n} \quad (14)$$

It follows from Jacobi's identity [Eq. (5)] that

$$c_n = \left(\frac{\lambda}{4\pi N} \right)^{1/2} q^{n^2} \quad (15)$$

where $q = \exp\{-\lambda/4N\}$ is the usual q variable of the Jacobi θ functions.

Summing up we have

$$Z(\lambda, N) = \left(\frac{\lambda}{4\pi N}\right)^{N/2} \det \left\| q^{(j-k)^2} \right\|_{j,k=1,\dots,N} \quad (16)$$

The determinant can easily be reduced to Vandermonde's (see Appendix 1) and we get

$$Z(\lambda, N) = \left(\frac{\lambda}{4\pi N}\right)^{N/2} \prod_{k=1}^{N-1} \left(1 - e^{-\frac{1}{2}\lambda(1-k/N)}\right)^k \quad (17)$$

The main difficulty in the case of Wilson's action was that Eq. (16) contained a determinant of Bessel functions which did not allow for a reduction to a simple product like Eq. (17). It is now an easy matter to apply the Euler-McLaurin summation formula to get the $1/N$ expansion for $\ln Z(\lambda, N)$, namely (see Appendix 2 for details):

$$\begin{aligned} \frac{1}{N^2} \ln Z(\lambda, N) &= \frac{1}{N^2} \ln \left[\frac{\prod_{v=1}^{N-1} v!}{(2\pi)^{N/2} N^{N^2/2}} \right] + \frac{1}{2} \ln \frac{\lambda}{2} - \frac{\lambda}{24} \\ &+ \int_0^1 (1-x) \ln \left[\frac{\operatorname{sh}(\lambda x/4)}{\lambda x/4} \right] dx + \frac{1}{N^2} \left[\frac{\lambda}{24} - \frac{1}{12} \ln \frac{\operatorname{sh}(\lambda/4)}{\lambda/4} \right] \\ &- \frac{1}{720 N^4} \left\{ \left(\frac{\lambda}{4}\right)^2 - 3 \left[1 - \left(\frac{\lambda/4}{\operatorname{sh}(\lambda/4)}\right)^2 \right] \right\} \\ &+ \frac{1}{30240 N^6} \left\{ -\frac{2}{3} \left(\frac{\lambda}{4}\right)^4 - 10 \left(\frac{\lambda/4}{\operatorname{sh}(\lambda/4)}\right)^2 \left[\left(\frac{\lambda}{4}\right)^2 + 3 \left(\frac{\operatorname{sh}(\lambda/4)}{\lambda/4}\right)^2 - 3 \left(\frac{\lambda}{4} \coth \frac{\lambda}{4}\right)^2 \right] \right\} \\ &+ \dots \end{aligned} \quad (18)$$

We find that all the coefficients in the $1/N$ expansion are analytic in a neighbourhood of the real λ axis except for the trivial logarithmic singularity at $\lambda = 0$; all other singularities are confined to the imaginary axis [one may easily compare the singularities showing up in the $1/N$ expansion with the actual singularities of the exact expression Eq. (17)]. Already at this point we may notice that the limit $\lambda \rightarrow \infty$ cannot be taken in the $1/N$ expansion [Eq. (18)], since the coefficients are growing with λ . The actual limit must be taken on the exact expression and gives

$$\frac{1}{N^2} \ln Z(\lambda, N) \underset{\lambda \rightarrow \infty}{\sim} \text{constant} + \frac{1}{2N} \ln \lambda - \frac{N-1}{N^2} e^{-\lambda/2N} + \dots \quad (19)$$

The specific heat $C(\lambda, N)$ can now be obtained by a simple differentiation. The first two terms in its $1/N$ expansion are the following

$$\begin{aligned} \frac{1}{N^2} C(\lambda, N) &= \frac{1}{\lambda^2} \int_0^\lambda x dx \frac{x/2}{e^{x/2} - 1} \\ &+ \frac{1}{12 N^2} \left[\lambda - 2 \left(\frac{\lambda}{4} \coth \frac{\lambda}{4} - 1 \right) - 1 + \left(\frac{\lambda/4}{\sinh \lambda/4} \right)^2 \right] + O\left(\frac{1}{N^4}\right) \end{aligned} \quad (20)$$

The leading term was obtained in Ref. 4b). The asymptotic expansion of Villain's and Manton's action of course agree because the error is of order $\exp(-cN/\lambda)$ for some constant c .

To get the asymptotic $1/N$ expansion for the group $SU(N)$, we take advantage of the fact that $U(N) = U(1) \times SU(N)/Z_N$ and

$$\sum_i \theta_i^2 = \frac{1}{N} \sum_{i < j} (\theta_i - \theta_j)^2 + \frac{1}{N} \left(\sum_i \theta_i \right)^2$$

It follows^{*)} that

$$\begin{aligned} Z_{U(N)} &\sim Z_{SU(N)} \int d\alpha e^{-N\alpha^2/\lambda} + (\text{exponentially small terms}) \\ Z_{SU(N)} &\sim \text{constant} \times \lambda^{-1/2} Z_{U(N)} \end{aligned}$$

By inserting the asymptotic expansion Eq. (18) we get:

$$\begin{aligned} \frac{1}{N^2-1} \ln Z_{SU(N)} &= \frac{1}{2} \ln \frac{\lambda}{2} - \frac{\lambda}{24} + \int_0^1 (1-x) \ln \left[\frac{\sinh(\lambda x/4)}{\lambda x/4} \right] dx \\ &+ \frac{1}{N^2} \int_0^1 (1-x) \left[\ln \frac{\sinh(\lambda x/4)}{\lambda x/4} - x^2 \ln \frac{\sinh(\lambda/4)}{\lambda/4} \right] dx + O\left(\frac{1}{N^4}\right) \end{aligned} \quad (21)$$

[the perturbative expansion of $\ln Z_{SU(N)}$ was also given in Ref. 4b)]. A simple differentiation gives the specific heat. Notice that the $(1/N)^2$ correction starts with a factor λ^4 due to a cancellation of the quadratic term in the second integral appearing in Eq. (21): this explains the N independence of the first few perturbative coefficients noticed in Ref. 4).

^{*)} We will see in Section 4 that a stronger result (factorization in the strict sense) holds; but this is irrelevant for the $1/N$ expansion.

