Ref.TH.3204-CERN

SO(10) UNIFICATION FROM HIGHER DIMENSIONS

C. Wetterich
CERN -- Geneva

ABSTRACT

Gravity in 13 dimensions with ground state $M^4 \times S^9$ exhibits a local $SO(10)$ gauge group. We discuss the dimensional reduction for this model. The model is enlarged to 17 dimensions to include a generation group for four fermion generations.

Ref.TH.3204-CERN
27 November 1981

THEORY OF THE EARTH

CHAPTER I

The Earth is a sphere, and its surface is covered by water. The land is divided into continents and islands. The continents are the large masses of land, and the islands are the small pieces of land. The water is divided into oceans and seas. The oceans are the large bodies of water, and the seas are the smaller bodies of water. The land is divided into countries and states. The countries are the large political divisions, and the states are the smaller political divisions. The water is divided into rivers and lakes. The rivers are the large bodies of water that flow, and the lakes are the smaller bodies of water that do not flow. The land is divided into mountains and valleys. The mountains are the high, rugged areas, and the valleys are the low, flat areas. The water is divided into icebergs and glaciers. The icebergs are the large pieces of ice that float, and the glaciers are the smaller pieces of ice that do not float.

Recently, there has been renewed interest¹⁾⁻⁴⁾ in higher dimensional models of the Kaluza-Klein⁵⁾ type, which unify a local gauge group with gravity. Many of the recent approaches have linked the higher dimensional models with local supersymmetry⁴⁾. In this case the maximal dimension is eleven. Here we renounce supersymmetry and make a more modest attempt of first unifying only the bosonic part of the interactions, including gravity, gauge interactions and scalar interactions. This enables us to consider dimensions higher than eleven, where large orthogonal gauge groups can be easily implemented. For example, a 13 dimensional model with ground state $M^4 \times S^9$ has $SO(10)$ as a local gauge group. After a general dimensional reduction for an arbitrary gauge group we consider this model explicitly. We show that there is an ansatz for massless fermions, where the left-handed (right-handed) fermions transform as 16 ($\overline{16}$) with respect to $SO(10)$. We generalize this approach to 17 dimensions with ground state $M^4 \times S^9 \times S^4$. Here we obtain an ansatz for four massless fermion generations.

Consider the action for a spinor and gravity in d dimensions:

$$S = - \int d^4x \int \frac{d^{d-4}y}{N} \hat{e} \left\{ \frac{1}{4\kappa^2} (\hat{R} + \hat{\lambda}) - i \bar{\psi} \hat{\gamma}^{\hat{m}} D_{\hat{m}} \psi \right\} \quad (1)$$

The d dimensional curvature $\hat{R}$ can be expressed in terms of the vielbein $\hat{e}_{\hat{\mu}}^{\hat{m}}$ using the spin connections $\hat{\omega}_{\hat{m}\hat{n}\hat{p}}$:

$$\begin{aligned} \hat{R} &= \hat{\omega}_{\hat{n}\hat{p}}^{\hat{m}} \hat{\omega}_{\hat{m}}^{\hat{n}\hat{p}} - \hat{\omega}_{\hat{m}\hat{p}}^{\hat{n}} \hat{\omega}_{\hat{n}}^{\hat{m}\hat{p}} - 2 \hat{e}_{\hat{m}}^{\hat{\mu}} \partial_{\hat{\mu}} \hat{\omega}_{\hat{n}}^{\hat{m}} \\ \hat{\omega}_{\hat{m}\hat{n}\hat{p}} &= -\hat{\Omega}_{\hat{m}\hat{n}\hat{p}} + \hat{\Omega}_{\hat{n}\hat{p}\hat{m}} - \hat{\Omega}_{\hat{p}\hat{m}\hat{n}} \\ \hat{\Omega}_{\hat{m}\hat{n}\hat{p}} &= \frac{1}{2} (\hat{e}_{\hat{m}}^{\hat{\mu}} \hat{e}_{\hat{n}}^{\hat{\nu}} - \hat{e}_{\hat{n}}^{\hat{\mu}} \hat{e}_{\hat{m}}^{\hat{\nu}}) \partial_{\hat{\mu}} \hat{e}_{\hat{\nu}}^{\hat{p}} \end{aligned} \quad (2)$$

