THE SUPERCURRENT IN $N = 1$ SUPERSYMMETRICAL YANG-MILLS THEORIES

II. Renormalization

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ABSTRACT

We show to all orders of perturbation theory that in a general $N = 1$ supersymmetrical Yang-Mills theory maintaining parity there exists a supercurrent. Since all Faddeev-Popov fields and gauge contributions are taken into account, the relation between supercurrent anomalies and the Callan-Symanzik equation can be established. The connection with the axial anomaly coefficient is somewhat obscured by the fact that the supercurrent cannot be improved.

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INTRODUCTION

In this paper we continue the study of the supercurrent in $N = 1$ supersymmetrical Yang-Mills (SYM) theories. Whereas in Ref. 1) (referred to as I) we have constructed the tree approximation, we show here the existence of the supercurrent to all (higher) orders of the perturbation series. The class of models we consider is rather large: the gauge group is assumed to be compact and semi-simple but otherwise arbitrary; the matter fields lie in arbitrary representations and may have self-interactions. Parity and charge conjugation invariance is assumed. As far as renormalization schemes are concerned, we only suppose that they allow for the action principle²⁾. The proof we shall present is algebraic in nature and does not rely on special invariances respected by the subtraction scheme.

Unfortunately, there is at present still one severe limitation, namely one due to the (off-shell!) infra-red divergences in supersymmetrical theories containing vector fields³⁾. In order to avoid them we put masses for the fields and consider only the asymptotic theory at large Euclidean momenta. As a bonus, all our results hold true in the case where supersymmetry is softly (spontaneously or explicitly) broken. Another (trivial) omission is that we suppress additional symmetries (and their breaking) commuting with supersymmetry and the gauge group. Including them would modify, e.g., the gauge fixing but not really alter our proof.

In Sect. 1 we recall, together with some notation, results of I and Ref. 4) needed for the sequel. In Sect. 2 we show how, on the basis of the action principle, the existence of the renormalized supercurrent can be proved. In Sect. 3 the relation between supercurrent and the Callan-Symanzik (CS) equation on the one hand, and the restriction on the anomalies provided by R invariance on the other, is established. It is shown that despite the fact that the supercurrent cannot be improved, the trace equation still governs the potential relations amongst the anomalies but in a rather indirect fashion. The Appendix contains some calculational details of Sect. 2.

1. THE RENORMALIZED MODEL

In I we have constructed the tree approximation supercurrent for a parity invariant, $N = 1$ SYM theory with compact, semi-simple gauge group G and matter fields in arbitrary representation. To be specific, we have considered the following field multiplets:

ϕ^i pseudoscalar, vector superfields in the adjoint representation of G

A^i chiral (matter) superfields in the adjoint representation of G

$A \pm a$ chiral (matter) superfields in an arbitrary unitary representation of G .

Notation:

$\phi \equiv \phi^i \tau^i$ τ^i generators in the fundamental representation of G

$\Phi \equiv \phi^i T^i$ T^i generators in an arbitrary unitary representation of G

$A \equiv A^i \tau^i$ $A_{\pm} \equiv (A \pm a)$

Furthermore, there have been introduced chiral Faddeev-Popov $(\phi\pi)$ superfields

$$C_{\pm} \equiv c_{\pm}^i \tau^i \quad \tilde{C}_{\pm} \equiv c_{\pm}^i T^i$$

and sources

$$\eta \equiv \eta^i \tau^i \quad Z \equiv Z^i \tau^i \quad Y \equiv Y^i \tau^i \quad Y_{\pm} \equiv (Y_{\pm} a)$$

$$\rho \equiv \rho^i \tau^i$$

(η, ρ pseudoscalar, vector superfields; $Z, Y, Y_{\pm}$ chiral superfields). The contributions of all of these fields to the supercurrent have been computed in the corresponding BRS-invariant model.

In Ref. 4) we have renormalized the model, i.e., shown that - up to a redefinition of the fields and a change of the gauge fixing - there exists uniquely a (one-particle, irreducible) functional $\Gamma = \Gamma(\phi, \eta, c_{\pm}, A, A_{\pm}, Z, Y, Y_{\pm})$ such that the following Ward Identities (WI) hold to all orders in perturbation theory:

$$W_{\alpha}^Q \Gamma = 0 = \bar{W}_{\dot{\alpha}}^Q \Gamma \quad (1.1)$$

$$W^R \Gamma \simeq 0 \quad (1.2)$$

$$\Delta(\Gamma) \equiv \text{Tr} \left(\int dV \frac{\delta \Gamma}{\delta \eta} \cdot \frac{\delta \Gamma}{\delta \phi} + \int dS \left(\frac{\delta \Gamma}{\delta Z} \cdot \frac{\delta \Gamma}{\delta c_{+}} + \frac{\delta \Gamma}{\delta Y} \frac{\delta \Gamma}{\delta A} \right) \right. \\ \left. + \int d\bar{S} \left(\frac{\delta \Gamma}{\delta \bar{Z}} \cdot \frac{\delta \Gamma}{\delta \bar{c}_{+}} + \frac{\delta \Gamma}{\delta \bar{Y}} \frac{\delta \Gamma}{\delta \bar{A}} \right) \right) \simeq 0 \quad (1.3)$$