2. THE LOOP VARIABLES $W_n(\lambda, N)$

To calculate $W_n(\lambda, N) = \langle 1/N \text{tr} U^n \rangle$ we need some properties of the orthogonal polynomials with respect to the measure e^{-A} . Since the factor $(\lambda/4\pi N)^{1/2}$ in Eq. (15) drops out in calculating averages, we can ignore it and consider the orthonormal polynomials with respect to the θ function $\sum_{-\infty}^{+\infty} q^{n^2} e^{in\theta}$; namely we introduce the polynomials $\phi_j(z) = k_j z^j + (\text{lower order terms})$ which satisfy^{*}

$$\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \sum_{-\infty}^{+\infty} q^{n^2} e^{in\theta} \overline{\phi_j(e^{i\theta})} \phi_k(e^{i\theta}) = \delta_{jk} \quad (22)$$

These polynomials were studied in detail by Rogers and Szegő; a thorough account of their properties can be found in a paper by Carlitz⁶⁾. They are given by^{6), 7)}

$$\phi_n(z) = \kappa_n \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} (-q)^j z^{n-j} \quad (23)$$

where

$$\kappa_n^{-2} = [n]! \equiv \prod_{j=1}^n (1 - q^{2j}) \quad (24)$$

(a "q factorial") and

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{[n]!}{[j]! [n-j]!} \quad (25)$$

(a "q binomial coefficient" or Gaussian polynomial). Actually in this Section we will only need the recursion relation

$$z \phi_j = (1 - q^{2j+2})^{1/2} \phi_{j+1} + q \phi_j - q(1 - q^{2j})^{1/2} z \phi_{j-1} \quad (26)$$

while Eq. (23) will be used in Section 3 to calculate the level density $\rho(\theta)$. To make the paper self-contained, Eqs. (23) - (26) are derived in Appendix 3.

^{*}) Equation (22) can be written as a Gaussian integral - in fact these polynomials are intimately related to Hermite polynomials (see Appendix 3).

By definition, W_n is formally identical to the ground state expectation value of the observable $w_n = \sum_{j=1}^N e^{in\phi_j}$ for a quantum system consisting of N non-interacting fermions, whose one particle eigenfunctions are precisely the polynomials $\phi_j(z)$. Hence we have

$$\langle \text{tr}(U^n) \rangle = \sum_{j=0}^{N-1} \langle \phi_j | z^n | \phi_j \rangle \quad (27)$$

By iterating Eq. (26) we easily get the matrix representation of z in the basis $\{\phi_j\}$ and a simple calculation leads to the first few W 's^{*}):

$$W_1 = \langle \frac{1}{N} \text{tr} U \rangle = \frac{1}{N} q \frac{1-q^{2N}}{1-q^2} \quad (28)$$

$$W_2 = \langle \frac{1}{N} \text{tr} U^2 \rangle = \frac{(1-q^6)(1-q^{4N})}{N(1-q^2)(1-q^4)} - \frac{(1-q^4)(1-q^{2N})}{N(1-q^2)^2} \quad (29)$$

$$W_3 = \frac{(1-q^8)(1-q^{10})(1-q^{6N})}{Nq^3(1-q^2)(1-q^4)(1-q^6)} - \frac{(1-q^8)(1-q^6)(1-q^{4N})}{Nq^3(1-q^2)^2(1-q^4)} + \frac{(1-q^6)(1-q^{2N})}{Nq(1-q^2)^2} \quad (30)$$

(recall that $q \equiv \exp\{-\lambda/4N\}$). The next W is already too long to be reported here. To get a closed form expression for W_n the iteration of the recurrence relation seems impractical. Before introducing our method, let us however examine the large N properties of the first few W 's calculated so far. By inserting the value of q we get

$$W_1 = \frac{1-e^{-\lambda/2}}{\lambda/2} \frac{\lambda/4N}{\text{sh}(\lambda/4N)} \quad (28')$$

the second factor can be expanded in powers of $(\lambda/4N)^2$; the first factor gives the leading $N \rightarrow \infty$ behaviour⁴⁾. The $1/N$ expansion turns out to be convergent for $|\lambda/4N| < \pi$. For the second loop variable we get:

$$W_2 = 3 \frac{1-e^{-\lambda}}{\lambda} \frac{\text{sh} \frac{3\lambda}{4N}}{\frac{3\lambda}{4N}} \left[\frac{\text{sh} \frac{\lambda}{4N}}{\frac{\lambda}{4N}} \right]^{-1} \left[\frac{\text{sh} \frac{\lambda}{2N}}{\frac{\lambda}{2N}} \right]^{-1} - 2 \frac{1-e^{-\lambda/2}}{\lambda/2} \frac{\text{sh} \frac{\lambda}{2N}}{\frac{\lambda}{2N}} \left[\frac{\text{sh} \frac{\lambda}{4N}}{\frac{\lambda}{4N}} \right]^{-2} \quad (29')$$

*) In the notation of Ref. 3) we have: $R_j = q^{2j+1} \sim \exp\{-\lambda x/2\}$; $S_j = -q^{2j}(1-q^{2j}) \sim -e^{-\lambda x/2}(1-e^{-\lambda x/2})$; $f_j = 1 - q^{2j} \approx 1 - e^{-\lambda x/2}$ which always satisfy the bounds $0 < R < 1$, $S < 0$, $0 < f < 1$.

The $1/N$ expansion is also convergent, but in a smaller radius, $|\lambda/4N| < \pi/2$. The general expression of W_n is expected to be

$$W_n = \sum_{p=1}^n \Lambda_{n,p} \frac{1 - e^{-\lambda p/2}}{\lambda p/2} \quad (31)$$

with $\Lambda_{n,p}$ being a suitable product of $[\text{sh}(k\lambda/4N)/(k\lambda/4N)]^{\pm 1}$. It is this general form that we are now going to calculate.

We introduce the linear operator A and its adjoint $A^\dagger$ such that

$$A \phi_n = \begin{cases} (1 - q^{2n})^{1/2} \phi_{n-1} & (n \geq 1) \\ 0 & (n = 0) \end{cases} \quad (32)$$

$$A^\dagger \phi_n = (1 - q^{2n+2})^{1/2} \phi_{n+1} \quad (33)$$

The recurrence relation Eq. (26) can be written as follows:

$$z \phi_n = A^\dagger \phi_n + q \phi_n - q z A \phi_n \quad (34)$$

or equivalently

$$z(1 + qA) = q + A^\dagger \quad (35)$$

Since A is a bounded operator with norm one and $q < 1$, $1 + qA$ has a bounded inverse and we finally get

$$z = (q + A^\dagger)(1 + qA)^{-1} \quad (36)$$

From Eq. (27) what we need are the diagonal elements of z^n which means that we can expand the inverse operator $(1 + qA)^{-1} = \sum_{\ell=0}^{\infty} (-qA)^\ell$ and keep only the factors in z^n with an equal number of A 's and $A^\dagger$'s. We are facing here a Wick-ordering type problem with rational functions of $A, A^\dagger$. This would perhaps appear a hopeless task; let us however have a look at the commutation relations between $A, A^\dagger$:

$$[A, A^\dagger] = (1 - q^2)(1 - A^\dagger A) \quad (37)$$

As probably expected, the commutator is down by a factor $1 - q^2 \sim \lambda/2N$ with respect to $A^\dagger A$. Hence if we succeed in expanding $z^k = (q + A^\dagger)(1 + qA)^{-1}$ in powers of the commutator we will get the $1/N$ expansion. We find once more that the $N \rightarrow \infty$ limit is analogous to a semiclassical limit, $\lambda/2N$ playing the role of Planck's constant in the fundamental commutation relations. The leading behaviour of W_n as $N \rightarrow \infty$ is therefore quite simple: since we can freely commute A and $A^\dagger$, we get

$$\begin{aligned} \langle \phi_j | z^n | \phi_j \rangle &\simeq \langle \phi_j | (q + A^\dagger)^n (1 + qA)^{-n} | \phi_j \rangle \\ &\simeq q^n \sum_{p=0}^n \binom{n}{p} \binom{-n}{p} \langle \phi_j | (A^\dagger A)^p | \phi_j \rangle \\ &\simeq \sum_{p=0}^n \binom{n}{p} \binom{-n}{p} (1 - q^2)^p = {}_2F_1(n, -n; 1; 1 - q^2) \end{aligned} \quad (38)$$

A standard transformation formula of the hypergeometric functions ${}_2F_1(\dots, z)$ into ${}_2F_1(\dots, 1 - z)$ gives

$$\langle \phi_j | z^n | \phi_j \rangle \simeq \frac{1}{n} (-1)^n \sum_{p=1}^n \frac{(-n)_p (n)_p}{p! (p-1)!} q^{2jp} \quad (39)$$

The sum $\sum_{m=0}^{N-1}$ can now be performed explicitly to yield

$$W_n(\lambda, \infty) = -\frac{2}{n\lambda} (-1)^n {}_2F_1(n, -n; 1; e^{-\lambda/2}) \quad (40)$$

Notice that ${}_2F_1(n, -n; 1; 1) = 0$ ensures that Eq. (40) defines an entire function of λ .

Higher order corrections can now be considered. The idea is to disentangle the product $[(q + A^\dagger)(1 + qA)^{-1}]^n$ by substituting $(1 + qA)^{-1} = \int_0^\infty ds e^{-s(1+qA)}$ and by expanding in powers of $\epsilon = 1 - q^2$, each commutator bringing a power of ϵ . We stop at ϵ^2 which is sufficient to get the first correction to W_n . But the trend in the formulae suggests a general form valid to all orders which turns out to be valid for $n = 1, 2, 3, 4$ when compared to the direct calculation [Eqs. (28)-(30)]. The $O(\epsilon^2)$ formula which is derived in Appendix 4 is the following

$$W_n = \sum_{p=1}^n \frac{1 - e^{-\lambda p/2}}{\lambda p/2} (-1)^{n+p} \binom{n}{p} \binom{n+p-1}{n} \left[1 + \frac{\lambda^2}{96 N^2} ((2p-1)n^2 - 2p^2) + O(N^{-4}) \right] \quad (41)$$

It seems plausible that a closed form of W_n should be expressible in terms of Gaussian polynomials $\begin{bmatrix} n \\ p \end{bmatrix}$ [Eq. (25)]; moreover, one has $\begin{bmatrix} n \\ p \end{bmatrix} \sim \binom{n}{p}$ for $q \rightarrow 1$. This makes us conjecture that the exact expression of W_n should be given by one of the following equivalent expressions^{*)}:

$$W_n = \frac{1}{N} \sum_{p=1}^n (-1)^{n+p} \begin{bmatrix} n \\ p \end{bmatrix} \begin{bmatrix} n+p-1 \\ n \end{bmatrix} q^{p(p+1)-n(2p-1)} \frac{1-q^{2Np}}{1-q^{2p}} \quad (42.a)$$

$$\equiv -\frac{1}{N} \frac{(-q)^n}{1-q^{2n}} \sum_{p=0}^n \begin{bmatrix} n \\ p \end{bmatrix} \begin{bmatrix} -n \\ p \end{bmatrix} q^{2Np+2p^2} \quad (42.b)$$

$$\equiv -\frac{1}{N} \frac{(-q)^n}{1-q^{2n}} {}_2\phi_1 \left[\begin{matrix} q^{2n}, q^{-2n} \\ q^2 \end{matrix}; q^2; q^{2N+2} \right] \quad (42.c)$$

$$\equiv \text{Eq. (2)}$$

where ${}_2\phi_1 \left[\begin{matrix} a, b; q^2; z \end{matrix} \right]$ is a "basic hypergeometric function"⁷⁾. This is very likely to be the exact answer. It is verified to hold for $n = 1, 2, 3$ [Eqs. (28) - (30)] and $n = 4$ (not reported here), and it holds up to $O[(\lambda/N)^4]$ for $N \rightarrow \infty$ for every n : actually this is enough for our purposes. A general proof would probably involve some remote identity for basic hypergeometric functions. (Further support is given by examining the $N \rightarrow \infty$, $\lambda/N = g^2$ fixed limit - see next section.) The loop variables for $SU(N)$ are easily obtained by multiplying Eqs. (41) and (42) by $\exp\{\lambda n^2/4N^2\} = q^{-n^2/N}$. For instance we have $\langle \text{tr} U \rangle_{SU(N)} = q^{1-1/N} (1 - q^{2N})/(1 - q^2)$ which is also the first coefficient in the character expansion (see first footnote, Section 4).

We did some numerical analysis to test the accuracy of the $1/N$ expansion of W_n . We report here just one example ($N = 3$, $n = 4$) (Table 1). Being $n > N$ it is obvious that we are pushing the expansion too much. Actually, provided we include higher corrections, the expansion turns out to be in good agreement with the exact result (seven figure accuracy at $\lambda = 0.1$ down to two figure accuracy at $\lambda = 3$).

*) The factor $q^{p(p+1)-n(2p-1)}$ is uniquely determined by the requirement that the $1/N$ expansion contains only even powers $N-2k$.

Notice that the loop variables W_n are entire functions of g^2 (despite their appearance they are polynomials in $q = e^{-\lambda/4N} = e^{-g^2/4}$); it is the expansion in $1/N$ at λ fixed which generates singularities in the $(2/N)$ complex plane where the expansion diverges. The only divergent factor is actually $(1 - q^{2p})^{-1}$ in Eq. (42.a) since the other terms are (Gaussian) polynomials in q ; the divergence is in fact there, because the Gaussian polynomials do not vanish as $q \rightarrow 1$. The nearest singularity occurs for $p = n$, i.e., the $1/N$ series for W_n converges for $|\lambda/4N| < \pi/n$, which explains why the planar approximation gets worse and worse for increasing n .

