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MASSLESS BARYONS AND ANOMALIES IN CHIRAL (QCD)₂

D. Amati, Kuang-Chao Chou^{*)}

and

S. Yankielowicz

CERN -- Geneva

ABSTRACT

The spectrum of chiral $SU(N)$ gauge theories in two dimensions is shown to consist of free massless mesons and baryons that may be visualized as composites of free massless quarks and antiquarks. Baryons do not contribute to anomalies which are saturated by the mesons. The singular nature of the chiral limit is recognized.

^{*)} On leave from Institute of Theoretical Physics, Academia Sinica, Beijing, People's Republic of China.

A dynamical framework is needed in order to understand how a confining theory treats chirality and anomalies: whether by realizing chirality and saturating anomalies with massless bound fermions¹⁾ (baryons) or by saturating them with Goldstone bosons if chiral symmetry is spontaneously broken.

In a $1/N$ approach the second scenario is realized²⁾, but this is no surprise because that approach is unwarranted for a dynamics with massless fermion bound states. Attention was then directed to two-dimensional models ($SU(N)_2$) where the dynamics is accessible, hoping of course to be able to distinguish peculiarities of two dimensions (particularly pathological as far as symmetry breaking is concerned) from dynamical features expected to survive in four dimensions.

't Hooft did³⁾ the first analysis of $SU(N)_2$ with a $1/N$ expansion in a light cone gauge. He identified a colour singlet meson bound state that in the chiral limit ($m = 0$ where m is the quark mass) becomes massless and free. This massless meson saturates the $U(1)$ anomaly as expected in a spontaneously broken scenario. But as previously mentioned, the $1/N$ expansion is unsuitable for detecting massless fermions.

For $m = 0$ and for any N , the conservation of both vector and axial currents implies $\square j_\mu = 0$ and thus the existence of a corresponding massless decoupled meson that saturates trivially the appropriate anomaly. In a recent paper⁴⁾ these states have been explicitly constructed and it has been argued that no massless baryon states could exist because they would spoil the anomaly condition.

On the other hand, working in a gauge first proposed by Baluni⁵⁾ the spectrum for finite m was identified⁶⁾ and shown to contain both bound mesons and baryons with masses going to zero for $m \rightarrow 0$. This is a common feature of two-dimensional models⁷⁾. It was also recognized⁷⁾ that these baryons would necessarily spoil the anomaly condition (as a function of N) in the chiral limit if baryon form factors would satisfy some smoothness conditions.

We therefore find a few apparent contradictions. On the one hand the baryons present for arbitrarily small m could not disappear from the Hilbert space and therefore it should be possible to find them also in the strict $m = 0$ chiral theory. On the other hand, if they are there they should not contribute to anomalies and this implies a singular behaviour of form factors such as $m \rightarrow 0$. We shall indeed show that both statements are correct and that there is no contradiction left.

The picture that arises in the chiral limit is very trivial (free massless mesons and baryons) and is totally due to the two dimensionality that allows free massless quarks to produce free massless bound states. These two-dimensional gauge theories realize chirality in such a trivial way that they shed no light on dynamical mechanisms for chirality realization or breaking in four dimensions.

In what follows we shall discuss the single flavour case, the generalization to arbitrary N_f being straightforward.

Let us first identify the massless baryon for the $m = 0$ case following a method analogous to that used in Ref. 4) to identify the boson. We shall therefore work in the $\hat{A}_- = 0$ light cone gauge in which the gauge field $\hat{A}_+$ may be totally eliminated in terms of a Coulomb interaction and in which only the left-handed (actually a left mover) quark field $\psi_+ = \frac{1}{2}(1+\gamma_5)\psi$ is dynamical in the chiral $m = 0$ case under discussion. In this gauge, the rôle of the Hamiltonian is played by the generator of displacements of $x^+ = 1/\sqrt{2}(x^0+x^1)$ given by

$$\hat{P}_+ = -\frac{1}{4} \int dx^- dy^- dy^+ J^{+a}(x) s(x^+-y^+) |x^--y^-| J^{+a}(y) \quad (1)$$

with

$$J^{+a} = \sqrt{2} g \psi_+^\dagger I^a \psi_+ \quad (2)$$

where I^a , $a = 1, \dots, N^2 - 1$ are the $SU(N)$ group generators. Energy momentum conservation ensures that $\hat{P}_+$ is independent of x^+ and one may take $x^+ = 0$ in studying the bound state spectrum.