Here $W_{\alpha}^Q, \bar{W}_{\dot{\alpha}}^Q, W^R$ denote the WI operators for supersymmetry and R transformations, respectively; $W \equiv -i \int dz \delta \phi \delta_{\phi}$. Equation (1.3) is the Slavnov identity expressing invariance under BRS-transformations. The $\simeq$ sign means that the right-hand side is soft, i.e., vanishes in the deep Euclidean region. In fact, as mentioned in the Introduction, in order to ensure infra-red convergence we have added mass terms which break R and BRS-symmetry softly.

The field transformations are given by

$$\delta_\alpha \Phi = \left(\frac{\gamma}{\gamma \theta^\alpha} + i \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu \right) \Phi$$

$$\delta_{\dot{\alpha}} \Phi = \left(-\frac{\gamma}{\gamma \bar{\theta}^{\dot{\alpha}}} - i \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \right) \Phi$$

$$\delta^R \Phi = i(n + \theta \partial_\theta + \bar{\theta} \partial_{\bar{\theta}}) \Phi$$

(n: R weight of Φ , Φ any field of the theory)

(1.4)

BRS:

$$b\phi = Q \equiv R\{c_+\} - \bar{R}\{\bar{c}_+\} \quad (\text{cf. I})$$

$$b c_+ = -c_+ c_+$$

$$b A = [A, c_+]$$

$$b A_+ = -\tilde{c}_+ A_+$$

$$b A_- = A_- \tilde{c}_+$$

We also collect the quantum numbers of all required quantities in the Table:

	θ	$\bar{\theta}$	D	$\bar{D}$	ϕ	A	c_+	c_-	η	ρ	Z	Y
d	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	0	1	0	1	2	3	2	
n	-1	+1	+1	-1	0	$-\frac{2}{3}$	0	-2	0	-2	$-\frac{4}{3}$	
$Q_{\phi\pi}$	0	0	0	0	0	0	+1	-1	-1	-2	-1	

$Q_{\phi\pi}$: $\phi\pi$ charge, d: dimension, n: R weight

The only other ingredients needed from other papers are some properties of the contact terms w_α governing the so-called trace equation of the supercurrent (I, Eq. (3.6)):

$$\bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} \simeq -2 w_\alpha \Gamma - D_\alpha S \quad (1.5)$$

As derived in Sect. 2 of I, the contact terms $w_\alpha \Gamma$ have dimension $7/2$, R weight -1 and commute with BRS-transformations. Furthermore, their explicit expression (2.15) of I yields

$$\overline{D} D w_\alpha = 0 \quad (1.6)$$

2. BRS-INVARIANCE OF CURRENT AND BREAKING

Once the symmetries (1.1), (1.2) are established on the quantum level - translation invariance being trivial - one can, in fact, follow quite closely the arguments of the tree approximation for the construction of the supercurrent. Exactly as in I the equations

$$w \Gamma \simeq [\partial^\mu V'_\mu] \cdot \Gamma \quad (2.1)$$

$$\begin{aligned} 2 w_\alpha \Gamma &\simeq [-\overline{D}^\alpha V'_{\alpha\dot{\alpha}} + \overline{D} D \partial_\alpha B + 2 B'_\alpha] \cdot \Gamma \\ &\equiv 4 \Delta_\alpha \cdot \Gamma \end{aligned} \quad (2.2)$$

with

$$\begin{aligned} B &= \overline{B} & \partial^\alpha B'_\alpha &= 0 \\ V'_\mu & \text{ parity odd} \end{aligned} \quad (2.3)$$

hold, now as a consequence of the renormalized action principle^{2),3)} and of locality of the insertions. The R quantum numbers are also as in the tree approximation ($V' : 0, B : 0, B' : -1$) whereas the naive dimensions represent now the power counting assignment of the insertions ($V' : 3, B : 2, B' : 7/2$).

In principle, one could now proceed by listing all terms Δ_α with the above quantum numbers and check whether they can be split into current + breaking with $B'_\alpha = 0$. But in practice, this is not feasible so instead we shall prove separate BRS-invariance for current and breaking since this is the only property still lacking.