Here $\hat{e}_{\hat{m}}^{\hat{\mu}}$ is the inverse vielbein and $\hat{e} = \det \hat{e}_{\hat{\mu}}^{\hat{m}}$. The Latin and Greek indices of hatted objects are lowered and raised with $\hat{\eta}_{\hat{m}\hat{n}} = \text{diag}(1, -1, -1, \dots, -1)$ and $\hat{e}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}^{\hat{m}} \hat{e}_{\hat{\nu}}^{\hat{n}}$ and their inverses. The integration involves a four-dimensional integration over Minkowski co-ordinates x^{μ} and a $d-4$ dimensional integration over the internal co-ordinates y^{α} . The normalization factor N is chosen in order to give the usual normalization for the four-dimensional gravitational interactions. This action has only two free parameters, Newton's constant $\kappa^2/4\pi$ and a cosmological constant $\hat{\lambda}$. The covariant derivative of the spinor is given by

$$D_{\hat{m}} \psi = \hat{e}_{\hat{m}}^{\hat{\alpha}} \partial_{\hat{\alpha}} \psi - \frac{1}{2} \hat{\omega}_{\hat{m}\hat{n}\hat{p}} \hat{\Sigma}^{\hat{n}\hat{p}} \psi$$

$$\hat{\Sigma}^{\hat{n}\hat{p}} = \frac{1}{4} [\hat{\gamma}^{\hat{n}}, \hat{\gamma}^{\hat{p}}] , \quad \{ \hat{\gamma}^{\hat{n}}, \hat{\gamma}^{\hat{p}} \} = 2 \hat{\eta}^{\hat{n}\hat{p}} \quad (3)$$

Now we suppose that the ground state of this theory is given by the direct product of the four-dimensional Minkowski space and a $d-4$ dimensional compact space K^{d-4} and that K^{d-4} is invariant under some internal co-ordinate transformations forming a Lie group G . The G transformations are local in the Minkowski co-ordinates:

$$y^{\alpha} \rightarrow y'^{\alpha} = y^{\alpha} + \xi^{\alpha} = y^{\alpha} + \theta^z(x) K_z^{\alpha}(y) \quad (4)$$

The Killing vectors K_z^{α} are related to the infinitesimal generators L_z of G by $L_z = K_z^{\alpha} \partial_{\alpha}$ and obey the commutation relation

$$K_x^{\alpha} \partial_{\alpha} K_y^{\beta} - K_y^{\alpha} \partial_{\alpha} K_x^{\beta} = f_{xy}^z K_z^{\beta} \quad (5)$$

where the f_{xy}^z are the (y independent) structure constants of G .

We know that the theory contains one massless graviton and a massless gauge boson for every generator L_z of G . Thus the first step in our dimensional reduction consists of making an ansatz for the massless spin 1 and spin 2 bosons, neglecting all other spin 1 and spin 2 excitations expected to have a mass of the order of the Planck mass. The ansatz for the vielbein is:

$$\hat{e}_{\mu}^m = e_{\mu}^m(x)$$

$$\hat{e}_{\alpha}^m = 0$$

$$\hat{e}_{\mu}^{\alpha} = g A_{\mu}^z(x) K_z^{\alpha}(y) e_{\alpha}^{\alpha}(y, x)$$

$$\hat{e}_{\alpha}^{\alpha} = e_{\alpha}^{\alpha}(y, x) \quad (6)$$

Here the early Greek and Latin indices refer to the internal $d-4$ dimensional space and the late Greek and Latin indices refer to the usual four-dimensional space. The indices of unhatted objects are lowered and raised by $g_{\mu\nu} = e_{\mu}^m e_{\nu m}$ (the ordinary metric tensor), $g_{\alpha\beta} = e_{\alpha}^a e_{\beta a}$, $\eta_{mn} = \text{diag}(1, -1, -1, -1)$, $\eta_{ab} = -\delta_{ab}$

and their inverses. It is easy to work out the transformation properties under G transformations:

$$\begin{aligned}
 \delta g_{\mu\nu} &= 0 \\
 \delta A_\mu^z &= -\frac{1}{g} \partial_\mu \theta^z - f^z_{xy} \theta^x A_\mu^y \\
 \delta g_{\gamma\delta} &= -\theta^z (T_z)_{\gamma\delta}^{\gamma\delta} g_{\gamma\delta} ; (T_z)_{\gamma\delta}^{\gamma\delta} = \delta_\rho^\delta \partial_\alpha K_z^\gamma + \delta_\alpha^\gamma \partial_\rho K_z^\delta + \delta_\alpha^\gamma \delta_\rho^\delta K_z^\epsilon \partial_\epsilon \\
 \delta \psi &= -\theta^z S_z \psi ; S_z = K_z^\alpha \partial_\alpha - \frac{1}{4} (\tilde{e}_a^\alpha (V_z)_\alpha \tilde{e}_{\gamma b} - \tilde{e}_b^\alpha (V_z)_\alpha \tilde{e}_{\gamma a}) \Sigma^{ab}
 \end{aligned}
 \tag{7}$$

