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HOPPING PARAMETER EXPANSION FOR THE MESON SPECTRUM
IN SU(3) LATTICE QCD

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ABSTRACT

The low-lying pseudoscalar and vector mesons containing u,d,s and c quarks are studied by the method of the hopping parameter expansion combined with a Monte Carlo simulation at $1/g^2 = 0.0, 0.7, 0.9, 0.925, 0.95$ and 1.0 . The effect of virtual quark loops is taken into account. Meson masses, quark masses, the lattice scale and K_u , K_s and K_c as functions of g^2 are determined.

In statistical physics the high temperature expansion in powers of $1/T$ is a valuable tool for investigating critical phenomena¹⁾. It has a non-vanishing radius of convergence and therefore the result is certainly correct at high enough temperatures. Close to the critical point, however, some information on the singularity structure is necessary. Based on that information (or assumption) some extrapolation method might be applied.

The hopping parameter expansion in lattice gauge theories with fermions in Wilson's formulation²⁾⁻¹⁰⁾ is analogous to the high temperature expansion in many respects. The amplitude of moving a quark by one lattice unit is proportional to the hopping parameter K . An expansion in K is equivalent to an expansion in the length of quark paths in configuration space. In their propagation the quarks are constrained by the maximum order of the expansion but they should still gather the essential information on the hadron's structure. This defines the conditions under which the expansion to a given order will be reliable. The size of the hadron measured in lattice units should not be too large, it has to be comparable to the regions covered by possible quark paths. Or, alternatively, the lattice distance a measured in physical units cannot be too small.

The hadron mass is determined by the long-distance behaviour of the propagators. An exponential decay in configuration space implies a pole in momentum space. As we discuss below, this singularity corresponds to a pole in the hopping parameter K . Of course a finite expansion cannot produce a pole. Padé approximants will be used to identify this pole in K . This procedure establishes a relationship between the pole position K and the mass of the hadron for each value of the gauge coupling $1/g^2$.

In Wilson's formulation the lattice action has the following form:

$$S = \sum_n \left\{ -\bar{\psi}_\alpha^{a,i}(n) \psi_\alpha^{a,i}(n) + \sum_\mu K_i \bar{\psi}_\alpha^{a,i}(n) (1-\gamma_\mu)_{\alpha\beta} U_{n\mu}^{ab} \psi_\beta^{b,i}(n+\hat{\mu}) + \sum_\mu K_i \bar{\psi}_\alpha^{a,i}(n+\hat{\mu}) (1+\gamma_\mu)_{\alpha\beta} U_{n\mu}^{+ab} \psi_\beta^{b,i}(n) \right\} + \frac{1}{g^2} \sum_{\text{plaquettes}} (\text{Tr } U_p + \text{cc.}) \quad (1)$$

where a, b, α, β and i are colour, Dirac and flavour indices respectively.

In the continuum limit there is a simple relation between the hopping parameter K_i and the more familiar mass parameter m_i :

$$K_i^{-1} = 2 m_i a + 8, \quad i = u, d, s, c, \dots \quad (2)$$

S is not chiral invariant; there is no symmetry which would prevent the occurrence of mass counterterms even for massless quarks. The mass counterterm receives perturbative and non-perturbative contributions and thus K_i should be renormalized. $K_i = K_i(g^2)$ is not known a priori.

The action is quadratic in the fermion fields and one may integrate over these fields. The result is an effective gauge theory. One possibility is to study this effective gauge theory by direct numerical methods¹¹⁾⁻¹⁴⁾. Preliminary studies indicated that it is very time-consuming to take the contribution of virtual quark loops in the effective action into account. However, there are qualitative arguments supporting the approximation where the virtual quark loops are neglected. Recently exciting results have been published in this approximation¹⁵⁾⁻¹⁷⁾.

Another possibility for studying the effective gauge theory is the hopping parameter expansion combined with a Monte Carlo simulation^{7),8)}. In this paper we report on a calculation for the spectrum of low-lying s wave, pseudo-scalar and vector mesons containing u,d,s and c quarks^{*)}. It is based on an expansion of tenth order in K and includes the contribution of virtual quark loops. The gauge group is SU(3)_{colour}. Our resolution at $1/g^2 = 0.90, 0.925, 0.95$ and 1.0 (these are the points we studied) is crude for mesons containing c quarks (especially concerning the perturbative contribution to the hyperfine splitting), therefore our results on these mesons should be considered as a first approximation.