In order to do so we infer from the Abelian limit⁵⁾ that the above B version of the breaking will, in general, not allow the proof of gauge invariance. Indeed [cf. (3.28) of Ref. 5)], the axial anomaly contributes to the breaking

$$S_\alpha = \tau_k \bar{\mathcal{D}}\mathcal{D}\phi \bar{\mathcal{D}}\mathcal{D}\phi \quad (2.4)$$

and although this can be written as a B type term

$$S_\alpha = \tau_k \bar{\mathcal{D}}\mathcal{D} B_\alpha = \tau_k \bar{\mathcal{D}}\mathcal{D} (\mathcal{D}\phi \bar{\mathcal{D}}\mathcal{D}\phi + \bar{\mathcal{D}}\phi \mathcal{D}\mathcal{D}\bar{\mathcal{D}}\phi + \phi \mathcal{D}\bar{\mathcal{D}}\mathcal{D}\phi) \quad (2.5)$$

we are in conflict with gauge invariance for this B_α . So we had better cast (2.2) into the form

$$\begin{aligned} \mathcal{L} u_\alpha \Gamma &\simeq [-\bar{\mathcal{D}}^\alpha V_{\alpha\dot{\alpha}} - 2\mathcal{D}_\alpha S + 2B'_\alpha] \cdot \Gamma \\ V &= V' + \frac{2}{3} [\mathcal{D}, \bar{\mathcal{D}}] B, \quad S = \frac{1}{6} \bar{\mathcal{D}}\mathcal{D} B \end{aligned} \quad (2.6)$$

Using now (1.6) we find that

$$B'_\alpha = \mathcal{D}^\beta S_{(\alpha\beta)} \quad (2.7)$$

$[(\alpha\beta)]$ symmetrization symbol] with $S_{(\alpha\beta)}$ being chiral. This yields the final form to be used:

$$\mathcal{L} u_\alpha \Gamma \simeq [-\bar{\mathcal{D}}^\alpha V_{\alpha\dot{\alpha}} + 2\mathcal{D}^\beta S_{\alpha\beta}] \cdot \Gamma \quad (2.8)$$

with

$$S_{[\alpha\beta]} = \varepsilon_{\alpha\beta} S \quad (2.9)$$

and

$$S_{\alpha\beta} = S_{(\alpha\beta)} + S_{[\alpha\beta]} \quad (2.10)$$

We shall now, indeed, prove in this form that the current V and the breaking $S_{\alpha\beta}$ are separately BRS-invariant, where S in (2.9) need not necessarily be of the form $\bar{\mathcal{D}}\mathcal{D}B$ with B invariant.

We decompose into invariant and variant parts:

$$\begin{aligned}
V_{\alpha\dot{\alpha}} &= V_{\alpha\dot{\alpha}}^H + V_{\alpha\dot{\alpha}}^b & \mathcal{L} X^H &= 0 \\
S_{\alpha\beta} &= S_{\alpha\beta}^H + S_{\alpha\beta}^b & \mathcal{L} X^b &\neq 0
\end{aligned}
\quad X=V,S \quad (2.11)$$

and obtain from BRS-invariance of (2.8) [using $\Delta_\alpha \cdot \Gamma = \Delta_\alpha + o(\hbar \Delta_\alpha)$]

$$\bar{\mathcal{D}} \mathcal{L} V^b = \mathcal{D} \mathcal{L} S^b \quad (2.12)$$

i.e., one has to solve a type of homogeneous trace equation. Let us assume, for the moment, that

$$\begin{aligned}
\mathcal{L} V^b &= \mathcal{D}_V u \\
\mathcal{L} S^b &= \mathcal{D}_S u
\end{aligned} \quad (2.13)$$

($\mathcal{D}$ being some differential operator) are the solutions of (2.12) and furthermore, that

$$\begin{aligned}
\mathcal{D}_V u &= \mathcal{L} \mathcal{D}_V \hat{u} \\
\mathcal{D}_S u &= \mathcal{L} \mathcal{D}_S \hat{u}
\end{aligned} \quad (2.14)$$

are indeed variations. Then

$$\begin{aligned}
V^b &= \mathcal{D}_V \hat{u} \\
S^b &= \mathcal{D}_S \hat{u}
\end{aligned} \quad (2.15)$$

and fulfil themselves, Eq. (2.12). Thus

$$2 \kappa_\alpha \Gamma \simeq [-\bar{\mathcal{D}}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} + 2 \mathcal{D}^\beta S_{\alpha\beta}] \cdot \Gamma$$

reduces to

$$2 \kappa_\alpha \Gamma \simeq [-\bar{\mathcal{D}}^{\dot{\alpha}} V_{\alpha\dot{\alpha}}^H + 2 \mathcal{D}^\beta S_{\alpha\beta}^H] \cdot \Gamma \quad (2.16)$$

as desired.