3. THE LEVEL DENSITY

The level density, i.e., the probability of finding an invariant angle, say θ_1 , in the interval $(\theta, \theta+d\theta)$ when the probability measure in the group is $e^{-A}(dU)$ is given by the well-known formula¹¹⁾

$$\rho_N(\theta) = \frac{1}{N} e^{-A(\theta)} \sum_{j=0}^{N-1} |\phi_j(e^{i\theta})|^2 \quad (43)$$

By a summation formula analogous to that of Christoffel-Darboux¹⁰⁾ we obtain

$$\rho_N(\theta) = e^{-A(\theta)} |\phi_N(e^{i\theta})|^2 \left\{ 1 - \frac{2}{N} \operatorname{Re} \left(z \frac{d}{dz} \ln \phi_N^*(z) \right) \right\} \Big|_{z=e^{i\theta}} \quad (44)$$

where ϕ_N^* is the reciprocal polynomial of ϕ_N , i.e., $\phi_N^*(z) = z^N \phi_N(z^{-1})$. The level density is easily given in terms of the loop variables, namely

$$\rho_N(\theta) = \sum_{-\infty}^{+\infty} W_n(\lambda, N) e^{in\theta} \quad (45)$$

Equation (43) is not convenient for estimating the asymptotic $N \rightarrow \infty$ behaviour of $\rho_N(\theta)$. In fact, the known asymptotic properties of orthogonal polynomials¹⁰⁾ refer to a fixed measure, while in our case the measure itself depends on N . It is instructive, however, to apply Szegő's theorem to our case and compare it with the result obtained from Eq. (45). We are taking the limit $N \rightarrow \infty$ with $\lambda/N = g^2$ fixed. Szegő's theorem gives

$$\phi_N^*(z) \sim \exp \left\{ \frac{1}{2} A(\theta) + i \gamma(\theta) \right\} \quad (46)$$

where

$$\gamma(\theta) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \cot \frac{\theta-\omega}{2} \ln \left(\sum_{n=-\infty}^{+\infty} q^{n^2} e^{in\omega} \right) d\omega \quad (47)$$

Taking the derivative and inserting the Fourier expansion of the logarithmic derivative of Jacobi's θ function¹²⁾, we get

$$\gamma'(\theta) = \sum_{n=1}^{\infty} \frac{(-q)^n}{1-q^{2n}} \cos n\theta \quad (48)$$

Plugging the result into Eq. (43) we find

$$\rho_N(\theta) \underset{\substack{N \rightarrow \infty \\ g^2 \text{ fixed}}}{\sim} 1 - \frac{2}{N} \sum_{n=1}^{\infty} \frac{(-q)^n}{1-q^{2n}} \cos n\theta \quad (49)$$

which agrees with Eq. (42.c) ($N \rightarrow \infty$, q fixed).

The asymptotic expansion of $\rho_N(\theta)$ at λ fixed can be obtained by starting from Eq. (45) and inserting the $1/N$ expansion of $W_n(\lambda, N)$. The leading term of W_n [Eq. (40)] can be more conveniently expressed in terms of Legendre polynomials as follows:

$$W_n(\lambda, \infty) = -\frac{1}{n\lambda} \left[P_n(\cos \theta_c) - P_{n-1}(\cos \theta_c) \right] \quad (50)$$

where

$$\cos \theta_c = 2 e^{-\lambda/2} - 1 \quad (51)$$

It is sufficient to recall the generating function of Legendre polynomials to obtain:

$$\rho_{\infty}(\theta) = \frac{4}{\lambda} \operatorname{Re} \left\{ \ln \left[\frac{\cos \theta/2}{\cos \theta_c/2} + \sqrt{\frac{\cos^2 \theta/2}{\cos^2 \theta_c/2} - 1} \right] \right\} \quad (52)$$

This is the same result as the one obtained in Ref. 4) by the saddle point method. The level density is vanishing outside the interval $|\theta| < |\theta_c|$ and has an infinite derivative at the end points. The higher order corrections to $\rho_N(\theta)$ can be obtained through Eq. (41). It can be shown that they too vanish outside the interval $|\theta| < |\theta_c|$ and they can be given in terms of differential operators

acting on the leading term ρ_∞ . This implies that higher order corrections are divergent at the end points. $\theta_c(\lambda)$ appears here as a "classical turning point" and the semiclassical wave function ρ_∞ is ill-defined near θ_c .

It may be interesting to compare the planar level density with the exact one obtained from Eqs. (23), (43) and (44). The main features emerging from this analysis are the following:

- i) $\rho_N(\theta)$ satisfies an approximate scaling property for $0 < \lambda \leq 1$, namely

$$\rho_N(\theta, \lambda) \sim \rho_\infty(\theta, \lambda) \Xi_N\left(\frac{\theta}{\theta_c(\lambda)}\right) \quad (53)$$

where $\theta_c(\lambda)$ is given by Eq. (51) and the universal functions $\Xi_N(x)$ are displayed in Fig. 1. In particular,

$$\rho_\infty(\theta, \lambda) \sim \rho_\infty(0, \lambda) \sqrt{1 - \theta^2/\theta_c^2} \quad (54)$$

with a maximum percentage error of $\sim 10^{-4}$ for all $\lambda \leq 1$; this is quite reasonable since for λ and θ small the Rogers-Szegö polynomials are essentially Hermite polynomials in the variable $x = (N/\lambda)^{1/2}\theta$ and Eq. (54) is the well-known "Wigner's law"¹¹⁾.

- ii) The level density $\rho_N(\theta, \lambda)$ has rather wide oscillations around ρ_∞ which, because of the scaling property, are not negligible even for $\lambda \rightarrow 0$.
- iii) There is a systematic deviation of ρ_N from ρ_∞ in the region $\theta \sim \theta_c(\lambda)$ with a tail in the "classically forbidden region" $\theta > \theta_c(\lambda)$ which is (roughly) exponentially decreasing with N .

All these properties insofar as they refer to the weak coupling limit $\lambda \rightarrow 0$ should hold in general for any other choice of the action. In fact, the presence of a "turning point" $\theta_c(\lambda)$ is necessary in order to have a weak coupling limit to Wigner's law. This was noticed for the first time for Wilson's action¹⁾: in this case

$$\rho_\infty(\theta) = \frac{4}{\lambda} \cos \frac{\theta}{2} \sqrt{\frac{\lambda}{2} - \sin^2 \frac{\theta}{2}} \quad (55)$$

which is also well approximated by Eq. (54) for $\lambda \ll 2$ with $\theta_c(\lambda) = 2\sin^{-1}\sqrt{\lambda/2}$. The approach to this scaling limit is however much slower for Wilson's action than for Villain's.