where $\tilde{e}_\alpha^a \equiv \tilde{e}_\alpha^a(y)$ is the internal vielbein of the ground state K^{d-4} and $(V_z)_\alpha^\gamma = \partial_\alpha K_z^\gamma + \delta_\alpha^\gamma K_z^\beta \partial_\beta$.^{*)} Of course, the scalar part $g_{\alpha\beta}$ and the spinor part ψ consist so far of infinitely many irreducible representations, most of which, however, are expected to have masses of the order of the Planck mass. We have to calculate six Latin connections

$$\begin{aligned}
 \hat{\omega}_{mnp} &= \omega_{mnp} \\
 \hat{\omega}_{mna} &= \frac{1}{2} e_{\alpha a} e_m^\alpha e_n^\beta F_{\beta\mu}^\alpha \\
 \hat{\omega}_{amn} &= -\frac{1}{2} e_{\alpha a} e_m^\alpha e_n^\beta F_{\beta\mu}^\alpha \\
 \hat{\omega}_{amb} &= -\frac{1}{2} e_m^\alpha (e_a^\alpha D_\mu e_{\alpha b} + e_b^\alpha D_\mu e_{\alpha a}) \\
 \hat{\omega}_{mab} &= \frac{1}{2} e_m^\alpha (e_a^\alpha D_\mu e_{\alpha b} - e_b^\alpha D_\mu e_{\alpha a}) \\
 \hat{\omega}_{abc} &= \omega_{abc}
 \end{aligned}
 \tag{8}$$

where ω_{mnp} and ω_{abc} are the connections of the four and the $d-4$ dimensional spaces, respectively,

$$F_{\beta\mu}^\alpha = g (\partial_\beta A_\mu^z - \partial_\mu A_\beta^z - g f^z_{xy} A_\beta^x A_\mu^y) K_z^\alpha = g F_{\beta\mu}^z K_z^\alpha \tag{9}$$

and

$$D_\mu e_\alpha^a = \partial_\mu e_\alpha^a - g A_\mu^z (V_z)_\alpha^\gamma e_\gamma^a . \tag{10}$$

The above ansatz then yields the following action:

*) We did not introduce new indices for the objects related to the ground state. The early Greek indices of these objects should, of course, be lowered and raised with the ground state metric $\tilde{g}_{\alpha\beta}$ instead of $g_{\alpha\beta}$.

$$\begin{aligned}
 S &= - \int d^4x e(x) \int \frac{d^{d-4}y}{N} E(y,x) \left\{ \frac{1}{4\kappa^2} (\hat{R} + \hat{\lambda}) - \mathcal{L}_\psi \right\} \\
 e(x) &= \det e_\mu^m \\
 E(y,x) &= \det e_\alpha^a \\
 \hat{R} &= R - \frac{1}{4} g^2 g_{\mu\nu} K_\mu^\alpha K_\nu^\beta F_{\mu\nu}^2 F^{\mu\nu} \\
 &\quad - \frac{1}{4} g^{\mu\nu} (g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma}) D_\mu g_{\gamma\delta} D_\nu g_{\alpha\beta} - \frac{1}{e} D_\mu (e g^{\mu\nu} g^{\alpha\beta} D_\nu g_{\alpha\beta}) + W \\
 \mathcal{L}_\psi &= i \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{i}{2} \bar{\psi} \gamma^m \omega_{mnp} \Sigma^{np} \psi - i g \bar{\psi} \gamma^\mu A_\mu^z S_z \psi \\
 &\quad + i \bar{\psi} \gamma^\alpha \partial_\alpha \psi - \frac{i}{2} \bar{\psi} \gamma^a \omega_{abc} \Sigma^{bc} \psi \\
 &\quad - i g \bar{\psi} \Sigma^{\mu\nu} F_{\mu\nu}^z W_z \psi \\
 &\quad - \frac{i}{4} \bar{\psi} \gamma^\mu \psi g^{\alpha\beta} D_\mu g_{\alpha\beta} \\
 &\quad - \frac{i}{2} \bar{\psi} \gamma^\mu \Sigma^{ab} \bar{e}_a^\alpha \bar{e}_b^\beta \psi \bar{e}_\alpha^c e_c^\gamma D_\mu (e_\gamma^d \bar{e}_{d\beta})
 \end{aligned} \tag{11}$$