We are left with the proof of the intermediate steps. To begin with we note that the c_- dependence of Δ_α and thus of V and S can be explicitly determined due to the equations of motion

$$\frac{\delta \Gamma}{\delta c_-} = 0 \quad \frac{\delta \Gamma}{\delta \bar{c}_-} = 0 \quad (2.17)$$

for $\Gamma = \Gamma(\eta, \dots)$ [cf. Ref. 4)]. They tell that Γ does not depend on $c_-(\bar{c}_-)$ and therefore in

$$\begin{aligned} (\Delta_\alpha \cdot \Gamma)(c_-, \eta, \dots) &= \frac{1}{2} \text{Tr} \Gamma \equiv \frac{1}{2} \text{Tr} (\dots - \overline{\partial\partial} \partial_\alpha \eta \delta_\eta \\ &+ \partial_\alpha \eta \overline{\partial\partial} \delta_\eta + \frac{2}{\alpha} \partial_\alpha \overline{\partial\partial} \bar{c}_- \overline{\partial\partial} \delta_\eta - c_- \partial_\alpha \delta_{c_-} + \frac{2}{\alpha} c_- \partial_\alpha \overline{\partial\partial} \partial\partial \delta_\eta) \Gamma \end{aligned} \quad (2.18)$$

the c_- dependence of $\Delta_\alpha \cdot \Gamma$ is given by

$$\begin{aligned} (\Delta_\alpha \cdot \Gamma)(c_-) &= \frac{2}{\alpha} \text{Tr} (c_- \partial_\alpha \overline{\partial\partial} \partial\partial Q + \partial_\alpha \overline{\partial\partial} \bar{c}_- \overline{\partial\partial} Q \\ &+ \text{terms rendering } (\Delta_\alpha \cdot \Gamma)(c_-) \text{ } \mathcal{L}\text{-invariant}) \\ &=: \Delta_\alpha^4 \cdot \Gamma \end{aligned}$$

$$\begin{aligned} \Delta_\alpha^4 \Big|_{\text{tree}} &= \frac{2}{\alpha} \text{Tr} (c_- \partial_\alpha \overline{\partial\partial} \partial\partial Q + \partial_\alpha \overline{\partial\partial} \bar{c}_- \overline{\partial\partial} Q - \overline{\partial\partial} \phi \partial_\alpha \overline{\partial\partial} \phi) \\ &= -\frac{4}{\alpha} (\overline{\partial\partial} \partial_\alpha B^2 - \overline{\partial}^{\dot{\alpha}} (V_{\alpha\dot{\alpha}}^2 + V_{\alpha\dot{\alpha}}^3)) \end{aligned} \quad (2.19)$$

$$B^2 = \text{Tr} (\partial\partial \phi \overline{\partial\partial} \phi + \overline{\partial\partial} Q \bar{c}_- + \partial\partial Q c_-)$$

$$V_{\alpha\dot{\alpha}}^2 = \text{Tr} (\partial_\alpha \overline{\partial\partial} \phi \overline{\partial}_{\dot{\alpha}} \partial\partial \phi + \partial_\alpha \overline{\partial\partial} Q \overline{\partial}_{\dot{\alpha}} \bar{c}_- - \overline{\partial}_{\dot{\alpha}} \partial\partial Q \partial_\alpha c_-)$$

$$\begin{aligned} V_{\alpha\dot{\alpha}}^3 &= \text{Tr} (\partial_\alpha \overline{\partial}_{\dot{\alpha}} \partial\partial \phi \overline{\partial\partial} \phi - \overline{\partial}_{\dot{\alpha}} \partial_\alpha \overline{\partial\partial} \phi \partial\partial \phi + \partial_\alpha \overline{\partial}_{\dot{\alpha}} \partial\partial Q c_- \\ &\quad - \overline{\partial}_{\dot{\alpha}} \partial_\alpha \overline{\partial\partial} Q \bar{c}_- + \overline{\partial\partial} Q \partial_\alpha \overline{\partial}_{\dot{\alpha}} \bar{c}_- - \partial\partial Q \overline{\partial}_{\dot{\alpha}} \partial_\alpha c_-) \end{aligned}$$

$$\mathcal{L} B^2 = \mathcal{L} V^2 = \mathcal{L} V^3 = 0$$

(Recall: $Q \equiv \delta_\eta \Gamma$ in the variable $\Gamma = \Gamma(\eta, \dots)$. Note that we have displayed this explicit term in the form B in order to show its tree approximation value.)

The remainder of $\Delta_0 \cdot \Gamma$ does not depend on c_- and therefore makes $\mathcal{L}$ into a nilpotent operator $\mathcal{L}^2 = 0$ - an extremely useful property.