Consider a propagator in momentum space: $D(p;K,g)$ and assume it has a pole. Close to this pole we have in the continuum limit

$$D(p_0 = iE, \mathbf{p} = 0; K, g) \sim \frac{1}{E^2 - M^2(K, g)} \quad (3)$$

Consider now D as a function of K at a fixed E (and g). Let us define K^* by the equation $E^2 = M^2(K^*, g)$. Then by using the expansion

$$M^2(K, g) = E^2 + \frac{\partial}{\partial K} M^2(K, g) \Big|_{K^*} (K - K^*) + \dots \quad (4)$$

one finds that a pole in momentum space implies a pole in K.

*) We consider only contributions where the valence quarks do not annihilate which is well-justified for the particles discussed.

Any operator with the correct quantum numbers may serve to obtain the masses of the lowest lying states. For pseudoscalar and vector mesons we use $\bar{\Psi}(x) \Gamma \Psi(x)$, where Γ is a matrix in Dirac and flavour space, e.g., $\gamma_5 \sigma_a$ for the $\pi(0^{--})$ and $\gamma_\mu \sigma_a$ for the $\rho(1^{--})$. The corresponding local operators for the other meson states [e.g., $\delta(0^{++})$, $B(1^{+-})$, $A_1(1^{++})$] lead to vanishing expansion co-efficients at $1/g^2 = 0$, and for larger $1/g^2$ the results change significantly as the order of the calculation is increased (from eighth to tenth order for instance). One possibility for studying these states in our framework is to go to higher orders. Another way would be to define spatially extended meson operators which might lead to faster convergence. We discuss only the results for pseudoscalar and vector mesons.

The propagator $D_\Gamma(n) = \langle \bar{\Psi}(0) \Gamma \Psi(0) \bar{\Psi}(n) \Gamma \Psi(n) \rangle$ can be written in the following way^{*)}

$$D_\Gamma(n; K, g) = \frac{\int DU \sum_{\text{closed paths connecting 0 with n}} e^{\frac{1}{g^2} \sum_p (\text{Tr } U_p + \text{c.c.}) + \sum_{\text{all closed paths}} \text{loop diagrams}}}{\int DU e^{\frac{1}{g^2} \sum_p (\text{Tr } U_p + \text{c.c.}) + \sum_{\text{all closed paths}} \text{loop diagrams}}} \quad (5)$$

where $\xrightarrow{m \rightarrow m+\hat{\mu}}$ contributes $K(1-\gamma_\mu)U_{m\mu}$, the matrix Γ is included at the points 0 and n and finally the trace is taken in flavour, Dirac and colour space. The Boltzmann factor $e^{S_{\text{eff}}}$ is to be expanded in terms of K , therefore (5) gives $D(n)$ as a systematic power series in K :

$$D(n; K, g) = \sum_{\ell=\ell_{\min}}^{\infty} c_\ell^{(n)}(g) K^{2\ell}. \quad (6a)$$

Consider first the contribution to $c_\ell^{(n)}(g)$ from those graphs which do not contain vacuum loops. All the closed loops of total length 2ℓ running through 0 and n should be constructed. By calculating the Dirac trace one is left with a trace over the product of U matrices along the closed loop, i.e., a Wilson loop. There are many different paths and corresponding Wilson loops. However, the number of topologically different Wilson loops is much less; all the others are the translated, rotated or reflected versions of these independent ones.

*)

For more details on the graphical rules see Refs. 2), 6)-8).

The calculation of Dirac traces and the topological classification of loops is done on a computer on a lattice of infinite extension. This part of the calculation is independent of the gauge group (and the coupling constant of course) such that

$$D(n; K, g) = \sum_{\substack{\text{topologically inequivalent} \\ \text{closed loops } C}} d^{(n)}(K, C) \text{Tr } U_C \quad (6b)$$

Here d is independent of the gauge group and exact up to the given order in K ; U_C is the product of link variables along the path C .