4. THE CHARACTER EXPANSION

Let $\chi_{(\ell_1, \dots, \ell_N)}$ be the fundamental characters of $U(N)$ given by Weyl's formula¹³⁾. To get the character expansion of Villain's action we have to calculate the integral:

$$c_{(\ell_1, \dots, \ell_N)} = \frac{1}{N! Z} \int_{U(N)} \bar{\chi}_{(\ell_1, \dots, \ell_N)} |\Delta(e^{i\theta})|^2 \prod_{j=1}^N e^{-A(\theta_j)} \frac{d\theta_j}{2\pi} \quad (56)$$

By inserting the explicit form of χ and after expanding the determinants we are left with Gaussian integrals which can be easily calculated. The result is the following:

$$c_{(\ell_1, \dots, \ell_N)} = \det \| q^{(\ell_j - k + 1)^2} \| / \det \| q^{(j - k)^2} \|_{j, k=1, \dots, N} \quad (57)$$

Hence we have:

$$\begin{aligned} \frac{e^{-A}}{Z} &= \sum_{\ell_1 < \ell_2 < \dots < \ell_N} \chi_{(\ell_1, \dots, \ell_N)} \frac{\det \| q^{(\ell_j - k + 1)^2} \|_1^N}{\det \| q^{(j - k)^2} \|_1^N} \\ &= \sum_{\ell_1 < \ell_2 < \dots < \ell_N} \chi_{(\ell_1, \dots, \ell_N)} q^{\sum_j (\ell_j - j + 1)^2} \prod_{j > k} \left[\frac{1 - q^{2(\ell_j - \ell_k)}}{1 - q^{2(j - k)}} \right] \end{aligned} \quad (58)$$

(see Appendix 1 for details). To get the character expansion for $SU(N)$, we recall⁵⁾ the following property of the heat kernel, which also holds for Villain's action ($K \equiv e^{-A}$):

$$K_{U(N)}(\theta_j + \alpha, \lambda) = \sum_{\nu \in \mathbb{Z}_N} K_{SU(N)}(\theta_j^\nu, \lambda) K_{U(1)}(\alpha^\nu, \frac{\lambda}{N}) \quad (59)$$

where

$$\sum \theta_j = 0, \quad \theta_j^\nu = \theta_j - 2\pi\nu/N + 2\pi\nu\delta_{jN}, \quad \alpha^\nu = \alpha + 2\pi\nu/N$$

Multiplying Eq. (59) by $\bar{\chi}_{(\ell_1, \dots, \ell_N)}(\theta_j + \alpha)$ and integrating over $SU(N) \times U(1)$ we get:

$$N \int_{U(N)} K_{U(N)} \bar{\chi} [dU] = \sum_{v \in \mathbb{Z}_N} \int_{SU(N)} \int_{U(1)} K_{SU(N)}(\theta^v, \lambda) K_{U(1)}(\alpha^v, \frac{\lambda}{N}) \bar{\chi}(\theta + \alpha) [d\theta] d\alpha \quad (60)$$

But $\chi(\theta + \alpha) = \chi(\theta^v + \alpha^v)$, hence the integrals in the right-hand side are independent of v and we get ^{*)}

$$\int_{U(N)} K \bar{\chi} [dU] = \int_{SU(N)} K \bar{\chi} [dU] \int_{-\infty}^{+\infty} e^{-N\alpha^2/\lambda - i\alpha \sum_j (\ell_j - j + 1)} d\alpha \quad (61)$$

or finally (putting $\ell_j = m_j + j - 1$)

$$\left[\frac{e^{-A}}{Z} \right]_{SU(N)} = \sum_{0 = m_1 \leq m_2 \leq \dots \leq m_N} \chi_{(m_1, \dots, m_N)} q^{\sum m_j^2 - \frac{1}{N} (\sum m_j)^2} \prod_{j > k} \left[\frac{1 - q^{2(m_j - m_k + j - k)}}{1 - q^{2(j - k)}} \right] \quad (62)$$

$$\equiv \sum_{\{m\}} q^{2C_{\{m\}}} \chi_{\{m\}}(U) \chi_{\{m\}}(T_q) \quad (63)$$

where $C_{\{m\}}$ is the quadratic Casimir and $T_q \in SL(N, \mathbb{C})$ is given by $\text{diag}[q^{N-1}, q^{N-3}, \dots, q^{3-N}, q^{1-N}]$. Other choices of T_q may provide interesting variations on Villain's action [for instance $T_q \equiv \mathbb{1}$ is the heat kernel of Ref. 5)].

The coefficient $d_{\{\ell\}} a_{\{\ell\}}(q)$ of $\chi(U)$ in Eq. (62) can be easily calculated in the same way as one calculates the dimensions of the representations of $SU(N)$. We give the coefficients which enter into the strong coupling expansion of the string tension ^{14c)} in Table 2.

Since the character coefficients are much simpler than the corresponding coefficients of Wilson's action, it is quite straightforward to investigate their behaviour as $N \rightarrow 0$. One finds that the same cancellations which

^{*)} This factorization property also applies to $\langle \text{tr} U^n \rangle$ which gives $\langle \text{tr} U^n \rangle_{SU(N)} = \langle \text{tr} U^n \rangle_{U(N)} q^{-n^2/N}$ (two dimensions).

occur with Wilson's action^{*)} give a finite limit for the string tension as $N \rightarrow \infty$, with a noticeable difference namely that the $1/N^2$ corrections to $d_{\{l\}}^a \{l\}$ are non-vanishing.

A detailed discussion about the $N \rightarrow \infty$ behaviour of the strong coupling expansion cannot be given here. We shall content ourselves with observing that, unlike the case of Wilson's action, the corrections of order $1/N^2$ can indeed fill the gap between $N = \infty$ and N finite, but the expansion breaks down for λ too large. The first few terms in the expansion are as follows (four dimensions):

$$\begin{aligned} \sigma = & -\ln u_\infty - 4 u_\infty^4 (1 - e^{-\lambda/2})^2 - 8 u_\infty^6 - 56 u_\infty^8 + O(u_\infty^{10}) \\ & - \frac{1}{N^2} \left\{ \frac{1}{4} \left(\lambda - \frac{\lambda^2}{4} \right) \left[1 + 16 u_\infty^4 (1 - e^{-\lambda/2})^2 + 48 u_\infty^6 + 448 u_\infty^8 \right] \right. \\ & \left. + 4 u_\infty^4 (1 - e^{-\lambda/2}) + O(u_\infty^{10}) \right\} + O\left(\frac{1}{N^4}\right) \end{aligned} \quad (64)$$

where $u_\infty = (1 - e^{-\lambda/2})/(\lambda/2)$.