We expand $\psi_+(x)$ in terms of the creation and annihilation operators:

$$\psi_+(x) = \frac{1}{2^{1/4}} \int \frac{dP_-}{\sqrt{2\pi}} (e^{-iP_-x^-} \hat{b}(P_-) + e^{iP_-x^-} \hat{d}^\dagger(P_-)) \quad (3)$$

and write $\hat{P}_+$ in the form:

$$\begin{aligned} \hat{P}_+ = \int d\mathbf{q}_- d\mathbf{p}_- \int d\mathbf{k}_- V(k_-^2) & \left[\hat{b}_i^\dagger(\mathbf{q}_-) \hat{b}_j(\mathbf{q}_- + \mathbf{k}_-) + \hat{b}_i^\dagger(\mathbf{q}_-) \hat{d}_j^\dagger(-\mathbf{q}_- \right. \\ & \left. - \mathbf{k}_-) + \hat{d}_i(\mathbf{q}_-) \hat{b}_j(-\mathbf{q}_- + \mathbf{k}_-) + \hat{d}_i(\mathbf{q}_-) \hat{d}_j^\dagger(\mathbf{q}_- - \mathbf{k}_-) \right] (I^a)_{ij} (I^a)_{mn} \end{aligned} \quad (4)$$

$$\begin{aligned} & \left[\hat{b}_m^+(p_-) \hat{b}_n(p_- - k_-) + \hat{b}_m^+(p_-) \hat{d}_n^+(-p_- + k_-) + \hat{d}_m(p_-) \hat{b}_n(-p_- - k_-) \right. \\ & \left. + \hat{d}_m(p_-) \hat{d}_n^+(p_- + k_-) \right] \end{aligned} \quad \begin{array}{l} (4) \\ \text{cont.} \end{array}$$

The specific form of the Coulomb potential

$$V(k_-) = \frac{g^2}{4\pi} P\left(\frac{1}{k_-^2}\right) \quad (5)$$

where P stands for the principle value, plays no rôle in the following calculations. It is easily shown that the physical vacuum is the usual perturbative state $|0\rangle$ defined by $\hat{b}_i(p_-)|0\rangle = \hat{d}_i(p_-)|0\rangle = 0$ with the vacuum energy normalized to zero.

An eigenstate of $\hat{P}_+$ with eigenvalue P_+ and fixed P_- will have mass

$$M^2 = 2P_+P_- \quad (6)$$

The exact zero mass boson state found in Ref. 4) has the form

$$|\varphi, P_-\rangle = \frac{P_-}{\sqrt{N}} \int_0^1 d\alpha \sum_{i=1}^N \hat{b}_i^+(\alpha P_-) \hat{d}_i^+((1-\alpha)P_-) |0\rangle \quad (7)$$

with a normalization condition

$$\langle \varphi, P'_- | \varphi, P_- \rangle = P_- \delta(P_- - P'_-) \quad (7')$$

Similarly, the massless baryon state we are looking for will be given by

$$|B, P_-\rangle = N_B \int_0^1 d\alpha_1 \dots d\alpha_N \hat{b}_1^+(\alpha_1 P_-) \dots \hat{b}_N^+(\alpha_N P_-) \delta(1 - \alpha_1 - \dots - \alpha_N) |0\rangle \quad (8)$$

with the normalization constant

$$N_B = \left(P_-^N / \int_0^1 d\alpha_1 \dots d\alpha_N \delta(1 - \alpha_1 - \dots - \alpha_N) \right)^{1/2} \quad (9)$$

Indeed, using the identities

$$(I^a)_{ij} (I^a)_{mn} = A \delta_{ij} \delta_{mn} + B \delta_{in} \delta_{jm} \quad (10)$$

with

$$NA + B = 0 \quad (11)$$

and

$$A - B = -\frac{1}{N-1} C_2(N) \quad (12)$$

where $C_2(N)$ is the second Casimir invariant, one can easily prove that

$$\hat{P}_+ |B, P_- \rangle = 0 \quad (13)$$

Therefore $|B, P_- \rangle$ represents an exact massless baryon state.

For quarks not in the fundamental representation of the $SU(N)$ group, it is still possible to have massless bound baryon states provided the relations (11) and (12) are satisfied.

A direct calculation for the matrix elements of the singlet $U(1)$ current yields

$$\langle 0 | J^+(q) | \varphi, P_- \rangle = \sqrt{N} q_- \delta(q_- - P_-) \delta(q_+) \quad (14)$$

and

$$\langle B, P'_- | J^+(q) | B, P_- \rangle = N \left(\frac{\text{Min}(P_-, P'_-)}{\text{Max}(P_-, P'_-)} \right)^{\frac{N}{2}} (P_- - P'_-)^{\frac{1}{2}} \delta(q_- - P_- + P'_-) \cdot \delta(q_+) \quad (15)$$

The spectrum as well as the matrix elements found above are not a surprise. In order to see why, let us first recall that the light cone gauge $\hat{A}_- = 0$ implies that the right-handed quark field $\psi_- = \frac{1}{2}(1 - \gamma_5)\psi$ remains free. Moreover, in two dimensions a system of free massless quarks moving in the same direction has zero mass.

Had we chosen the opposite light cone gauge condition $\hat{A}_+ = 0$ the roles would be reversed, ψ_+ would be free and the gauge invariant states (7) and (8) would appear as massless mesons and baryons constituted of free quarks. Their singlet current matrix elements are those of a free theory and being gauge invariant quantities should coincide with those calculated before in the $\hat{A}_- = 0$ gauge. And, indeed, Eqs. (14) and (15) represent free theory matrix elements. Besides

$$\langle 0 | J^\dagger(q) | B, p_- ; \tilde{B}, p'_- \rangle = 0 \quad (16)$$

even if this is not trivial to obtain from the interaction formalism developed before. Equation (16) implies of course that baryons do not contribute to the anomaly.