Here R is the four-dimensional curvature. The scalar interactions ($W = \omega_{bc}^a \omega_{a\alpha}^{bc} - \omega_{ac}^a \omega_{b\alpha}^{bc} - 2e_a^\alpha \partial_\alpha \omega_{b\alpha}^{ba}$) are so far undetermined. The action contains the required gauge invariant kinetic terms for the gauge bosons, spinors and scalars ($D_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - g A_\mu^z (T_z)_{\alpha\beta}^{\gamma\delta} g_{\gamma\delta}$). It also contains mass terms, Yukawa couplings and possible Pauli interactions ($W_z = \frac{1}{4} K_z^\alpha e_{\alpha a} \gamma^a$) for the spinors. Due to the scalar potential the symmetry G may be broken spontaneously. If $\bar{E}(y)$ and $\bar{g}_{\alpha\beta}(y)$ are the determinant of the vielbein and the metric of the final ground state of the theory, we normalize $N = \int d^{d-4}y \bar{E}(y)$. Then normal gravity is obtained except x dependent deviations of $E(y,x)$ from $\bar{E}(y)$ which could be absorbed by a redefinition of e_μ^m ³⁾. However, the corresponding interactions are suppressed by inverse powers of the Planck mass and we neglect them here. The gauge coupling g can be determined by demanding the usual normalization of the kinetic term for the gauge bosons:

$$\int \frac{d^{d-4}y}{N} \bar{E}(y) \frac{1}{4\kappa^2} \bar{g}_{\mu\nu}(y) K_\mu^\alpha(y) K_\nu^\beta(y) = - \frac{1}{g^2} \delta_{\mu\nu} \tag{12}$$

Typically, $1/g^2$ is of the order $M_p^2 L^2$ where L is a characteristic length of the internal space. Note that small gauge couplings are obtained for L larger than the Planck length (and inversely).

In order to proceed further, we need an ansatz for the spinors ψ and scalars $g_{\alpha\beta}$. The classical scalar potential $V_c \sim \int d^{d-4}y E(W+\hat{\lambda})$ has to be calculated. (However, in order to check if the assumed ground state is stable, quantum fluctuations may have to be included in the effective scalar potential.) In the following, we are mainly interested in the renormalizable interactions contained in S since at mass scales $M \ll M_p$ the non-renormalizable interactions are supposed to be suppressed by powers of M/M_p . (Here we assume that L is not very far from the Planck length.) Since the renormalizable interactions are already largely determined by the G transformation properties of the spinors and scalars, we only need to calculate the gauge coupling g , the normalization for the scalar kinetic term, the renormalizable part of V_c and the mass and Yukawa coupling of the spinors contained in $\bar{\Psi} \gamma^\alpha \partial_\alpha \psi - \frac{1}{2} \bar{\Psi} \gamma^a \omega_{abc} \Sigma^{bc} \psi$.

To be more concrete, we next discuss a 13 dimensional model with ground state $\mathcal{M}^4 \times S^9$. Many things can be learned about this model by embedding it in a flat 14 dimensional model. Thus we first consider a ground state $\mathcal{M}^4 \times B^{10}$ where B^{10} is a flat 10 dimensional manifold ($g_{\alpha\beta}^0 = -\delta_{\alpha\beta}$) with a sharp boundary given by $-y_\alpha y^\alpha \leq L^2$. The internal symmetry group is $SO(10)$. (The 10 dimensional ball is not a very attractive ground state and we consider this example mainly as a technical step. Finally we want to obtain $SO(10)$ from 13 dimensions with ground state $\mathcal{M}^4 \times S^9$.) The 45 Killing vectors are

$$K_z^\alpha = (L_z)^\alpha{}_\beta y^\beta$$

$$(L_z)^\alpha{}_\beta \equiv (L_{AB})^\alpha{}_\beta \equiv -(L_{BA})^\alpha{}_\beta = \delta_B^\alpha \delta_{\beta A} - \delta_A^\alpha \delta_{\beta B}, \quad A, B = 1 \dots 10 \quad (13)$$

and

$$g_{\alpha\beta} K_{AB}^\alpha K_{CD}^\beta = \delta_{AD} y_B y_C - \delta_{AC} y_B y_D - \delta_{BD} y_A y_C + \delta_{BC} y_A y_D.$$