In order to proceed one now uses dimensionality and chirality, as well as R weight arguments to show that $S_{\alpha\beta}^b$ and $V_{\alpha\dot{\alpha}}^b$ in (2.12) have only the following four types of solutions:

$$\begin{aligned}
 \text{A)} \quad & \mathcal{L} S_{\alpha\beta}^b = \overline{\mathcal{D}\mathcal{D}} \overline{T}_{\alpha\beta} \quad T_{\dot{\alpha}\dot{\beta}} \text{ chiral} \\
 & \mathcal{L} V_{\alpha\dot{\alpha}}^b = 2 (\mathcal{D}^{\dot{\rho}} \overline{\mathcal{D}}_{\dot{\alpha}} \overline{T}_{\alpha\beta} + \overline{\mathcal{D}}^{\dot{\rho}} \mathcal{D}_{\alpha} T_{\dot{\alpha}\dot{\beta}}) \\
 \text{B)} \quad & \mathcal{L} S_{\alpha\beta}^b = \overline{\mathcal{D}\mathcal{D}} \mathcal{D}_{\rho} U_{\alpha} \\
 & \mathcal{L} V_{\alpha\dot{\alpha}}^b = -\mathcal{D}\mathcal{D} \overline{\mathcal{D}}_{\dot{\alpha}} U_{\alpha} + \overline{\mathcal{D}\mathcal{D}} \mathcal{D}_{\alpha} \overline{U}_{\dot{\alpha}} \\
 \text{C)} \quad & \mathcal{L} S_{\alpha\beta}^b = \overline{\mathcal{D}\mathcal{D}} \mathcal{D}^{\gamma} U_{(\alpha\beta\gamma)} \quad (2.21) \\
 & \mathcal{L} V_{\alpha\dot{\alpha}}^b = 2 \mathcal{D}^{\dot{\rho}} \overline{\mathcal{D}}_{\dot{\alpha}} \mathcal{D}^{\gamma} U_{(\alpha\beta\gamma)} - 2 \overline{\mathcal{D}}^{\dot{\rho}} \mathcal{D}_{\alpha} \overline{\mathcal{D}}^{\dot{\gamma}} \overline{U}_{(\dot{\alpha}\dot{\beta}\dot{\gamma})} \\
 \text{D)} \quad & \mathcal{L} S_{\alpha\beta}^b = 2 \overline{\mathcal{D}\mathcal{D}} \mathcal{D}_{\alpha} \overline{\mathcal{D}}_{\dot{\alpha}} W_{\beta}^{\dot{\alpha}} \quad \overline{W}_{\dot{\rho}}^{\dot{\alpha}} = -W_{\rho}^{\dot{\alpha}} \\
 & \mathcal{L} V_{\alpha\dot{\alpha}}^b = (\mathcal{D}\mathcal{D}\overline{\mathcal{D}\mathcal{D}} + \overline{\mathcal{D}\mathcal{D}}\mathcal{D}\mathcal{D} - 6\mathcal{D}\mathcal{D}\overline{\mathcal{D}\mathcal{D}}) W_{\alpha\dot{\alpha}} \\
 & \quad + 4 \overline{\mathcal{D}}_{\dot{\alpha}} \mathcal{D}_{\alpha} \overline{\mathcal{D}}_{\dot{\beta}} \mathcal{D}^{\dot{\rho}} W_{\rho}^{\dot{\beta}} + 4 \mathcal{D}_{\alpha} \overline{\mathcal{D}}_{\dot{\alpha}} \mathcal{D}^{\dot{\rho}} \overline{\mathcal{D}}_{\dot{\beta}} W_{\rho}^{\dot{\beta}} \\
 & \quad - 4 \overline{\mathcal{D}}_{\dot{\alpha}} \mathcal{D}\mathcal{D} \overline{\mathcal{D}}_{\dot{\beta}} W_{\alpha}^{\dot{\beta}} - 4 \mathcal{D}_{\alpha} \overline{\mathcal{D}\mathcal{D}} \mathcal{D}^{\dot{\rho}} W_{\rho}^{\dot{\alpha}}
 \end{aligned}$$

This establishes (2.13). Proving (2.14) is a cohomology problem: does

$$\begin{aligned}
 \mathcal{L}^2 V^b &= 0 = \mathcal{D}_V \mathcal{L} U \\
 \mathcal{L}^2 S^b &= 0 = \mathcal{D}_S \mathcal{L} U
 \end{aligned} \quad (2.22)$$

imply that

$$\begin{aligned}
 \mathcal{D}_V U &= \mathcal{L} \mathcal{D}_V \hat{U} \\
 \mathcal{D}_S U &= \mathcal{L} \mathcal{D}_S \hat{U} \quad ?
 \end{aligned} \quad (2.23)$$

The proof that the answer is positive is given in the Appendix.