Let us now consider the simple case $N = 2$. By denoting $m_2 = 2j$ ($m_1 = 0$) we have

$$(2j+1) a_{2j} = q^{2j^2} \frac{1 - q^{2(2j+1)}}{1 - q^2} \quad (65)$$

These coefficients can be inserted into Münster's expansion for the string tension^{14a)} $[u = 1/2 q^{1/2}(1 + q^2), \text{ etc.}]$. The result is reported in Fig. 2 together with Monte Carlo data taken from Ref. 9). (The data refer to Manton's action which, however, is indistinguishable from Villain's in the range $\beta = 4g^{-2} \gtrsim 0.5$.)

Notice that the tangency point to the asymptotic freedom fit occurs at $u \sim 0.46$ which is the same value (in the "universal parameter" u) found for the heat kernel^{**) action}. When expressed in terms of the respective u parameters, the Monte Carlo data for Manton's and the heat kernel action agree with one single exception. If one corrects for this "event", one would find a different value $\Lambda_0^{(V)} \sim 0.0224$ [10% higher than the value given in Ref. 9)].

^{*)} Notice that there is a misprint in Eq. (8) of Ref. 14c) which should read $d_{i_1} \sim V_1 N^2 u^2$, $D_{i_1 i_1} \sim W_1 N^3 u^3$. These limits hold for Villain's action too, but the corrections are down by a factor N^{-2} only, instead of $(2u)^N$.

^{**) The heat kernel is called "Villain" in Ref. 9).}

5. CONCLUDING REMARKS

We have considered the properties of lattice gauge theory which arise when one substitutes Wilson's action by the simplest non-Abelian generalization of Villain's model. The two dimensional case can be studied in the greatest detail, in particular:

- i) a closed form (not involving N dimensional determinants) was derived for the partition function, the specific heat and for the multiple Wilson loops;
- ii) the $1/N$ expansion turns out to be convergent and its coefficients are analytic in λ in a whole neighbourhood of the positive λ real axis, but
- iii) $\lambda \rightarrow \infty$ and $N \rightarrow \infty$ do not commute;
- iv) the level density has big "non-perturbative" (in $1/N$) contributions, its behaviour being in several respects similar to that of a semiclassical wavefunction. Finally the character expansion which allows us to calculate strong coupling expansion was exhibited for both $SU(N)$ and $U(N)$; since there is no Gross - Witten singularity, we are always in the "weak coupling phase" and the coefficients of the strong coupling expansion for the string tension admit a $1/N$ expansion: next-to-leading corrections are, however, rather large.

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APPENDIX 1 - D_N

We have to calculate the determinant

$$D_N(q) = \det \left\| q^{(j-k)^2} \right\|_{j,k=0,1,\dots,N} \quad (A.1)$$

More generally, let $m_1, \dots, m_N$ be any complex numbers and let

$$\begin{aligned} D_{N-1}(q, m) &= \det \left\| q^{(m_j + j - k)^2} \right\|_{j,k=1,\dots,N} \\ &= q^{\sum m_j^2} \det \left\| q^{(s_j - k)^2 - (s_j - j)^2} \right\| \\ &= q^{\sum m_j^2} \det \left\| q^{2s_j(j-k)} \right\| \quad (s_j = m_j + j) \end{aligned} \quad (A.2)$$

This is reduced to Vandermonde's determinant and we then obtain

$$D_{N-1}(q, m) = q^{\sum m_j^2} \prod_{j>k} (1 - q^{2(m_j - m_k + j - k)}) \quad (A.3)$$

In particular, for $m_1 = m_2 = \dots = m_N = 0$ we get [Eq. (17)]:

$$D_{N-1}(q) = \prod_{j>k} (1 - q^{2(j-k)}) = \prod_{j=1}^{N-1} [j]! \quad (A.4)$$

where $[n]! = \prod_{k=1}^n (1 - q^{2k})$. Equation (A.3) for $m_j = \ell_j - j + 1$ ($m_1 \leq m_2 \leq \dots, \leq m_N$ integers) gives the character coefficients [Eq. (58)].

APPENDIX 2 - $1/N$ EXPANSION OF $\ln Z(\lambda, N)$

We shall now derive the $1/N$ expansion of $\ln Z(\lambda, N)$. Starting from Eq. (17) we have

$$\frac{1}{N^2} \ln Z(\lambda, N) = \frac{1}{2N} \ln \frac{\lambda}{4\pi N} + \sum_{k=1}^{N-1} \frac{k}{N^2} \ln \left[1 - e^{-\frac{\lambda}{2}(1-\frac{k}{N})} \right] \quad (A.5)$$

This is already in a form of the type $\sum_{k=1}^{N-1} 1/N F(k/N)$, but $F(x)$ diverges at $x = 1$. Before applying the Euler-McLaurin summation formula, it is therefore convenient to regularize the function $F(x)$ as follows:

$$F(x) = x \ln(1 - e^{-\frac{1}{2}\lambda(1-x)}) \equiv x \ln \left[\frac{1 - e^{-\frac{1}{2}\lambda(1-x)}}{\frac{1}{2}\lambda(1-x)} \right] + x \ln \left[\frac{\lambda}{2}(1-x) \right] \quad (A.6)$$

$$\equiv x \ln \left[\frac{\text{sh } \frac{\lambda}{4}(1-x)}{\frac{\lambda}{4}(1-x)} \right] - \frac{\lambda x}{4}(1-x) + x \ln \left[\frac{\lambda}{2}(1-x) \right]$$

The last two terms in Eq. (A.6) can be summed directly to give

$$-\frac{\lambda}{24} \left(1 - \frac{1}{N^2}\right) + \frac{N(N-1)}{2N^2} \ln \frac{\lambda}{2} + \frac{1}{N^2} \ln \left[\frac{\prod_{\nu=1}^{N-1} \nu!}{N^{N(N-1)/2}} \right] \quad (A.7)$$

The functions appearing in the first term of Eq. (A.6), i.e.,

$$f(x) = x \ln \left[\frac{\text{sh } \frac{\lambda}{4}(1-x)}{\frac{\lambda}{4}(1-x)} \right] \quad (A.8)$$

is analytic near the interval $0 < x < 1$; the nearest singularity occurs at $\lambda(1-x)/4 = \pm i\pi$, i.e., $x = 1 \pm 4\pi i/\lambda$. The Euler-McLaurin summation formula can therefore be applied and we get Eq. (18). The remaining term is of $O[(\lambda/N)^{2p}]$ for λ small but it grows for $\lambda \rightarrow +\infty$ because of the end point singularities.