With $\bar{E}(y) = 1$, N is the volume of the 10 dimensional ball and the gauge coupling is

$$\frac{1}{g^2} = \frac{1}{24} \frac{L^2}{K^2} \quad (14)$$

It is easy to make an ansatz for massless scalars:

$$g_{\alpha\beta} = g_{\alpha\beta}(x) = -\delta_{\alpha\beta} - \frac{2\kappa}{\sqrt{55}} \delta_{\alpha\beta} h(x) - 2\kappa h_{\alpha\beta}(x); \quad h_{\alpha\beta} = h_{\beta\alpha}, \quad \text{Tr } h = 0. \quad (15)$$

Since $\partial_\alpha K_Z^\gamma = (L_Z)^\gamma_\alpha$, $k(x)$ is an $SO(10)$ singlet and $h_{\alpha\beta}(x)$ transforms as a second rank symmetric tensor. For this ansatz, one has $\omega_{abc} = 0$, $W = 0$ and for $\hat{\lambda} = 0$ there is no scalar potential. It is also easy to see that the fermion mass vanishes for a y independent ansatz $\psi = \psi(x)$. However, the non-vanishing fermion mass is not accidental as in the scalar case, but rather due to a chiral symmetry. In fact, one has

$$S_Z \equiv S_{AB} \equiv -S_{BA} = \sum_{AB} \quad (16)$$

and $\psi(x)$ therefore transforms as $(16, 2, 1) + (\overline{16}, 1, 2)$ with respect to $SO(10) \times \mathcal{L}$ (Lorentz group). Since the left-handed and the right-handed spinors transform differently with respect to $SO(10)$, no $SO(10)$ invariant mass term or Yukawa coupling to a $SO(10)$ singlet or second rank symmetric tensor is possible^{*)}. To resume this 14 dimensional example we give the Lagrange density for the renormalizable interactions:

$$\begin{aligned} \mathcal{L} = & i \bar{\psi} \gamma^\mu (\partial_\mu - g A_\mu^Z S_Z) \psi - \frac{1}{4} F_{\mu\nu}^Z F^{\mu\nu Z} \\ & + \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{1}{4} D_\mu h_{\alpha\beta} D^\mu h^{\alpha\beta} \end{aligned} \quad (17)$$

We now turn to the more interesting case of 13 dimensions with ground state $\mathcal{M}^4 \times S^9$ and symmetry group $SO(10)$. Since S^9 is contained in B^{10} some results of the above 14 dimensional case can be used without explicit calculation. Since $SO(10) \times \mathcal{P}$ (Poincaré group) is a subgroup of gen_{13} (we design by gen_d the group of local co-ordinate transformations and Lorentz rotations in d dimensions) and gen_{13} is a subgroup of gen_{14} , all the $SO(10) \times \mathcal{P}$ representations $g_{\mu\nu}(x)$, $A_\mu^Z(x)$, $k(x)$, $h_{\alpha\beta}(x)$ and $\psi(x)$ we found in the $d = 14$ example can also be obtained from gen_{13} representations. However, the action of the 13 dimensional model does not contain all the gen_{13} representations we would obtain by reducing the action in 14 dimensions with respect to gen_{13} . In other words, we have to subtract from $\mathcal{L}$ in Eq. (17) all the renormalizable pieces which come from gen_{13} representations contained in the 14 dimensional action but not contained in the 13 dimensional action. The reduction from gen_d to gen_{d-1} can be done by two steps: the $d-1$ dimensional manifold on which gen_{d-1} acts is a submanifold of a d dimensional manifold characterized by $f(x^{\hat{\mu}}) = 0$. Any co-ordinate dependent field $\phi(x^{\hat{\mu}})$ can be written as $\phi(\tilde{x}^{\hat{\mu}}, f)$, where $\tilde{x}^{\hat{\mu}}$ are co-ordinates of the $d-1$ dimensional manifold. Then by reduction from gen_d to gen_{d-1}

^{*)} In our case the Pauli term also vanishes.