Having established separate BRS-invariance for the current V and the breaking $S_{\alpha\beta}$ we have to list all of $S_{\alpha\beta}$ since the breaking terms are of interest later on. Symmetric parts of $S_{\alpha\beta}$ correspond to B'_α , antisymmetric parts show up as admissible breaking terms. $S_{\alpha\beta}$ obeys

$$\begin{aligned} \bar{D}_\alpha S_{\alpha\beta} &= 0 & \ell S_{\alpha\beta} &= 0 \\ n(S_{\alpha\beta}) &= -2 & d(S_{\alpha\beta}) &= 3 \\ S_{\alpha\beta} &= \varepsilon_{\alpha\beta} S + S(\alpha\beta) \end{aligned} \quad (2.24)$$

Result:

$$S(\alpha\beta) = 0 \quad \text{i.e.} \quad B'_\alpha = 0$$

This means that there exists a supercurrent to all orders. Absence of the B' term ensures that the current V has good R and supersymmetry currents amongst its components, as well as a conserved and symmetric energy momentum tensor [Appendix C of Ref. 6)]. The actual list of all terms $\epsilon_{\alpha\beta} S$ is given below, (3.2).

3. THE RENORMALIZED SUPERCURRENT

The result of the last section can be collected in the following form

$$[\bar{D}^\alpha V_{\alpha\dot{\alpha}}] \cdot \Gamma \simeq -2 \psi_\alpha \Gamma - 2 [D_\alpha S] \cdot \Gamma \quad (3.1)$$

where V and S are BRS-invariant operators with quantum numbers:

V : parity-odd vector superfield, $\dim = 3$, R weight = 0

S : chiral superfield, $\dim = 3$, R weight = -2

V and S can be expanded in their respective bases of BRS-invariant operators with the above quantum numbers. Since the explicit basis for V is not needed later on, we shall not elaborate on it. For S it is provided by a symmetric extension of the following tree approximation terms:

$$\begin{aligned} \bar{D}\bar{D} B_{(0)}^i : \quad B_{(0)}^1 &= \ell \text{Tr}(\eta\phi) & B_{(0)}^2 &= \text{Tr}(\bar{D}\bar{D}\phi \bar{D}\bar{D}\phi + \bar{D}\bar{D}Q\bar{c} + \bar{D}\bar{D}Q\bar{c}) \\ B_{(0)}^3 &= \bar{A}_+ e^{\tilde{\phi}} A_+ + A_- e^{-\tilde{\phi}} \bar{A}_- & B_{(0)}^4 &= \text{Tr} e^{-\phi} \bar{A} e^{\phi} A \\ B_{(0)}^n &= \ell \text{Tr}(\eta\phi^{2n+1}) \end{aligned} \quad (3.2)$$

$$S_{(0)}^5 = \text{Tr } F^\alpha F_\alpha \quad S_{(0)}^6 = \text{Tr } A^3 \quad S_{(0)}^7 = b \text{Tr}(z c_+) \quad (3.2) \text{ cont.}$$

This symmetric extension can be performed in different, though equivalent ways, adapted to specific aims.

Let us first establish the relation of the breaking S to the CS-equation. For this purpose it is most convenient to define symmetric extensions $\hat{S}^i$ of the above $S_{(0)}^i$ by postulating

$$[\int dS \hat{S}^i + \int d\bar{S} \hat{\bar{S}}^i] \cdot \Gamma = \nabla^i \Gamma \quad (3.3)$$

Here ∇^i stands symbolically for the symmetric differential operators constructed in Ref. 4), i.e.,

$$\begin{aligned} (\nabla^i) &= (\mathcal{N}_-, \mathcal{N}_\phi, \mathcal{N}_{A_+}, \mathcal{N}_A, \mathcal{N}_+, \frac{\partial}{\partial g}, \frac{\partial}{\partial h}, \frac{\partial}{\partial a_n}) \\ \mathcal{N}_- &\equiv N_\phi - N_\eta - 2\alpha \partial_\alpha + \frac{4}{\epsilon} \int dV (\mathcal{D}\mathcal{D}c_- + \overline{\mathcal{D}\mathcal{D}}\bar{c}_-) \frac{\delta}{\delta \eta} \\ \mathcal{N}_\phi &\equiv N_\phi - N_\eta + 2\alpha \partial_\alpha \\ \mathcal{N}_{A_+} &\equiv N_{A_+} + \bar{N}_{\bar{A}_+} + N_{A_-} + \bar{N}_{\bar{A}_-} - (N_{\gamma_-} + \bar{N}_{\bar{\gamma}_-} + N_{\gamma_+} + \bar{N}_{\bar{\gamma}_+}) \\ \mathcal{N}_A &\equiv N_A + \bar{N}_{\bar{A}} - (N_\gamma + \bar{N}_{\bar{\gamma}}) \\ \mathcal{N}_+ &\equiv N_{c_+} + \bar{N}_{\bar{c}_+} - (N_c + \bar{N}_{\bar{c}}) \end{aligned} \quad (3.4)$$

N_ϕ denotes the counting operator for the field ϕ ; $\mathcal{N}_-$ is the only c_- dependent symmetric operator, a_n are the parameters occurring in the higher orders via gauge fixing. In the tree approximation the connection of these differential operators with the left-hand side of (3.3) is easily established [cf. (3.15) - (3.19) of the first of Ref. 4)].