APPENDIX 3 - ROGERS-SZEGÖ POLYNOMIALS

Let $\phi_n(z) = k_n z^n + \dots$ be the polynomials orthonormal with respect to the measure $\sum_{n=0}^{\infty} q^{n^2} z^n dz / 2\pi i z$ on the unit circle. From the general theory¹⁰⁾ we have $k_n = (D_{n-1}/D_n)^{1/2} = ([n]!)^{-1/2}$ [Eq. (A.4)]. We also have the recurrence relations

$$k_n z \phi_n = k_{n+1} \phi_{n+1} - k_{n+1} \mu_{n+1} \phi_n^* \quad (\text{A.9})$$

$$k_n \phi_{n+1} = k_{n+1} z \phi_n + k_{n+1} \mu_{n+1} \phi_n^* \quad (\text{A.10})$$

where $\phi_n^*(z) = z^n \phi_n(z^{-1})$ and $\mu_n = k_n^{-1} \phi_n(0)$. By eliminating ϕ_n^* from the recurrence relations we find

$$z \phi_n = \frac{k_n}{k_{n+1}} \phi_{n+1} - \frac{\mu_{n+1}}{\mu_n} \phi_n + \frac{\mu_{n+1}}{\mu_n} \frac{k_{n-1}}{k_n} z \phi_{n-1} \quad (\text{A.11})$$

or

$$z \phi_n = \sqrt{(1-q^{2n+2})} \phi_{n+1} - \frac{\mu_{n+1}}{\mu_n} \phi_n + \frac{\mu_{n+1}}{\mu_n} z \sqrt{(1-q^{2n})} \phi_{n-1} \quad (\text{A.12})$$

We now need the explicit value of μ_n . From the explicit expression of $\phi_v(z)$ as a determinant¹⁰⁾ we have

$$\mu_n = (-)^n D_{n-1}^{-1} \det \| q^{(j-k-1)^2} \|_1^n = (-q)^n \quad (\text{A.13})$$

The fact that μ_{n+1}/μ_n is a constant enormously simplifies the recurrence relation Eq. (A.12) and leads to its final form Eq. (26). It can be easily shown that no other set of polynomials enjoys this property. Now, let $\phi_v(z) = k_v \sum_{j=0}^v a_j^v z^{v-j}$; the coefficients are determined through the recurrence relation:

$$a_k^{n+1} = a_k^n - q a_{k-1}^n + q(1-q^{2n}) a_{k-1}^{n-1} \quad (\text{A.14})$$

[which follows from Eq. (26)] and they are given by Eq. (23).

Rogers-Szegő polynomials are known to be related to Hermite polynomials⁶⁾. One way to see it is to scale the angle $\theta = gx = (\lambda/N)^{-1/2}x$ and take the limit $N \rightarrow \infty$ ($g \rightarrow 0$). Despite the fact that the normalization $k_n = (n!)^{-1/2}$ diverges like $N^{n/2}$, there are strong cancellations between various terms which give a finite limit, precisely

$$\lim_{N \rightarrow \infty} \phi_n(e^{i\sqrt{\lambda/N}x}) = \frac{i^n H_n(x)}{\sqrt{2^n n!}} \quad (\text{A.15})$$

This can be proved by induction through the recurrence relation, since it is obviously true for $n = 0, 1$. Notice, however, that the limit relation (A.15) is non-uniform in n since otherwise we would obtain Wigner's law for the level density instead of Eq. (52) (we do obtain Wigner's law only as $\lambda \rightarrow 0$).

APPENDIX 4 - THE DIAGONAL MATRIX ELEMENTS OF z^n

We start from Eq. (36) giving the operator $(Z\phi)(z) = z\phi(z)$ as a function of the bounded operators $A, A^\dagger$. To simplify the notation we shall define

$$a = qA, \quad a^* = q^{-1}A^\dagger, \quad 1 - q^2 = \epsilon \quad (A.16)$$

The operators a, a^* satisfy the commutation relations

$$[a, a^*] = \epsilon (1 - a^*a) \quad (A.17)$$

Our goal is to calculate the diagonal matrix elements of the operator

$$Z^n = q^n \{ (1 + a^*)(1 + a)^{-1} \}^n \quad (A.18)$$

By writing $(1 + a)^{-1} = \int_0^\infty ds e^{-s(1+a)}$ we get:

$$\left(\frac{Z}{q}\right)^n = (1 + a^*) \int_0^\infty ds_1 e^{-s_1(1+a)} (1 + a^*) \int_0^\infty ds_2 e^{-s_2(1+a)} \dots (1 + a^*) \int_0^\infty ds_n e^{-s_n(1+a)} \quad (A.19)$$

After introducing new integration variables $y_m = \sum_{k=1}^m s_k$, this becomes

$$\left(\frac{Z}{q}\right)^n = (1 + a^*) \int_0^{y_2} dy_1 \dots \int_0^{y_n} dy_{n-1} \int_0^\infty dy_n [1 + a^*(y_1)] \dots [1 + a^*(y_{n-1})] e^{-y_n(1+a)} \quad (A.20)$$

where

$$\begin{aligned} a^*(y) &= e^{-ya} a^* e^{ya} = a^* + (1 - a^*a) \frac{1 - e^{-\epsilon ya}}{\epsilon} \\ &= a^* - \epsilon y (1 - a^*a) - \frac{1}{2} \epsilon^2 y^2 (1 - a^*a) a + \dots \end{aligned} \quad (A.21)$$

Now we insert $a^*(y)$ into Eq. (A.20) and retain only the terms at most quadratic in ϵ . This gives

$$\left(\frac{Z}{q}\right)^n = \int \dots \int_{0 \leq y_1 \leq \dots \leq y_n} dy_1 \dots dy_n \left\{ (1 + a^*)^n - \epsilon \sum_{m=1}^{n-1} (1 + a^*)^m (1 + a^*a) (1 + a^*)^{n-m-1} \right\} \quad (A.22)$$

$$\begin{aligned}
 & -\frac{1}{2} \varepsilon^2 \sum_{m=1}^{n-1} y_m^2 (1+a^*)^{n-1} (1-a^*a) a + \frac{1}{2} \varepsilon^2 \sum_{m \neq m'}^{n-1} y_m y_{m'} (1+a^*)^{n-2} (1-a^*a)^2 \\
 & + O(\varepsilon^3) \Big\} e^{-y_n(1+a)} \\
 & = (1+a^*)^n (1+a)^{-n} - \varepsilon A_1 (1-a^*a) (1+a^*)^{n-1} (1+a)^{-n-1} \\
 & + \varepsilon^2 (1-a^*a) (1+a^*)^{n-2} \left[A_2 (1-a^*a) (1+a)^{-n-2} \right. \\
 & \left. - A_3 (1+a)^{-n-1} - A_4 a^* (1+a)^{-n-1} \right] + O(\varepsilon^3)
 \end{aligned} \tag{A.22}$$

cont.

where the constants $(A_j; j = 1, \dots, 4)$ are given as follows:

$$\begin{aligned}
 A_1 &= \frac{n(n-1)}{2} \\
 A_2 &= \frac{n(n^2-1)(3n-2)}{24} \\
 A_3 &= \frac{n(n^2-1)}{6} \\
 A_4 &= \frac{n(n-1)(2n-1)}{6}
 \end{aligned} \tag{A.23}$$