$\phi(x^{\hat{\mu}})$ is reduced to infinitely many representations $\phi_f(\tilde{x}^{\hat{\mu}})$, one for every value of f . In our case, this step of the reduction consists only in replacing $\int d^{10}y$ by $\int d^{10}y \delta(-y_\alpha y^\alpha - L^2)$ in Eq. (11) where L is now the "radius" of S^9 . The second step consists in the reduction of the metric (or equivalently the vielbein) and the spinor for a given value of f . The d dimensional metric reduces to the $d-1$ dimensional metric, a $d-1$ dimensional vector and a gen_{d-1} scalar. The reduction of the spinor depends on the dimension d . For d even, an irreducible spinor in d dimensions is also irreducible in $d-1$ dimensions. For d odd the spinor reduces to two different $2^{(d-3)/2}$ component spinors. In our case the 64 component spinor of gen_{14} is irreducible with respect to gen_{13} (for L fixed!) and $\psi(x)$ is therefore entirely contained in the spinor of the 13 dimensional interaction. This has an important consequence: since we know the $SO(10)$ transformation properties of $\psi(x)$ we again have an ansatz for a spinor not allowing a mass term because of chiral symmetry! Thus the 13 dimensional model with ground state $M^4 \times S^9$ not only gives rise naturally to the interesting unification group $SO(10)$ but it also has a good chance ^{*)} to have massless fermions in the wanted representations! On the other hand, the gauge bosons A_μ^z and the scalars k and $h_{\alpha\beta}$ are not completely contained in the 13 dimensional action but they also have contributions from the 13 dimensional vector and from the scalar contained in the 14 dimensional metric. The subtraction of these contributions from $\mathcal{L}$ (17) in general will lead to a different gauge coupling, a new normalization of the scalars and, most important, to a non-vanishing W . This has to be calculated explicitly. In order to carry out these calculations, we have to make a suitable projection from B^{10} objects to S^9 objects ^{**)}. The ansatz is easily found:

$$\begin{aligned} K_{\hat{z}}^{\hat{\alpha}} &= K_z^\alpha F_\alpha^{\hat{\alpha}} \\ g_{\hat{z}\hat{\beta}} &= F_z^\alpha F_\beta^\alpha g_{\alpha\beta}(x) \end{aligned} \tag{18}$$

*) Note that an ansatz for massless fermions is of course not sufficient to establish the content of massless fermions of the model. One also has to prove that there are no other massless fermions and that there is no mass term involving fermions contained in the ansatz and fermions not contained in the ansatz. This requires a more complete knowledge of the infinitely many representations contained in $\psi(y,x)$ than we have so far.

**) In this paper we use the tool of dimensional reduction for two different purposes. The reduction from 14 dimensions to 13 dimensions is purely a technical device and the fourteenth dimension has no physical significance. On the other hand, the 13 dimensions are assumed to correspond to reality. The reduction from 13 to 4 dimensions is not a technical step to obtain a four-dimensional model which could also have been obtained otherwise. It corresponds to the reduced ground state symmetry compared with the symmetry of the action in a 13 dimensional world.

Here α, β are the old B^{10} indices and $\tilde{\alpha}, \tilde{\beta} = 1 \dots 9$ are S^9 indices. The $F_{\tilde{\alpha}}^{\alpha}$ are given by ($\phi^{\tilde{\alpha}}$ are S^9 co-ordinates):

$$\begin{aligned} F_2^{\alpha} &= \frac{\partial y^{\alpha}}{\partial \phi^{\tilde{\alpha}}} & F_{\alpha}^{\tilde{\alpha}} &= \frac{\partial \phi^{\tilde{\alpha}}}{\partial y^{\alpha}} \\ F_{\tilde{\alpha}}^{\alpha} F_{\alpha}^{\tilde{\beta}} &= \delta_{\tilde{\alpha}}^{\tilde{\beta}} & F_{\alpha}^{\tilde{\alpha}} F_{\tilde{\alpha}}^{\beta} &= P_{\alpha}^{\beta} \end{aligned} \quad (19)$$

where P_{α}^{β} projects an infinitesimal co-ordinate difference dy^{α} from B^{10} on S^9 . We find

$$\dot{g}_{\tilde{\alpha}\tilde{\beta}} K_{AB}^{\tilde{\alpha}} K_{CD}^{\tilde{\beta}} = \delta_{AD} y_B y_C - \delta_{AC} y_B y_D - \delta_{BD} y_A y_C + \delta_{BC} y_A y_D$$