In the next step the expectation value of the topologically independent Wilson loops is calculated by a usual Monte Carlo calculation for pure gauge theory for different g^2 values^{*)}. The number of configurations at a given g^2 was chosen to be between 15 and 20 on an 8^4 lattice. We believe that the systematical errors (mainly due to the truncation of the K series) are larger than the statistical ones, therefore we did not attempt to further reduce the statistical errors.

In tenth order the algebraic part [determination of all the $d^{(n)}(K, C)$] required ~ 2500 seconds on the CDC 7600, while the Monte Carlo time was ~ 3 hours at a given g^2 on the CERN IBM.

Including the vacuum loop contributions up to tenth order is rather simple. These contributions can be expressed in terms of the derivatives (with respect to $1/g^2$) of the plaquette and of the three topologically independent Wilson loops of length six.

Tables 1 and 2 give the Fourier transform of the π and ρ propagators at $1/g^2 = 0.925$, taking $p = (iE, 0, 0, 0)$ ^{**)} . The Fourier transform is defined as

$$D(p) = \sum_n e^{-ipn} D(n) \quad (7)$$

It is straightforward to modify the calculation outlined above for mesons containing quarks with different K 's. (We did not distinguish K_u from K_d however.)

*) We are indebted to E. Pietarinen¹⁸⁾ for the computer programme we used to create equilibrium configurations.

**) Tables at other $1/g^2$ values and also the corresponding tables for mesons containing quarks with different K 's can be obtained from the authors on request.

Considerable effort has been spent on verifying the correctness of the expansion coefficients. The propagators can be calculated by other methods in the strong and weak coupling limits. The computer result also agrees with a sixth order calculation obtained "by hand".

ANALYSIS; RESULTS

The pole of the propagator in Fourier space is built up from the long-distance part of the propagator in configuration space. We decided to leave out the contribution of the origin $D(n=(0,0,0,0))$ in performing the Fourier transform. $D(0)$ has nothing to do with real propagation, in addition it is divergent in the continuum limit. [$D(0)$ is the expectation value of the product of four fermion fields taken at the same point.]

For every n there are special graphs contributing to $D(n)$ containing loops stuck to the end points and thus independent of n . These end point contributions factorize, giving rise to $D(n;K) \sim Z^2(K) \tilde{D}(n;K)$ where Z is independent of n . In order to cancel this factor we considered the ratio $D(p;K)/D(n=(1,0,0,0))$. [The distance = 1 propagator $D(1000)$ contains the Z factor up to the highest order in a K expansion.] $D(p;K)/D(1000)$ is then a series of eighth order.

In order to consider the analysis sensible, it should satisfy a few obvious requirements. First, it should work in the strong coupling limit, where the exact result is known (exact though without virtual quark loops). Second, the different Padé ratios must give consistent results and finally the results should change smoothly with a signal of convergence as the order of the calculation is increased.

Our analysis satisfies these consistency requirements. The Padé analysis for $D(p;K)/D(1000)$ works reasonably well in the strong coupling limit. We switched off the vacuum quark contribution in order to compare it with the exact results obtained in this limit. The different Padé's are almost identical. The predicted critical K^{crit} (where $m_\pi = 0$) is between 0.254 and 0.255 (the exact value is 0.25), while $(m_\rho \cdot a)$ is between 0.843 and 0.86 (the exact value is 0.867) in tenth order for that K^{crit} . The Padé table is also stable for non-zero $1/g^2$ values. The following tables and figures refer to the 3/1 Padé. Figure 1 gives the predicted $(m_\pi a)^2$ and $(m_\rho a)^2$ as the function of K_u at $1/g^2 = 0.925$. For comparison, the corresponding curves without vacuum loops are also

given. By taking the experimental value for m_π and m_ρ , $K_u = K_u(g^2)$ and $a = a(g^2)$ are obtained as given in Figs. 2 and 3 respectively. $K_s(g^2)$ and $K_c(g^2)$ are also plotted in Fig. 2. They are obtained by fixing the mass of $K(495)$ and $D(1868)$.