The diagonal part of Z^n is now obtained by expanding $(1+a)^{-n}$ and keeping only products with an equal number of a, a^* . It is convenient to order all terms in such a way that Z^n be expressed as a function of $1-a^*a$. To achieve this we have still to rearrange the operators occurring in the $O(1)$ and $O(\varepsilon)$ terms by making use of relations like

$$a^{*s} a^s = a^* a \prod_{p=1}^{s-1} \left[1 - \frac{1-a^*a}{(1-\varepsilon)^p} \right] \tag{A.24}$$

and of elementary binomial coefficient identities¹⁵⁾ like

$$\sum_{m=0}^n (-1)^m \binom{n}{m} \binom{s-m}{p} = \binom{s-n}{p-n} \tag{A.25}$$

We find

$$\langle \phi_j | z^n | \phi_j \rangle = q^n \sum_{p=1}^n C_{n,p} \langle \phi_j | (1-a^*a)^p | \phi_j \rangle \left(1 + \varepsilon \frac{n-p}{2} + \varepsilon^2 \eta_{n,p} + o(\varepsilon^3) \right) \quad (A.26)$$

where

$$C_{n,p} = (-1)^{n+p} \binom{n}{p} \binom{n+p-1}{n} \quad (A.27)$$

$$\eta_{n,p} = \frac{1}{12} [(p+1)n^2 - 3(p-1)n + p^2 - 3p] \quad (A.28)$$

Since

$$\langle \phi_j | (1-a^*a)^p | \phi_j \rangle = q^{2jp} \quad \sum_{j=0}^{N-1} \langle \phi_j | z^n | \phi_j \rangle$$

gives

$$\langle \text{tr } U^n \rangle = q^n \sum_{p=1}^n C_{n,p} \frac{1-q^{2Np}}{1-q^{2p}} \left[1 + \varepsilon \frac{n-p}{2} + \varepsilon^2 \eta_{n,p} + o(\varepsilon^3) \right] \quad (A.29)$$

The $O(\varepsilon)$ contribution from $q^n(1-q^{2p})^{-1}$ exactly cancels $(n-p)\varepsilon/2$ and we get the final result

$$\frac{1}{N} \langle \text{tr } U^n \rangle = \sum_{p=1}^n \frac{1-e^{-\lambda p/2}}{\lambda p/2} C_{n,p} \left\{ 1 + \frac{\lambda^2}{96N^2} [(2p-1)n^2 - 2p^2] + O\left(\left(\frac{\lambda}{N}\right)^4\right) \right\} \quad (A.30)$$

If our conjecture, Eq. (42), is true the next correction $[O(\lambda/N)^4]$ in Eq. (A.30) should be the following

$$\frac{1}{360} \left(\frac{\lambda}{4N}\right)^4 [(2p-1)(10p-7)n^4 - 4p(2p-1)(6p-1)n^2 + 24p^4] \quad (A.31)$$

λ	A	B	C	D
0.1	0.65244	0.06544125	0.65441381	0.65441385
0.2	0.39452	0.4004604	0.40047729	0.40047731
0.5	-0.01748	-0.0036947	-0.00336048	-0.00335826
1.0	-0.12730	-0.1291862	-0.12770676	-0.12765323
1.5	-0.04690	-0.0731760	-0.07213461	-0.07191318
3.0	0.06414	0.0548528	0.04190347	0.04223070

Table 1 : $W_q(\lambda, N = 3)$ in the leading $N \rightarrow \infty$ approximation (A), including the first $1/N^2$ correction (B), the $1/N^4$ correction (C); exact value (D).

Young Tableau	$d_{\{\ell\}}$	$d_{\{\ell\}}^a_{\{\ell\}}$	Notation of Ref. 14c)
$\square$	N	$q^{1-1/N} \begin{bmatrix} N \\ 1 \end{bmatrix}$	u
$\square \square$	$\binom{N+1}{2}$	$q^{4-4/N} \begin{bmatrix} N+1 \\ 2 \end{bmatrix}$	$d_1 v_1$
$\begin{array}{c} \uparrow \square \square \\ N-1 : \\ \downarrow \square \end{array}$	N^2-1	$q^2 \frac{(1-q^{2(N-1)})(1-q^{2(N+1)})}{(1-q^2)^2}$	$d_2 v_2$
$\begin{array}{c} \square \\ \square \end{array}$	$\binom{N}{2}$	$q^{2-4/N} \begin{bmatrix} N \\ 2 \end{bmatrix}$	$d_3 v_3$
$\square \square \square$	$\binom{N+2}{3}$	$q^{9(1-1/N)} \begin{bmatrix} N+2 \\ 3 \end{bmatrix}$	$D_1 w_1$
$\begin{array}{c} \uparrow \square \square \square \\ N-1 : \\ \downarrow \square \end{array}$	$\frac{1}{2}(N-1)N(N+2)$	$q^{5-1/N} \frac{(1-q^{2(N-1)})(1-q^{2N})(1-q^{2(N+2)})}{(1-q^2)^2(1-q^4)}$	$D_2 w_2$
$\begin{array}{c} \square \square \\ \square \end{array}$	$\frac{1}{3}N(N^2-1)$	$q^{5-9/N} \frac{(1-q^{2(N-1)})(1-q^{2N})(1-q^{2(N+1)})}{(1-q^2)^2(1-q^6)}$	$D_3 w_3$
$\begin{array}{c} \square \\ \square \\ \square \end{array}$	$\binom{N}{3}$	$q^{3-9/N} \begin{bmatrix} N \\ 3 \end{bmatrix}$	$D_4 w_4$
$\begin{array}{c} \uparrow \square \square \\ N-1 : \\ \downarrow \square \end{array}$	$\frac{1}{2}(N-2)N(N+1)$	$q^{3-1/N} \frac{(1-q^{2(N-2)})(1-q^{2N})(1-q^{2(N+1)})}{(1-q^2)^2(1-q^4)}$	$D_5 w_5$

Table 2 : The coefficients of the character expansion of Villain's action for $SU(N)$: $\binom{N}{k}$ are ordinary binomial coefficients, $\begin{bmatrix} N \\ k \end{bmatrix}$ are q binomial coefficients (Gauss polynomials) defined by Eq. (25).

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FIGURE CAPTIONS

- Figure 1 : The "universal" level density $E_N(x)$ [Eq. (53)]: E_∞ is given by the "semicircle-law" [Eq. (54)].
- Figure 2 : Strong coupling expansion for the string tension σ (solid line) compared to the Monte Carlo data of Ref. 9).

Fig. 1

FIG.2