and therefore

$$\frac{1}{g^2} = \frac{L^2}{20\kappa^2} \quad (20)$$

So far, we did not succeed to calculate the classical potential for the 54 representation in $g_{\alpha\beta}(x)$. The classical potential for the $SO(10)$ singlet field alone is easily obtained and can be used to express L as a function of $\hat{\lambda}$ and κ . The ansatz is

$$\begin{aligned} g_{\tilde{\alpha}\tilde{\beta}} &= \dot{g}_{\tilde{\alpha}\tilde{\beta}}(y) f(x) \\ f(x) &= 1 + \frac{2\kappa}{145} k(x) \end{aligned} \quad (21)$$

where the kinetic term of $k(x)$ is properly normalized. The singlet corresponds to x dependent fluctuations of the radius of S^9 and L is determined by requiring that the effective potential for k has its minimum at $k = 0$. The classical potential is given by

$$V_c = \frac{1}{4\kappa^2} f^{\frac{9}{2}} \left(-\frac{72}{L^2} f^{-1} + \hat{\lambda} \right). \quad (22)$$

Here we see the need of a positive $\hat{\lambda}$ in order to stabilize the potential. The minimum is at $f = 1$ provided

$$L^2 = \frac{56}{\hat{\lambda}} \quad (23)$$

corresponding to

$$g^2 = \frac{5}{56} (4\kappa^2 \hat{\lambda}) \quad (24)$$

It is interesting to note that for $4\kappa^2\hat{\lambda}$ of the order one (corresponding to only one relevant length scale) g^2 is predicted to be small. This is due to the appearance of a natural small parameter $1/D$, D (equal to nine in our case) being the dimension of the internal hypersphere S^D .

We may expand V_c for L taking its classical value. The renormalizable terms up to the order k^4 are given by

$$\begin{aligned} V_c^{(\kappa)} &= \lambda_0 + \frac{1}{2} \mu^2 k^2 + \frac{1}{6} \gamma k^3 + \frac{1}{24} \lambda k^4 \\ \lambda_0 &= -\frac{1}{5} g^2 / \kappa^4 \\ \mu^2 &= \frac{7}{25} g^2 / \kappa^2 \\ \gamma &= \frac{14}{15 \cdot 15} g^2 / \kappa \\ \lambda &= \frac{7}{25} g^2 \end{aligned} \quad (25)$$

We note the appearance of a cosmological constant with negative sign, only somewhat smaller than the Planck mass squared (for $g^2 < 1$). Such a large cosmological constant persisting at length scales much larger than the Planck length would be completely incompatible with observation. Furthermore, it implies that $\mathcal{M}^4 \times S^9$ is not a solution of Einstein's equation in 13 dimensions, since the scalar potential was derived assuming $R = 0$. We hope, however, that strong quantum fluctuations may renormalize λ_0 to a value consistent with observation. (Remember that the problem of a non-vanishing cosmological constant is common in unified gauge theories using spontaneous symmetry breaking.) The singlet mass term is somewhat smaller than the Planck mass and the quartic scalar coupling is of order g^2 .

Next we will show how the dimensional reduction approach using orthogonal gauge groups could in principle give a solution to the problem of different fermion generations. Consider the 17 dimensional case with ground state $\mathcal{M}^4 \times S^9 \times S^4$. The gauge group is $SO(10) \times SO(5)$ and the spinor has 256 components. We will show that ψ contains a massless spinor transforming as $(16, 4, 2, 1) + (\overline{16}, 4, 1, 2)$ with respect to $SO(10) \times SO(5) \times \mathbb{Z}_2$. We may again embed $\mathcal{M}^4 \times S^9 \times S^4$ in the flat 19 dimensional manifold $\mathcal{M}^4 \times B^{10} \times B^5$ where the mass term of the 512 component γ independent spinor $\psi(x)$ vanishes. With respect to $SO(10) \times SO(5)$ $\psi(x)$ transforms as $2 \times ((16, 4, 2, 1) + (\overline{16}, 4, 1, 2))$. However, the 19 dimensional spinor transforms as two identical 256 component spinors with respect to gen_{17} and we need to consider only one of them in the 17 dimensional action. Thus we

are left with only one $(16,4,2,1)+(\overline{16},4,1,2)$ representation and $SO(10)$ invariance guarantees a vanishing mass for four fermion generations. Note that our approach using orthogonal groups requires the existence of a fourth fermion generation!