Let us recapitulate the number of input parameters: one pseudoscalar meson mass for each of the different quark species [i.e., three masses for $K_u (=K_d)$, K_s and K_c] corresponding to the unknown quark mass, and one additional mass (the ρ mass in our case) in order to fix the scale. All other meson masses as well as $K_u(g^2)$, $K_s(g^2)$, $K_c(g^2)$, the critical $K^{\text{crit}}(g^2)$ (where $m_\pi = 0$) and the lattice spacing $a(g^2)$ in physical units are results.

The lattice distance a (Fig. 3) changes very slowly until the crossover and falls down rapidly beyond this point. Between 0.95 and 1.0 it breaks away to a slow variation again. This behaviour is similar to that of ratios of Wilson loops of a given size, and indicates the region of reliability of the results. Our tenth order calculation could not reproduce the slope expected from the renormalization group behaviour for $a(g^2)$. Correspondingly, the predicted value for Λ^{latt} changes with g^2 , giving $\Lambda^{\text{latt}} = 0.9 \text{ MeV}$ ($\Lambda_{\alpha=1}^{\text{MOM}} = 95 \text{ MeV}$ ¹⁹) at $1/g^2 = 0.925$ and $\Lambda^{\text{latt}} = 0.77 \text{ MeV}$ ($\Lambda_{\alpha=1}^{\text{MOM}} = 81 \text{ MeV}$) at $1/g^2 = 0.95$. Here $\Lambda_{\alpha=1}^{\text{latt}}$ ($\Lambda_{\alpha=1}^{\text{MOM}}$) is defined by using $n_f = 3$, assuming that the number of relevant flavour channels is three at the cut-off values entering our calculation.

The mass predictions are almost independent on the $1/g^2$ value taken. For the mass of K^* and ϕ we obtained:

$$m_{K^*(892)} = 860 \text{ MeV}, \quad m_{\phi(1020)} = 950 \text{ MeV} \quad (8)$$

Our lattice is too coarse grained for the charm sector where there are important perturbative contributions as well. Our mass predictions are:

$$m_{F(1300)} = 1960 \text{ MeV}, \quad m_{D^*(2009)} = 1910 \text{ MeV}, \quad m_{F^*(2140)} = 1980 \text{ MeV} \quad (9)$$

$$m_{\eta_c(1980)} = 2870 \text{ MeV}, \quad m_{\chi(3097)} = 2870 \text{ MeV}.$$

The effect of the statistical errors of the Wilson loops on these numbers is small ($\sim 1\%$). A conservative estimate of the systematic errors related to the Padé analysis is $\sim 5\%$. It is difficult to guess the systematic error due to the finite order of the expansion, which depends on the $1/g^2$ value taken and will be more important for the higher mass mesons.

Finally, let us collect the results on the quark masses. In Wilson's formulation the free quark propagator taken at $p = (im, \underline{0})$ has a pole determined by the equation

$$\frac{1}{2} \left(\frac{1}{K} - \frac{1}{K^{\text{crit}}} \right) = e^{ma} - 1, \quad (10)$$

where for the free case $K^{\text{crit}} = 1/8$. This equation can also be considered to be valid when, due to the gauge interaction, K^{crit} (the value of K when $m_\pi = 0$) is different from $1/8$. The mass parameters we obtained are summarized in Table 3.

The renormalization group invariant mass is obtained by multiplying these mass values by $A \equiv (\text{const} \cdot g^2)^{-\frac{4}{11} - 20}$. Choosing the constant is a matter of convention²¹⁾. For the possibility of comparison we choose $\text{const.} = 3/4\pi$ and replace g_{latt}^2 by g_{MOM}^2 as in Ref. 17)*). In the equation

$$\frac{1}{g_{\text{latt}}^2} - 2\beta_0 \ln \left(\frac{1}{\pi} \frac{\Lambda^{\text{MOM}}}{\Lambda^{\text{latt}}} \right) = \frac{1}{g_{\text{MOM}}^2} \quad (11)$$

the relating constant $c = 2\beta_0 \ln(\Lambda^{\text{MOM}}/\Lambda^{\text{latt}})$ depends weakly on the number of flavours ($c = 0.458$ for $n_f = 0$, $c = 0.400$ for $n_f = 3$ ¹⁹⁾). In our coupling constant range one obtains $A = 1.3$ which gives, for the renormalization group invariant $u(d)$, s and c masses ≈ 7.5 MeV, ≈ 190 MeV and ≈ 1600 MeV respectively.