The definition (3.3) actually determines the $\hat{S}^i$ uniquely: let

$$S'^i = \hat{S}^i + \sum_j x^{ij} \hat{S}^j \quad x = o(\epsilon) \quad (3.5)$$

be another extension with the property (3.3). Then we have

$$\sum_j x^{ij} [\int dS \hat{S}^j + \int d\bar{S} \hat{\bar{S}}^j] \cdot \Gamma = 0$$

i.e.,

$$\sum_j x^{ij} \nabla^j \Gamma = 0$$

Since the ∇^j form a basis,

$$x^{ij} = 0 \quad (3.6)$$

as claimed.

Expanding now S of (3.1) in this basis we can write

$$S = \frac{4}{3} \hat{S}_0 + \sum_k \frac{2}{3} \beta_k \hat{S}^k \quad \beta_k = o(\epsilon) \quad (3.7)$$

with $[\int dS \hat{S}_0 + \int d\bar{S} \hat{\bar{S}}_0] \Gamma = \mathcal{N}_- \Gamma$ according to (3.3). That means in (3.7) that we separate explicitly the symmetric extension of the tree approximation breaking $S_0 = \frac{1}{2} \bar{D} \bar{D} (B^1 + \frac{4}{\alpha} B^2)$. As discussed in I

$${}^{\text{W}}\Gamma \simeq [\int d\chi \tau_\chi] \cdot \Gamma \quad (3.8)$$

or

$${}^{\text{W}}\Gamma \simeq -\frac{3}{2} i [\int dS S + \int d\bar{S} \bar{S}] \cdot \Gamma \quad (3.9)$$

From I we also know that the true dilatation WI operator W^D is

$$W^D = {}^{\text{W}}\Gamma + 2i \mathcal{N}_- \quad (3.10)$$

Furthermore, from naive dimensional analysis

$$i m \partial_m \Gamma = W^D \Gamma \quad (3.11)$$

(m comprises all mass parameters of the theory), i.e.,

$$i(m \partial_m + \frac{3}{2} [\int dS (S - \frac{4}{3} \hat{S}_0) + \int d\bar{S} (\bar{S} - \frac{4}{3} \hat{\bar{S}}_0)]) \Gamma \simeq 0 \quad (3.12)$$

$$(m \partial_m + \sum_k \beta_k \nabla^k) \Gamma \simeq 0 \quad (3.13)$$

Since $m \partial_m \Gamma$ does not depend on c_- [cf. the explicit discussion in Ref.4)], one concludes that

$$\beta_0 = 0 \quad (3.14)$$

and one finally arrives at the CS-equation

$$(m \partial_m + \sum_{k \neq 0} \beta_k \nabla^k) \Gamma \simeq 0 \quad (3.15)$$

In this extension of the $S_{(0)}^i$ the restriction $\beta_0 = 0$ on the parameters in S is easily found. Unfortunately, it is only in another basis that the restrictions due to asymptotic R invariance (1.2) are quickly established. Let us turn now to those.

In this case we define the symmetric extension locally and extend first the terms $B_{(0)}^i$ of (3.2), $B_{(0)}^i \rightarrow B^i$ thus maintaining their reality. Next we extend $S_{(0)}^i \rightarrow S^i$, $i = 5, 6, 7$ (note that in S^i there will be contributions of the $B_{(0)}^i$ in general). We now expand S of (3.1) in this basis

$$S = \sum_i \tau_B^i \overline{D} B^i + \sum_i \tau_S^i S^i \quad \tau_S^i = o(\hbar) \quad (3.16)$$

$W_\Gamma^R \simeq 0$ means

$$[\int dS S - \int d\bar{S} \bar{S}] \cdot \Gamma \simeq 0 \quad (3.17)$$

i.e.,

$$\sum_i \tau_S^i [\int dS S^i - \int d\bar{S} \bar{S}^i] \cdot \Gamma \simeq 0 \quad (3.18)$$

and there are no restrictions of R symmetry on the parameters r_B^i . In the one-loop approximation of this equation $r_S^i = o(\hbar)$ and $S^i =$ tree approximation; due to $\text{Tr} (\int dS F_\alpha^\alpha F_\alpha - \int d\bar{S} \bar{F}_\alpha^\alpha \bar{F}_\alpha) = 0$ there follows

$$\tau_S^6 = o(\hbar^2) \quad \tau_S^7 = o(\hbar^2) \quad (3.19)$$

Since in higher order the coefficients r_B^i still do not contribute to (3.18), we can conclude from (3.19) that r_S^6, r_S^7 are functions of r_S^5 . We can therefore define a normal product $N[FF] \cdot \Gamma = [\text{Tr } FF + \text{correct.}] \cdot \Gamma$ by the property

$$\int ds N[FF] \cdot P - \int d\bar{S} N[\bar{F}\bar{F}] \cdot P \simeq 0 \quad (3.20)$$

and comprise in this definition the restrictions of (3.17) - the R invariance - on the breaking parameters in S.