By further exploiting this separation of left and right worlds, it is easy to see that the chiral theory consists wholly of free colourless mesons and baryons.

This very simple result leads to a fact that at first sight is puzzling. Usually the matrix elements of (15) and (16) are considered to be the same, each obtained by a different analytic continuation. We will show that this concept of analytic continuation is untenable in this two-dimensional chiral theory and we will identify the origin of this apparent contradiction. As already stressed, the left and right worlds are completely separated in the chiral theory and the physical states are composed of purely right movers or purely left movers, massless quarks or antiquarks. Therefore all physical states have zero square momentum and there is no invariant momentum transfer other than zero entering the theory. Thus the very concept of analytic continuation loses meaning.

This is well seen by considering the theory for $m \neq 0$ where analytic continuation of form factors has been established, and witnessing how the $m \rightarrow 0$ limit is non-analytic. Indeed, for $m/g \ll 1$ and working in the Baluni gauge⁵⁾ it has been established⁶⁾ that the spectrum is described by a Sine-Gordon theory with a Hamiltonian given, for the single flavour case we are discussing, by

$$H = \frac{1}{2} \pi^2 - \frac{1}{2} (\partial_x \varphi)^2 - \alpha (1 - \cos(\beta \varphi)) \quad (17)$$

where

$$\alpha = 4 m^2 N_c, \quad \beta = \sqrt{\frac{4\pi}{N_c}}. \quad (18)$$

The baryon is identified with the soliton state with mass

$$M_s = \frac{8\sqrt{\alpha}}{\beta} = \frac{16}{\sqrt{\pi}} m N_c \quad (19)$$

that vanishes for $m \rightarrow 0$.

The U(1) current is given by

$$j_\mu = -\frac{\beta}{2\pi} \varepsilon_{\mu\nu} \partial^\nu \phi \quad (20)$$

and its matrix element between soliton states defines a U(1) form factor which has been exactly calculated⁸⁾. It depends analytically on a parameter θ related to the momentum transfer t by $t = 4m^2 \operatorname{ch}^2 \theta / 2$. For $t = 0$ ($m \neq 0$) the form factor is equal to N (i.e., the charge). For $m \rightarrow 0$, $t \rightarrow 0$ irrespectively of θ which obviously implies a singular limit. We see therefore that for the chiral case the matrix elements (15) and (16) are not analytically related.

The $m = 0$ limit in the Baluni gauge is indeed subtle. The Hamiltonian (17) loses its non-linear term which generates the topological excitations. It acquires however a fermionic zero mode⁹⁾ whose contribution to the current matrix elements is proportional to $\delta(q)$. It ensures that the baryon charge is N [i.e., Eq. (15)] that does not contribute to the anomaly [i.e., Eq. (16)] which is a $\delta(q^2)$ effect.

In everything said before we considered $N > 1$. The singular behaviour of the $m \rightarrow 0$ limit is also manifested if we consider the continuation to $N \leq 1$. In this region, corresponding to $4\pi \leq \beta^2 \leq 8\pi$ [cf. Eq. (18)], the Sine-Gordon theory of Eq. (17) is equivalent to a repulsive massive Thirring model whose coupling g is related to β through¹⁰⁾

$$1 + \frac{2g}{\pi} \equiv \lambda = \frac{8\pi}{\beta^2} - 1$$

In this region ($\lambda < 1$, $g < 0$) there are no bound states and the soliton of the Sine-Gordon equation corresponds to the Thirring fermion. This is reflected in the observation that the time-like form factors of the Sine-Gordon theory in this region have a cut starting at $4m^2$, unlike the meson pole which was present for $\lambda > 1$ which goes into the unphysical sheet⁸⁾. Note that for $N = 1$ (i.e., $\lambda = 1$) we obtain a free fermion theory. Thus for $\lambda \leq 1$ the intermediate states contributing to the current correlation function will be fermion-antifermion pairs. Therefore, in the $m \rightarrow 0$ limit the anomaly is trivially saturated by the original quark. This apparent change of responsibility in going from $\lambda > 1$ to $\lambda \leq 1$ in the anomaly saturation has little meaning. As long as $m \neq 0$ a free $q\bar{q}$ system is basically different from a meson bound state, but for $m \rightarrow 0$ they become indistinguishable. This is obviously a peculiarity of two dimensions.

We see therefore that $SU(N)_2$, which has a rich light spectrum for $m \neq 0$, has a trivial chiral ($m = 0$) limit. There, the left and right moving worlds are separated and each includes free mesons and baryons which are trivial composites of massless quarks and antiquarks moving parallelly. As it is easy to visualize from this trivial picture, anomalies are saturated by mesons (colinear massless $q\bar{q}$ states!) while baryon-antibaryons do not contribute because they cannot be created by the quark currents from the vacuum. This trivial way of realizing chirality is clearly a characterization of two dimensions and is therefore of little help in understanding how chirality is realized or broken in four-dimensional gauge theories.

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