Some of our numbers can be compared with those obtained in Ref. 17) by switching off the virtual quark loop contribution. Unfortunately the only point investigated in Ref. 17) is $1/g^2 = 1.0$, where our series becomes too short. Without quark loops we get

$$K_u \left(\frac{1}{q^2} = 0.90 \right) = 0.201, \quad K_u \left(\frac{1}{q^2} = 0.925 \right) = 0.195, \quad K_u \left(\frac{1}{q^2} = 0.95 \right) = 0.186, \quad (12)$$

and by a smooth extrapolation one estimates $K_u(1/g^2 = 1.0) \sim 0.165 \pm 0.01$ which is slightly above the value $K_u = 0.156$ of Ref. 17). Our u quark mass also seems to be larger by almost a factor of two compared to the number obtained by Hamber and Parisi.

*) This last step is arbitrary. In the continuum limit $A \cdot m$ is both RG and scheme invariant.

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	C_0	C_1	C_2	C_3	C_4	C_5
K^2	144	24	—	—	—	—
K^4	2903.81 ± 0.72	869.95 ± 0.31	96	—	—	—
K^6	43558.58 ± 24.17	24465.93 ± 11.27	5866.21 ± 2.82	384	—	—
K^8	506003.7 ± 1771.8	616671.9 ± 769.5	242173.5 ± 81.3	34398.9 ± 19.2	1536	—
K^{10}	4429340 ± 147086	15061550 ± 65351	8390367 ± 10307	1945945 ± 563	184403 ± 116	6144

Table 1

The K expansion of the Fourier coefficients of the π propagator at $p = (iE, 0, 0, 0)$ is given for $1/g^2 = 0.925$ in the form $D_\pi = C_0 + \sum_{t=1}^5 C_t (e^{Et} + e^{-Et})$. The contribution of the origin has been discarded. The errors have been determined by treating all contributions to the coefficients as uncorrelated, which provides a conservative estimate.

	C_0	C_1	C_2	C_3	C_4	C_5
K^2	96	24	—	—	—	—
K^4	1355.90 ± 0.72	677.95 ± 0.31	96	—	—	—
K^6	4611.04 ± 23.41	13515.03 ± 10.67	5098.21 ± 2.82	384	—	—
K^8	-382531.1 ± 1746.0	197041.0 ± 762.2	175707.4 ± 75.74	31326.9 ± 19.2	1536	—
K^{10}	-15998000 ± 156294	1529173 ± 64254	4800184 ± 10270	1581902 ± 529	172115 ± 116	6144

Table 2

The same as Table 1, but for the ρ meson

$1/g^2$	$m_{u,d}$	m_s	m_c
0.925	5.6	144	1213
0.95	5.7	141	1209

Table 3: The mass parameters m_u (which is not distinguished from m_d), m_s and m_c in MeV for two different gauge couplings. (These quantities are not the renormalization group invariant masses.)

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FIGURE CAPTIONS

Fig. 1 $(m_\pi a)^2$ and $(m_\rho a)^2$ are plotted as the functions of K_u at $1/g^2 = 0.925$ (solid lines). For comparison, the corresponding curves without vacuum loop corrections are also given (dotted lines). In the insert the same curves are given over an extended K region.

Fig. 2 The hopping parameter $K_i = K_i(g^2)$, $i = \text{up, strange and charmed}$. They are obtained by fixing m_π/m_ρ , m_K/m_ρ and m_D/m_ρ . The curves have been drawn to guide the eye.

Fig. 3 The lattice distance a as the function of the gauge coupling $1/g^2$. a is determined by fixing the ρ meson mass. The straight line indicates the expected RG behaviour for $n_f = 3$.

- Figure 1 -

- Figure 2 -

- Figure 3 -