Clearly the operators $\hat{S}^i$ and $(\bar{D}\bar{D} B^i, S^i)$ are related by a change of basis but since it is not clear whether

$$[\int ds \hat{S}_0 - \int d\bar{S} \hat{\bar{S}}_0] \cdot P = 0 \quad (3.21)$$

this may complicate the relation of the β_k to the τ_i considerably. The presence of $\hat{S}_0$ is, of course, a consequence of the non-improved character of the supercurrent V. But still, we may conclude that the trace equation (3.1) provides all the links existing between the anomalies of the theory.

APPENDIX

It has to be shown that the equations

$$\begin{aligned} b^2 V^\nu &= 0 = \partial_\nu b U \\ b^2 S^\nu &= 0 = \partial_S b U \end{aligned} \quad (A.1)$$

imply

$$\begin{aligned} \partial_\nu U &= b \partial_\nu \hat{U} \\ \partial_S U &= b \partial_S \hat{U} \end{aligned} \quad (A.2)$$

There are two operations to be performed. First one has to commute b with derivatives ∂ and to deduce from the vanishing of $\partial b U$ the restrictions on $b U$. At some stage(s) of the argument a second step will then be necessary: to prove that $b F = 0$ implies $F = b \hat{F}$, this being the standard cohomology problem. We will discuss the steps separately, but treat explicitly only one example and quote the results for all other cases.

For notational convenience we write $S' \equiv b S$ and similarly for higher variations, i.e., the number of primes counts the $\phi\pi$ charge.

Case A:

$$\begin{aligned} S'_{\alpha\beta} &= \overline{\partial\partial} \bar{T}'_{\alpha\beta} & d(S') &= 3 & n(S') &= -2 \\ \text{i.e., } d(\bar{T}') &= 2 & n(\bar{T}') &= 0 & T &\text{ chiral} \end{aligned}$$

Since matter fields have $n = -\frac{2}{3}$, T' cannot contain them. Consequently

$$\bar{T}'_{\alpha\beta} = \partial\partial R'_{\alpha\beta} \quad d(R') = 1 \quad n(R') = -2$$

But then the symmetric part of R'

$$R'_{(\alpha\beta)} = f(\partial_\alpha, \partial_\beta, c_+, \bar{c}_+, \phi)$$

has necessarily $n = 2$ which is forbidden, hence

$$R'_{(\alpha\beta)} = 0$$

The antisymmetric part

$$\bar{T}'_{\alpha\beta} = \varepsilon_{\alpha\beta} \mathcal{D}\mathcal{D}R'$$

fulfils

$$\mathcal{L}\bar{T}' = 0 \quad \text{i.e.,} \quad \mathcal{D}\mathcal{D}\mathcal{L}R' = 0 \quad \text{or} \quad \mathcal{L}R' = \mathcal{D}^2 X_\alpha$$

which also cannot have $n = -2$. Therefore, $\mathcal{L}R' = 0$. In the second step we will show that this implies $R' = \mathcal{L}R$ and thus

$$\bar{T}'_{\alpha\beta} = \mathcal{L}\varepsilon_{\alpha\beta} \mathcal{D}\mathcal{D}R$$

as desired.

Going through Cases B, C and D in a similar manner one finds the results as needed. It is interesting to observe that in Case B there arise obstruction terms U_α^H , i.e.,

$$U'_\alpha = \mathcal{L}U_\alpha + U_\alpha^H, \quad U_\alpha^H \text{ chiral}$$

but that these do not contribute in

$$\partial_S U'_\alpha = \mathcal{L}\bar{\mathcal{D}}\mathcal{D}\mathcal{D}_\beta U_\alpha$$

The cohomologies met in the course of the analysis are the following:

1) $F = R'$	$d = 1, n = -2$	Case A
2) $F = A''_\alpha$	$d = \frac{1}{2}, n = 1$	} Case B
3) $F = U'_\alpha$	$d = \frac{3}{2}, n = -1$	
2)		} Case D
4) $F = W'_{\alpha\alpha}$	$d = 1, n = 0$	

The cohomology proof is quite analogous to the one treated in Appendix A of the second reference of Ref. 4). The lemma C given there shows that a sufficient condition for $F = \mathcal{L}\hat{F}$ being the general solution of $\mathcal{L}F = 0$ is that the Abelian cohomology equation $\mathcal{L}_\bullet F = 0$ has as general solution $F = \mathcal{L}_\bullet \hat{F}$. Once the problem is reduced to this point the above cohomologies can be proved exactly like the ones in Ref. 4).

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