If we make an ansatz for the scalars in 17 dimensions similar as in the previous 13 dimensional example, the scalars transform as $(54,1)+(1,14)+2(1,1)+(10,5)$ with respect to $SO(10) \times SO(5)$. It is interesting to note that a Yukawa coupling of the $(10,5)$ scalar to the fermions is allowed by $SO(10) \times SO(5)$ invariance. After spontaneous symmetry breaking of $SU(2)_L$ this could finally give masses to the fermions of order M_L or smaller.

We may again consider the interactions of the two $SO(10) \times SO(5)$ singlet scalars. The corresponding ansatz is

$$g_{\alpha\beta} = \dot{g}_{\alpha\beta} \cdot f(x)$$

$$g_{\bar{\alpha}\bar{\beta}} = 0$$

$$g_{\alpha\bar{\beta}} = \dot{g}_{\alpha\bar{\beta}} \cdot e(x)$$

(26)

where $\alpha, \bar{\beta}$ are indices related to the internal S^4 manifold. Here $f(x) = 1 + (2\kappa/\sqrt{45})k(x)$ and $e(x) = 1 + (2\kappa/\sqrt{10})s(x)$ correspond to x dependent fluctuations of the radii of S^9 and S^4 and the kinetic terms for $k(x)$ and $s(x)$ are normalized as usual. The classical potential is given by:

$$V_c = \frac{1}{4\kappa^2} f^{3/2} e^2 \left(-\frac{72}{L^2} f^{-1} - \frac{12}{L_4^2} e^{-1} + \hat{\lambda} \right) \quad (27)$$

where L_4 is the radius of S^4 . The classical situation is very different from the above 13 dimensional case: for small values of e (or f) the classical potential is unbounded from below for f (or e) going to infinity. In fact, the cosmological constant in the classical potential for the f field alone is given by $\hat{\lambda} - (12/L_4^2 e)$, which is negative for small values of $L_4^2 e$. However, we remind that the classical potential is a bad approximation in the presence of strong interactions. There are several possibilities for the stabilization of the scalar potential: one could arise from the fluctuations of the non-singlet scalar fields. It may also happen that the $SO(5)$ symmetry is broken at the

Planck scale which would even modify the form of the classical potential for f . Finally we note that the $SO(5)$ gauge coupling becomes strong for small values of L_4 . We think that the question of stability of a $M^4 \times S^9 \times S^4$ ground state is far from being answered.

In conclusion, a 17 dimensional model with ground state $M^4 \times S^9 \times S^4$ looks at first sight very promising as a candidate for the unification of gravity, gauge interactions and Yukawa interactions. It exhibits the interesting unification group $SO(10)$ with four generations of left (right) handed fermions transforming as 16 ($\overline{16}$). However, there are still many serious difficulties. First, it is not at all clear if the assumed ground state can be realized. Second, in order to get a realistic pattern of spontaneous symmetry breaking one needs some scalar masses much lower than the Planck mass, which is the intrinsic mass scale of the model. It may be that strong scalar or Yukawa couplings can induce such small masses. Finally, we have so far no ansatz for the scalar representations required to break local $B-L$ and $SU(4)_C$ symmetry contained in $SO(10)$ at a scale much above M_L . This is necessary to give a heavy mass to the right-handed neutrinos. As well, we need a suitable breaking pattern of $SO(5)$ in order to get the observed mass pattern for the fermions. An interesting possibility of a solution of some of these problems could be the formation of scalar fermion-fermion bound states which would have the required $SO(10)$ transformation properties (for example, a 126 dimensional representation). Thus, even if 17 dimensional gravity offers a very interesting structure, its usefulness for low energy predictions has still to be proved.

ACKNOWLEDGEMENTS

The author would like to thank J. Ellis for helpful discussions.

REFERENCES

- 1) E. Witten, Nucl.Phys. B186 (1981) 412.
This paper also gives a list of references for earlier research on the subject.
- 2) Y.M. Cho and P.G.O. Freund, Phys.Rev. D12 (1975) 1711;
Y.M. Cho and P.S. Jang, Phys.Rev. D12 (1975) 3789.
- 3) J. Scherk and J.H. Schwarz, Nucl.Phys. B153 (1979) 61.
- 4) E. Cremmer and B. Julia, Nucl.Phys. B159 (1979) 141.
- 5) Th. Kaluza, Sitzungsber.Preuss.Akad.Wiss.Berlin, Math.-Phys. K1 (1921) 966;
O. Klein, Z.Phys. 37 (1926) 895.